Four-dimensional Lattice Gauge Theory with ribbon categories and the Crane–Yetter state sum

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Abstract

Lattice Gauge Theory in 4-dimensional Euclidean space-time is generalized to ribbon categories which replace the category of representations of the gauge group. This provides a framework in which the gauge group becomes a quantum group while space-time is still given by the ‘classical’ lattice. At the technical level, this construction generalizes the Spin Foam Model dual to Lattice Gauge Theory and defines the partition function for a given triangulation of a closed and oriented piecewise-linear 4-manifold. This definition encompasses both the standard formulation of $d=4$ pure Yang–Mills theory on a lattice and the Crane–Yetter invariant of 4-manifolds. The construction also implies that a certain class of Spin Foam Models formulated using ribbon categories are well-defined even if they do not correspond to a Topological Quantum Field Theory.

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1 Introduction

The formulation of gauge theory on a lattice [1] combines manifest gauge symmetry with the path integral approach although space-time cannot be retained as a smooth manifold and is replaced instead by a discrete structure. In the present paper Lattice Gauge Theory (LGT) always refers to pure gauge theory in Euclidean space-time.

LGT offers a number of generalizations that do not have a naïve continuum analogy such as gauge theory with finite gauge groups. Furthermore in three dimensions it is possible to define LGT for quantum groups [2,3]. Combining the various actions and Boltzmann weights with suitable ‘gauge groups’ (finite groups, Lie groups or quantum groups), this model has several special cases that belong to different branches of physics and mathematics. It is at the centre of the relation between LGT with Yang–Mills [1] or with Chern–Simons action [4,5], the Turaev–Viro invariant of 3-manifolds [6,7], a purely algebraic construction of Topological Quantum Field Theory [4,6] and 3-dimensional Euclidean quantum gravity without or with cosmological constant [8].

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At least some of the above constructions are known to have analogies in four dimensions. Even though the question of which is a suitable unified model remains unsolved in full generality, some of the relations known from three dimensions persist also in four dimensions. In the present paper we concentrate one the standard formulation of LGT and on the Crane–Yetter state sum [9, 10]. We present a definition which encompasses both and generalizes four-dimensional LGT to quantum groups. Technically this is realized for ribbon categories which arise as the categories of representations of certain quantum groups and which replace the category of representations of the gauge group of LGT.

The main result of the present paper is the existence of such a generalized LGT in four-dimensional Euclidean space-time using ribbon categories. This model contains the Crane–Yetter state sum as a special case for a particular Boltzmann weight and agrees on the other hand with the Spin Foam Model which is strong-weak dual to LGT if the ribbon category is the category of finite-dimensional representations of a compact Lie group. Beyond this, it provides a definition of Spin Foam Models in \( d = 4 \) using ribbon categories which includes in particular the proof that this construction is well-defined even in cases in which the model does not correspond to a Topological Quantum Field Theory.

At the technical level, the construction of these Spin Foam Models using ribbon categories can be motivated from the following observations. From the study of non-perturbative quantum gravity it has emerged that LGT admits a reformulation as a Spin Foam Model — see, for example [11, 12, 8]. Many models of interest in quantum gravity are either Topological Quantum Field Theories and use `delta-functions' as Boltzmann weights (for example [13]), or they are topological up to constraints which do not change the weights, but restrict the set of admissible representations. This is the case for some versions of the Barrett–Crane model [14].

LGT, however, admits more general Boltzmann weights,

\[
    w: G \to \mathbb{R}, \quad g \mapsto \exp(-s(g)). \tag{1.1}
\]

Here the compact Lie group \( G \) is the gauge group, the (local) action \( s: G \to \mathbb{R} \) is a real, bounded and \( L^2 \)-integrable class function, and the Boltzmann weight \( w(g) \) is evaluated for each plaquette of the lattice. This model encompasses lattice Yang–Mills theory, but it is not restricted to this case. For general background on LGT the reader is referred to standard textbooks, for example [15, 16].

The Spin Foam Model corresponding to the standard formulation of LGT on a hypercubic lattice was constructed in detail in [17] where it was found that it generalizes the strong-weak dual of LGT which had been known only in the Abelian case [18, 19] and for \( SU(2) \) in \( d = 3 \) [20] before. The Boltzmann weight (1.1) enters the Spin Foam Model via the coefficients \( \tilde{w}_\rho \) of its character expansion,

\[
    w(g) = \sum_{\rho \in \mathcal{R}} \tilde{w}_\rho \chi(\rho)(g), \quad \tilde{w}_\rho = \dim V_\rho \int_G \chi(\rho)(g)w(g) \, dg. \tag{1.2}
\]

Here \( \chi(\rho): G \to \mathbb{C} \) denotes the character of the finite-dimensional irreducible representation \( V_\rho \), the sum is over equivalence classes of finite-dimensional irreducible representations of \( G \), and \( \int_G \) is the normalized Haar measure on \( G \).

The way the coefficients \( \tilde{w}_\rho \) appear in the Spin Foam Model dual to LGT [17] compared with the Ooguri state sum [13] indicates that there exists a unified construction encompassing both. In addition, the fact that the Crane–Yetter state sum [9, 10] can be understood as a generalization of the Ooguri model to quantum groups, suggests the construction given in the present paper.

The strategy for the definition of \( d = 4 \) LGT using ribbon categories is as follows. The construction is based on a triangulation of a closed and oriented piecewise-linear four-manifold...
which is specified by an abstract combinatorial complex. In the special case of a Lie group symmetry, the definition shall coincide with the Spin Foam Model dual to LGT if that LGT is formulated on the 2-complex dual to the triangulation (note that we formulate the Spin Foam Model on the triangulation itself following [10]). In the Lie group case, both pictures are available: the Spin Foam Model on the triangulation and LGT on the dual 2-complex. They are dual to each other in the sense of [17]. Physically this means strong-weak duality between LGT and the Spin Foam Model while on the mathematical side the two models are related by a Tannaka–Krein like reconstruction theorem relating LGT (formulated in terms of the gauge group $G$) with the spin foam model (formulated in terms of the category of representations $\text{Rep}G$). For details on quantum groups, ribbon categories and the reconstruction theorems, the reader is referred to standard textbooks such as [21,22].

The generalization takes place in the spin foam picture where the category $\text{Rep}G$ is replaced by a suitable ribbon category $\mathcal{C}$. Loosely speaking, using the reconstruction theorems, this provides a definition of LGT in which the gauge group is replaced by a quantum group. Technically, the notion of gauge group is lost, but one can think of replacing the algebra of representation functions $C_{\text{alg}}(G)$ by a non-commutative algebra (a suitable ribbon Hopf algebra) while space-time is still given by the ‘classical’ lattice.

The generalization from $\text{Rep}G$ to a generic ribbon category $\mathcal{C}$ involves choices of the ordering of tensor factors and choices of the braiding whenever tensor factors are exchanged. These choices are not at all obvious from the Lie group case which involves only the symmetric category $\text{Rep}G$.

The method to achieve a consistent definition in the Spin Foam picture is to choose a linear order of vertices for the combinatorial complex and to define the partition function in a way that employs special choices and that refers explicitly to that order. It is then possible to show in a second step that the partition function is actually independent of the order (combinatorially invariant) and is thus well-defined for a given triangulation. This approach can be seen as a generalization to four dimensions of the strategy which Barrett and Westbury [7] employ in their approach to the Turaev–Viro invariant [6].

Another point of view on the definition given in the present paper is related to the construction of the Crane–Yetter state sum in [10]. The authors of [10] first show that the state sum is independent of the triangulation which in our terminology relies on the choice of a particular Boltzmann weight. Triangulation independence then implies combinatorial invariance and thus establishes that the state sum is well-defined. As an alternative proof, it is conceivable to show combinatorial invariance in the first step. This holds for any choice of Boltzmann weights. One could then prove in a second step that the choice of special Boltzmann weights implies triangulation independence by standard arguments as in [13,9,10]. The construction presented in the present paper can be viewed as the first of these two steps.

Finally, we would like to mention D. V. Boulantov’s approach to LGT for quantum groups in 3 dimensions [2]. His construction makes use of the general result of Reshetikhin and Turaev [23] establishing a functor from the category of ribbon graphs in $\mathbb{R}^3$ to the ribbon category $\mathcal{C}$. The strategy in [2] is to construct a suitable ribbon graph in the triangulated manifold which then yields a well-defined partition function as the quantum trace of a ribbon morphism.

A related definition of $d = 3$ LGT for quantum groups was developed by R. Oeckl [3] in which the duality between LGT and its dual Spin Foam Model is understood entirely in terms of manipulations of ribbon graphs. The approach of [3] also develops the correspondence of ribbon categories with suitable quantum groups, namely coribbon Hopf algebras, in a way that transparently generalizes the duality transformation of the Lie group case.

However, since the Reshetikhin–Turaev functor is available only for ribbon graphs in $\mathbb{R}^3$, these approaches do not have a direct generalization to higher dimension. In the present
paper, we use the functor mainly to justify diagrammatic calculations.

The present paper is organized as follows. In Section 2, we review some mathematical background on the Peter–Weyl theory for compact Lie groups and on ribbon categories, and we introduce our notation for combinatorial and simplicial complexes. The duality transformation for LGT with Lie gauge groups which was derived in [17] on a cubic lattice is reviewed in Section 3 and formulated there on a 2-complex. In Section 4, we define the Spin Foam Model generalizing the dual of LGT to suitable ribbon categories. This section contains the definition of the partition function, the proof that it is well-defined and comments on the construction of observables and on the role played by the gauge transformations in the Spin Foam Model. In Section 5, we indicate how these definitions specialize to the standard formulation for LGT with Lie gauge groups which was derived in [17] on a cubic lattice is reviewed. In Section 6, we briefly summarize definitions and basic statements related to the algebra of representation functions \( C_{\text{alg}}(G) \) of \( G \) where \( G \) is a compact Lie group (or a finite group). These results are needed in Section 3 in order to present the duality transformation relating LGT and the Spin Foam Model. For more details, the reader is referred to the introduction of [17] or to textbooks such as [24, 25].

## 2 Preliminaries

### 2.1 Peter–Weyl Theory

In this section, we briefly summarize definitions and basic statements related to the algebra of representation functions \( C_{\text{alg}}(G) \) of \( G \) where \( G \) is a compact Lie group (or a finite group). These results are needed in Section 3 in order to present the duality transformation relating LGT and the Spin Foam Model. For more details, the reader is referred to the introduction of [17] or to textbooks such as [24, 25].

#### 2.1.1 Representation functions

Finite-dimensional complex vector spaces on which \( G \) is represented are denoted by \( V_\rho \) and by \( \rho : G \to \text{Aut}V_\rho \) the corresponding group homomorphism. Let \( \bar{\mathcal{R}} \) denote a set containing one unitary representative of each class of finite-dimensional representations and \( \mathcal{R} \) the subset of irreducible representations. For a representation \( \rho \in \bar{\mathcal{R}} \), the dual representation is denoted by \( \rho^* \), and the dual vector space of \( V_\rho \) by \( V_{\rho^*} \). The dual representation is given by \( \rho^* : G \mapsto \text{Aut}V_{\rho^*} \), where \( \rho^*(g) : V_{\rho^*} \to V_{\rho^*}, \eta \mapsto \eta \circ \rho(g^{-1}) \), i.e. \( (\rho^*(g)\eta)(v) = \eta(\rho(g^{-1})v) \) for all \( v \in V_{\rho^*} \). There exists a one-dimensional ‘trivial’ representation of \( G \) which is denoted by \( V_{[1]} \cong \mathbb{C} \).

For the unitary representations \( V_\rho, \rho \in \bar{\mathcal{R}} \), there exist standard sesquilinear scalar products \( \langle \cdot, \cdot \rangle \) and orthonormal bases \( (v_j) \) in such a way that the basis \( (v_j) \) of \( V_\rho \) is dual to the basis \( (\eta^j) \) of \( V_{\rho^*} \); i.e. \( \eta^j(v_k) = \delta_k^j \). Duality is here given by the scalar product, i.e. \( \langle v_j, v_k \rangle = \eta^j(v_k) \) and \( \langle \eta^j, \eta^k \rangle = \eta^k(v_j), 1 \leq j, k \leq \dim V_\rho \).

The complex-valued functions

\[
t^\rho_{\eta,v} : G \to \mathbb{C}, \quad g \mapsto \eta(\rho(g)v),
\]

where \( \rho \in \bar{\mathcal{R}}, v \in V_\rho \) and \( \eta \in V_{\rho^*} \), are called representation functions of \( G \). They form a commutative and associative unital algebra over \( \mathbb{C} \),

\[
C_{\text{alg}}(G) := \{ t^\rho_{\eta,v} : \rho \in \bar{\mathcal{R}}, v \in V_\rho, \eta \in V_{\rho^*} \},
\]

whose operations are given by

\[
(t^\rho_{\eta,v} + t^\sigma_{\theta,w})(g) := t^\rho_{\eta+\theta,v+w}(g),
\]

\[
(t^\rho_{\eta,v} \cdot t^\sigma_{\theta,w})(g) := t^\rho_{\eta \otimes \theta,v \otimes w}(g),
\]

for all \( \rho, \sigma \in \bar{\mathcal{R}}, v, w \in V_{\rho}, \eta, \theta \in V_{\rho^*} \).
for \( \rho, \sigma \in \mathcal{R} \) and \( v \in V_\rho, w \in V_\sigma, \eta \in V_\rho^*, \vartheta \in V_\sigma^* \) and \( g \in G \). The zero element of \( C_{\text{alg}}(G) \) is 
\[ t_{00}^{[1]}(g) = 0 \]
and its unit element \( t_{11}^{[1]}(g) = 1 \) where the normalization is such that \( \eta(v) = 1 \).

The algebra \( C_{\text{alg}}(G) \) is furthermore equipped with a Hopf algebra structure employing the coproduct \( \Delta : C_{\text{alg}}(G) \to C_{\text{alg}}(G) \otimes C_{\text{alg}}(G) \cong C_{\text{alg}}(G \times G) \), the co-unit \( \varepsilon : C_{\text{alg}}(G) \to \mathbb{C} \) and the antipode \( S : C_{\text{alg}}(G) \to C_{\text{alg}}(G) \) which are defined by
\[
\Delta t_{\eta,v}^{(\rho)}(g, h) := t_{\eta,v}^{(\rho)}(g \cdot h),
\]
\[
\varepsilon t_{\eta,v}^{(\rho)} := t_{\eta,v}^{(\rho)}(1),
\]
\[
S t_{\eta,v}^{(\rho)}(g) := t_{\eta,v}^{(\rho)}(g^{-1}),
\]
for \( \rho \in \tilde{\mathcal{R}} \) and \( v \in V_\rho, \eta \in V_\rho^* \) and \( g, h \in G \). For unitary representations, the antipode relates a representation with its dual which is just the conjugate representation,
\[
S t_{nm}^{(\rho)}(g) = t_{nm}^{(\rho^*)}(g) = t_{nm}(g).
\]
The bar denotes complex conjugation.

### 2.1.2 Peter–Weyl decomposition and theorem

The structure of the algebra \( C_{\text{alg}}(G) \) can be understood if \( C_{\text{alg}}(G) \) is considered as a representation of \( G \times G \) by combined left and right translation of the function argument.

**Theorem 2.1 (Peter–Weyl decomposition).** Let \( G \) be a compact Lie group (or a finite group).

1. There is an isomorphism
\[
C_{\text{alg}}(G) \cong_{G \times G} \bigoplus_{\rho \in \mathcal{R}} (V_\rho^* \otimes V_\rho),
\]
of representations of \( G \times G \). Here the direct sum is over one unitary representative of each equivalence class of finite-dimensional irreducible representations of \( G \). The direct summands \( V_\rho^* \otimes V_\rho \) are irreducible as representations of \( G \times G \).

2. The direct sum in (2.6) is orthogonal with respect to the \( L^2 \)-scalar product on \( C_{\text{alg}}(G) \) which is formed using the Haar measure on \( G \) on the left hand side, and using the standard scalar products on the right hand side, namely
\[
\langle t_{\eta,v}^{(\rho)}; t_{\vartheta,w}^{(\sigma)} \rangle_{L^2} = \int_G t_{\eta,v}^{(\rho)}(g) \cdot t_{\vartheta,w}^{(\sigma)}(g) \, dg = \frac{1}{\text{dim} V_\rho} \delta_{\rho\sigma} \langle \eta; \vartheta \rangle \langle v; w \rangle,
\]
where \( \rho, \sigma \in \mathcal{R} \) are irreducible. The Haar measure is denoted by \( \int_G \) and normalized such that \( \int_G \, dg = 1 \).

Each representation function \( f \in C_{\text{alg}}(G) \) can thus be decomposed according to (2.6) such that its \( L^2 \)-norm is given by
\[
||f||^2_{L^2} = \sum_{\rho \in \mathcal{R}} \frac{1}{\text{dim} V_\rho} ||f_\rho||^2,
\]
where \( f_\rho \in V_\rho^* \otimes V_\rho \cong \text{Hom}(V_\rho, V_\rho), \rho \in \mathcal{R}, \) and all except finitely many \( f_\rho \) are zero.

**Theorem 2.2 (Peter–Weyl theorem).** Let \( G \) be a compact Lie group. Then \( C_{\text{alg}}(G) \) forms a dense subset of \( L^2(G) \).
The characters \( \chi^{(\rho)} : G \to \mathbb{C} \) associated with the finite-dimensional unitary representations \( \rho \in \tilde{\mathcal{R}} \) of \( G \) are given by the traces,

\[
\chi^{(\rho)} := \dim V_{\rho} \sum_{j=1}^{\text{dim } V_{\rho}} t_{jj}^{(\rho)}.
\]

(2.9)

Each class function \( f \in C_{\text{alg}}(g) \) can be character-decomposed

\[
f(g) = \sum_{\rho \in \tilde{\mathcal{R}}} \chi^{(\rho)}(g) \hat{f}_\rho,
\]

where \( \hat{f}_\rho = \dim V_{\rho} \int_G \overline{\chi^{(\rho)}(g) f(g)} \, dg \),

(2.10)

such that the completion of \( C_{\text{alg}}(G) \) to \( L^2(G) \) is compatible with this decomposition.

### 2.1.3 The Haar measure

The Haar measure on \( G \) can be understood in terms of the Peter–Weyl decomposition (2.6) as follows.

**Proposition 2.3.** Let \( G \) be a compact Lie group (or a finite group) and \( \rho \in \tilde{\mathcal{R}} \) be a finite-dimensional unitary representation of \( G \) with the orthogonal decomposition

\[
V_{\rho} \cong \bigoplus_{j=1}^{k} V_{\tau_j}, \quad \tau_j \in \mathcal{R}, k \in \mathbb{N},
\]

(2.11)

into irreducible components \( \tau_j \). Let \( P^{(j)} : V_{\rho} \to V_{\tau_j} \subseteq V_{\rho} \) be the \( G \)-invariant orthogonal projectors associated with the above decomposition. Assume that precisely the first \( \ell \) components \( \tau_1, \ldots, \tau_\ell, 0 \leq \ell \leq k \), are equivalent to the trivial representation. Then the Haar measure of a representation function \( t_{mn}^{(\rho)} \), \( 1 \leq m,n \leq \dim V_{\rho} \), is given by

\[
\int_G t_{mn}^{(\rho)}(g) \, dg = \sum_{j=1}^{\ell} P^{(j)m} P^{(j)n}, \quad P^{(j)m} = \eta^m(w^{(j)}), \quad P^{(j)n} = \vartheta^{(j)}(v_n).
\]

(2.12)

Here \((v_n)\) and \((\eta^m)\) are dual bases of \( V_{\rho} \) and \( V_{\rho}^* \), the \( w^{(j)} \) are normalized vectors in \( V_{\tau_j} \subseteq V_{\rho} \), and \( \vartheta^{(j)} \) denotes the linear form dual to \( w^{(j)} \).

### 2.2 Ribbon Categories

Ribbon categories are used in the present paper as a generalization of the category of representations of the gauge group. A ribbon category is a braided monoidal category with some additional structure. In this section, we summarize the basic definitions with emphasis on a convenient diagrammatic notation. We refer the reader to the literature for more details, for example, to the textbooks [21, 22]. Our presentation is similar to that of [3]; we essentially follow [22], but use the diagrams of [21]. Also relevant in the context of the present paper are the results of Reshetikhin and Turaev [23, 26].

#### 2.2.1 Basic definitions

**Definition 2.4.** A **strict monoidal category** is a category \( \mathcal{C} \) together with a covariant functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and a **unit object** \( 1 \) such that

\[
U \otimes (V \otimes W) = (U \otimes V) \otimes W, \quad (2.13a)
\]

\[
V \otimes 1 = V = 1 \otimes V, \quad (2.13b)
\]

for all objects \( U, V, W \).
Definition 2.5. A strict braided monoidal category is a strict monoidal category with natural isomorphisms (the braiding),
\[ \psi_{V,W} : V \otimes W \to W \otimes V, \]  
(2.14)
such that
\[ \psi_{U \otimes V, W} = (\psi_{U,W} \otimes \id_V) \circ (\id_U \otimes \psi_{V,W}), \]  
(2.15a)
\[ \psi_{U,V \otimes W} = (\id_V \otimes \psi_{U,W}) \circ (\psi_{U,V} \otimes \id_W), \]  
(2.15b)
\[ \psi_{V,1} = \id_V = \psi_{1,V}, \]  
(2.15c)
for all objects \( U, V, W \). The category is called symmetric if in addition
\[ \psi_{W,V} \circ \psi_{V,W} = \id_V \otimes W. \]  
(2.16)

Definition 2.6. A strict monoidal category \( \mathcal{C} \) is called rigid if for each object \( V \) there exists an object \( V^* \) (the left dual) and if there are natural isomorphisms
\[ \text{ev}_V : V^* \otimes V \to 1, \quad (\text{evaluation}) \]  
(2.17a)
\[ \text{coev}_V : 1 \to V \otimes V^*, \quad (\text{co-evaluation}) \]  
(2.17b)
which satisfy for all objects \( V \),
\[ \id_V = (\id_V \otimes \text{ev}_V) \circ (\text{coev}_V \otimes \id_V), \]  
(2.18a)
\[ \id_{V^*} = (\text{ev}_V \otimes \id_{V^*}) \circ (\id_{V^*} \otimes \text{coev}_V). \]  
(2.18b)

For a given morphism \( f : V \to W \), the dual morphism \( f^* : W^* \to V^* \) is defined by
\[ f^* := (\text{ev}_W \otimes \id_{V^*}) \circ (\id_{V^*} \otimes f \otimes \id_{V^*}) \circ (\id_{V^*} \otimes \text{coev}_V). \]  
(2.19)
Left duality thus defines a contravariant functor \( * : \mathcal{C} \to \mathcal{C} \).

Definition 2.7. A strict ribbon category \( \mathcal{C} \) is a strict rigid braided monoidal category with natural isomorphisms (the twist),
\[ \nu_V : V \to V, \]  
(2.20)
such that for all objects \( V, W \),
\[ \nu_{V \otimes W} = (\nu_V \otimes \nu_W) \circ \psi_{W,V} \circ \psi_{V,W}; \]  
(2.21a)
\[ (\nu_V)^* = \nu_{V^*}, \]  
(2.21b)
\[ \nu_1 = \id_1. \]  
(2.21c)

It is now possible to construct right duals \( *V \) from the braiding, the twist and the left duals. The right dual objects agree in this case with the left duals, \( *V = V^* \), and right evaluation and right co-evaluation are given by,
\[ \tilde{\text{ev}}_V : V \otimes V^* \to 1, \quad \tilde{\text{ev}}_V := \text{ev}_V \circ \psi_{V,V^*} \circ (\nu_V \otimes \id_{V^*}), \]  
(2.22a)
\[ \tilde{\text{coev}}_V : 1 \to V^* \otimes V, \quad \tilde{\text{coev}}_V := (\nu_{V^*} \otimes \id_V) \circ \psi_{V,V^*} \circ \text{coev}_V. \]  
(2.22b)

Finally, right and left duals can be employed in order to define the analogues of trace and dimension.

Definition 2.8. Let \( \mathcal{C} \) be a strict ribbon category, \( V \) an object of \( \mathcal{C} \) and \( f : V \to V \).
1. The quantum trace of $f$ is defined by
\[ \text{qtr}(f) := \tilde{ev}_V \circ (f \otimes \text{id}_{V^*}) \circ \text{coev}_V. \] (2.23)

2. The quantum dimension of $V$ is defined by
\[ \text{qdim} V := \text{qtr} \circ (\text{id}_V) = \tilde{ev}_V \circ \text{coev}_V. \] (2.24)

Note that the quantum trace satisfies $\text{qtr}(g \circ f) = \text{qtr}(f \circ g)$ for $f: V \to W$ and $g: W \to V$. Furthermore, for $h: V \to V$ and $k: W \to W$, $\text{qtr}(h \otimes k) = \text{qtr}(h) \circ \text{qtr}(k)$ and $\text{qdim}(V \otimes W) = \text{qdim} V \circ \text{qdim} W$, where the compositions are in $\text{Hom}(1, 1)$.

All monoidal categories defined above, starting from Definition 2.4, are strict. If a non-strict category is given, there exists an equivalent strict version [23] which can be used instead. As a consequence of the coherence conditions on associativity and unit constraints in the definition of a (non-strict) monoidal category, it would also be possible to make a choice of parentheses in all definitions and to insert the constraints in a consistent way in all equations. The same would apply to the calculations and results presented in the following sections of this paper.

Furthermore, all categories of interest in this paper are $\mathbb{C}$-linear (for details see, for example, [21, 27]). This means that there is the notion of a (finite) direct sum of objects, that furthermore for given objects $V, W$ the sets $\text{Hom}(V, W)$ form $\mathbb{C}$-vector spaces and that composition of morphisms is $\mathbb{C}$-bilinear. Additionally, there are notions of monomorphism and epimorphism which have the usual properties known from linear algebra. The reader might think of the case where all objects are $\mathbb{C}$-vector spaces. Finally, the additional structures such as tensor product, braiding, duality and twist are required to be compatible with the $\mathbb{C}$-linear structure, in particular $\text{Hom}(1, 1) \cong \mathbb{C}$ such that composition corresponds to multiplication.

As a consequence, $\text{Hom}(U \otimes V, W) \cong \text{Hom}(V, W \otimes U^*)$ are isomorphic as $\mathbb{C}$-vector spaces. We also need the dual space of $\text{Hom}(V, W)$. One can make use of a non-degenerate $\mathbb{C}$-bilinear pairing
\[ \text{ev}_W \circ (g \otimes f) \circ \tilde{\text{coev}}_V, \] (2.25)
in order to define the dual space $\text{Hom}(V, W)^*$ up to isomorphism. Here we use $\text{Hom}(V^*, W^*)$ rather than $\text{Hom}(W, V)$ because some diagrams in the following sections are then related by a mirror symmetry.

All conditions that are required for the construction of the Spin Foam Model are summarized in the following definition.

**Definition 2.9.** An admissible ribbon category is a $\mathbb{C}$-linear strict ribbon category which satisfies the following conditions,

1. For all objects $V, W$ of $\mathcal{C}$ the space $\text{Hom}(V, W)$ is finite-dimensional as a $\mathbb{C}$-vector space.

2. The pairing (2.25) is non-degenerate for all objects $V, W$ of $\mathcal{C}$.

A set of colours $\mathcal{C}_0$ is a countable set of objects of $\mathcal{C}$ such that

1. No two elements of $\mathcal{C}_0$ are isomorphic,

2. For each object $V \in \mathcal{C}_0$, the set $\mathcal{C}_0$ also contains an object which is isomorphic to $V^*$.

There are two cases in which one wants to require stronger conditions. Firstly, in order to have a correspondence of the Spin Foam Model with LGT, one seeks a categorical version of the Peter–Weyl Theorem and of the Haar measure.

The role of the irreducible representations in the Peter–Weyl theory is now played by the simple objects:
Definition 2.10. Let $C$ be a $\mathbb{C}$-linear strict ribbon category.

1. An object $V$ of $C$ is called simple if each non-zero monomorphism $f: U \to V$ is an isomorphism and each non-zero epimorphism $f: V \to W$ is an isomorphism.

2. $C$ is called semi-simple if each object $V$ of $C$ is isomorphic to a (finite) direct sum of simple objects.

3. $C$ is called finitely [countably] semi-simple if $C$ is semi-simple and if there are only finitely [countably] many simple objects up to isomorphism.

Corollary 2.11. Let $C$ be a countably semi-simple and admissible ribbon category such that the unit object $1$ is simple and such that for each simple object $J$ we have Hom$(J, J) \cong \mathbb{C}$. Let $C_0$ denote a set containing one representative per equivalence class of simple objects of $C$ and $V, W$ be objects of $C$. Then the natural map given by composition of morphisms,

$$
\bigoplus_{J \in C_0} \text{Hom}(V, J) \otimes \text{Hom}(J, W) \to \text{Hom}(V, W),
$$

(2.26)
is an isomorphism of $\mathbb{C}$-vector spaces.

The direct sum in (2.26) plays the role of the Peter–Weyl decomposition (2.6) in the categorical framework.

Remark 2.12. Under the conditions of Corollary 2.11, there exists furthermore for each natural transformation $f_V: V \to V$ another natural transformation $(Tf)_V: V \to V$ which is defined by the projection onto the direct summand labelled by $J = 1$ in (2.26),

$$
T: \text{Hom}(V, V) \to \text{Hom}(V, 1) \otimes \text{Hom}(1, V) \subseteq \text{Hom}(V, V).
$$

(2.27)

These $(Tf)_V$ satisfy, for example, $(Tf)_J = 0$ for all simple objects $J$ which are not isomorphic to the unit object $1$.

The projection $T$ can be viewed as the translation of the Haar measure $\int: C_{\text{alg}}(G) \to \mathbb{C}$ into the categorical language; compare (2.27) with (2.12).

A detailed explanation of how Corollary 2.11 and Remark 2.12 are related with Peter–Weyl decomposition and Haar measure in the Lie group case can be found in [3]. In the picture of [3], the algebra of representation functions $C_{\text{alg}}(G)$ of the Lie group $G$ co-acts on the vector spaces dual to the representations $V_\rho$ of $G$.

A second situation in which one sometimes requires semi-simplicity is the case when the Spin Foam Model defines a topological invariant, see, for example [10]. Recall, however, that the categories of finite-dimensional representations of the quantum groups $U_q(\mathfrak{g})$, $q$ a root of unity, which form important examples, are not semi-simple [21]. I thank R. Oeckl for pointing out that the weaker notions of quasi-dominance and dominance (Chapter XI of [26]) can be used to establish a uniform treatment of all interesting cases.

The problem of the definition of the Spin Foam Model with ribbon categories which is the subject of the present paper is, however, not affected by these subtleties since it relies only on Definition 2.9.

2.2.2 Ribbon diagrams

There exists a very convenient notation for morphisms of a ribbon category $C$ in terms of ribbon diagrams (Figure 1). The diagrams consist of ribbons which have a white side (normally facing up) and a black side (facing down). They are directed which is denoted by arrows, and they are labelled with objects of $C$. 
The identity morphism \( \text{id}_V \) is represented by a ribbon labelled \( V \) with the arrow pointing down. The identity morphism \( \text{id}_{V^*} \) of the dual object has the same label \( V \), but an arrow pointing up. The diagrams are generally read from top to bottom. Putting diagrams below each other denotes composition of morphisms while putting them next to each other denotes the tensor product of morphisms. Figure 1 also shows the natural isomorphisms \( \text{ev}_V \), \( \text{coev}_V \) of (2.17), the braiding \( \psi_{V,W} \) of (2.14), the twist \( \nu_V \) of (2.20) and their inverses, respectively. The unit object \( 1 \) is invisible in the diagrams which is justified by (2.13b), (2.15c) and (2.21c).

Figure 2 shows the conditions (2.18a) and (2.18b) on evaluation and co-evaluation in (a) and (b). Morphisms \( f: V \rightarrow W \) are represented by a coupon labelled \( f \) with an incoming and an outgoing ribbon as in (c). Figure 2 also shows the definition of the dual morphism \( f^* \) as given by (2.19).

The main purpose of the ribbon diagrams presented in this section is that they have an immediate translation into algebraic language in terms of morphisms of the ribbon category \( \mathcal{C} \) and at the same time provide an intuitive way of dealing with the algebraic manipulations. One can imagine that the ribbons shown in the diagrams are embedded in \( \mathbb{R}^3 \). The obvious isotopies then correspond to relations in \( \mathcal{C} \). This is a direct consequence of the functor constructed by Reshetikhin and Turaev in [23] where more details can be found. In the following, we present many calculations in the diagrammatic language. If required, they can
be translated at any stage into the corresponding algebraic expressions.

In the remaining parts of the paper, we employ a simplified notation in which the ribbons are represented by single directed lines, and it is understood that their white side always faces up. This is known as blackboard framing. It is particularly convenient here because it turns out that the relevant diagrams in the following sections can be drawn without twists.

2.2.3 Quantum groups and ribbon categories

The ribbon categories arising in [23] are constructed as the categories of finite-dimensional representations of suitable ribbon Hopf algebras, see also [21, 22, 26].

An alternative picture is developed in [3]. It is dual to the former in the sense that it uses the dual Hopf algebra co-acting on the dual spaces of the representations. It is thus based on coribbon Hopf algebras and their ribbon category of corepresentations. This point of view is much closer to the duality transformation for LGT with Lie groups (see Section 3 or [17]) since the algebra of representation functions $C_{\text{alg}}(G)$ of the gauge group naturally co-acts on the dual spaces of the representations of $G$ and can be replaced by a suitable coribbon Hopf algebra [3].
2.3 Combinatorial and Simplicial Complexes

2.3.1 Triangulations

For the construction of the Spin Foam Model using ribbon categories, we need combinatorial complexes and simplicial complexes. Combinatorial complexes contain the information of which simplices are contained in the boundary of a given simplex while simplicial complexes also provide a linear order of the vertices and keep track of all relative orientations. This terminology follows [7]. In order to construct a Spin Foam Model for ribbon categories, we aim for a definition of the partition function which takes the relative orientations into account, but which does not depend on the linear order of vertices.

For the purpose of the present paper, it is furthermore sufficient to deal with abstract complexes. The details of how their simplices are mapped to the given manifold are not discussed here except for a few restrictions that apply if the complex corresponds to a closed and oriented manifold.

**Definition 2.13.** For a given set Λ of vertices, a combinatorial complex Λ(∗) is a non-empty set of subsets of Λ,

\[ \emptyset \neq \Lambda^{(\ast)} \subseteq \mathcal{P} \Lambda, \tag{2.28} \]

such that for each \( v \in \Lambda \), \( \{v\} \in \Lambda^{(\ast)} \) and for each set \( X \in \Lambda^{(\ast)} \), all its non-empty subsets are also contained in \( \Lambda^{(\ast)} \), i.e.

\[ X \in \Lambda^{(\ast)} \quad \text{and} \quad \emptyset \neq Y \subseteq X \quad \implies \quad Y \in \Lambda^{(\ast)}. \tag{2.29} \]

The sets \( X \in \Lambda^{(\ast)} \) are called simplices. The subsets \( \emptyset \neq Y \subseteq X \) are the faces of \( X \). The elements of the set

\[ \Lambda^{(k)} := \{ X \in \Lambda^{(\ast)} : \ |X| = k + 1 \}, \quad k \in \mathbb{N} \tag{2.30} \]

are called k-simplices. Here \( |·| \) denotes the cardinality of a set. A combinatorial k-complex is a combinatorial complex for which \( \Lambda^{(j)} = \emptyset \) for all \( j > k \). For each k-simplex \( X \in \Lambda^{(k)} \), its boundary is defined as the collection of \((k - 1)\)-simplices,

\[ \partial X := \{ Y \subseteq X : \ |Y| = k \}. \tag{2.31} \]

A combinatorial complex is called finite if \( \Lambda^{(\ast)} \) is a finite set.

**Definition 2.14.** A simplicial complex \((\Lambda^{(\ast)}, <)\) is a combinatorial complex \( \Lambda^{(\ast)} \) with a linear order \((<)\) of the vertices \( \Lambda \). The \( k \)-simplices \( X \in \Lambda^{(k)} \) can then be represented by \((k+1)\)-tuples \((v_0, v_1, \ldots, v_k)\) of vertices \( v_j \in \Lambda \) in standard order \( v_0 < v_1 < \cdots < v_k \). In the free \( \mathbb{Z} \)-module generated by \( \Lambda^{(\ast)} \), the boundary of a \( k \)-simplex is given as a sum over the \((k - 1)\)-simplices in \( \partial X \),

\[ \partial(v_0, v_1, \ldots, v_k) := \sum_{j=0}^{k} (-1)^j (v_0, v_1, \ldots, \hat{v}_j, \ldots, v_k), \tag{2.32} \]

where the hat (\( \hat{·} \)) indicates that a symbol is omitted. An abbreviated notation is

\[ (01\cdots k) := (v_0, v_1, \ldots, v_k). \tag{2.33} \]
In the following we also use the notation \((v_0, v_1, \ldots, v_k)\) with arbitrary vertex order. In the simplicial complex this denotes an oriented \(k\)-simplex \(\text{sgn} \cdot (v_{\tau(0)}, \ldots, v_{\tau(k)})\) where the \(\text{sgn}\) depends on the sign of the permutation \(\tau \in S_{k+1}\) which is required to sort the vertices such that \(v_{\tau(0)} < \cdots < v_{\tau(k)}\).

The triangulations of a compact piecewise-linear \(k\)-manifold \(M\) can be chosen to have only finitely many simplices. In this case their combinatorics are described by a finite combinatorial \(k\)-complex for which there always exists a linear order of vertices.

For a simplicial \(k\)-complex which corresponds to the triangulation of a closed and oriented \(k\)-manifold \(M\), the relative orientation of each simplex \(\sigma\) with respect to \(M\) is given, i.e. whether \(+\sigma\) or \(-\sigma\) is isomorphic to a simplex in \(M\). Observe further that in this case each \((k-1)\)-simplex is contained in the boundary of exactly two \(k\)-simplices: once with positive and once with negative relative orientation.

### 2.3.2 The dual \(2\)-complex

In the present paper the Spin Foam Model is defined on a combinatorial complex \(\Lambda^{(\ast)}\). This point of view agrees with [7, 10], but is dual to the definition given in [12].

In order to compare the Spin Foam Model on \(\Lambda^{(\ast)}\) with LGT, this LGT has to be formulated on the \(2\)-complex dual to \(\Lambda^{(\ast)}\). In this section, we define a generalized notion of \(2\)-complexes which includes polygons rather than just triangles and which makes the cyclic ordering of edges around the polygons explicit. This ordering is necessary to arrange the factors of the group products which are used in the definition of LGT.

**Definition 2.15.** A finite generalized \(2\)-complex with cyclic structure \(\mathcal{(V,E,F)}\) consists of finite sets \(\mathcal{V}\) (vertices), \(\mathcal{E}\) (edges) and \(\mathcal{F}\) (polygons) together with maps

\[
\partial^+ : E \rightarrow V, \quad \text{(end point of an edge)} \quad (2.34a)
\]
\[
\partial^- : E \rightarrow V, \quad \text{(starting point of an edge)} \quad (2.34b)
\]
\[
N : F \rightarrow \mathbb{N}, \quad \text{(number of edges in the boundary of a polygon)} \quad (2.34c)
\]
\[
\partial_j : F \rightarrow E, \quad \text{(the } j\text{-th edge in the boundary)} \quad (2.34d)
\]
\[
\varepsilon_j : F \rightarrow \{-1,+1\}, \quad \text{(its orientation)} \quad (2.34e)
\]

such that

\[
\partial^- \varepsilon_j \partial_j f = \partial^+ \varepsilon_{j+1} \partial_{j+1} f, \quad 1 \leq j \leq N(f) - 1, \quad (2.35a)
\]
\[
\partial^- \varepsilon_{N(f)} \partial_{N(f)} f = \partial^+ \varepsilon_1 \partial_1 f, \quad (2.35b)
\]

for all \(f \in \mathcal{F}\).

The conditions \((2.35)\) state that the edges in the boundary of a polygon \(f \in \mathcal{F}\) are in cyclic ordering from \(\partial_1 f\) to \(\partial_{N(f)} f\) where one encounters the edges with a relative orientation given by \(-\varepsilon_j f\), see Figure 5. Observe that \((2.35)\) can be used to generalize the condition \(\partial \circ \partial = 0\) to the situation where the edges are labelled with non-commutative variables.

Given a finite simplicial \(k\)-complex \(\Lambda^{(\ast)}\), one can construct the dual \(2\)-complex \((\mathcal{V,E,F})\) in the standard way: The dual vertices are just the \(k\)-simplices, \(\mathcal{V} := \Lambda^{(k)}\). The dual edges are the \((k-1)\)-simplices, \(\mathcal{E} := \Lambda^{(k-1)}\), and the dual polygons are given by the \((k-2)\)-simplices, \(\mathcal{F} := \Lambda^{(k-2)}\). Observe that the \((k-2)\)-simplices \(\Lambda^{(k-2)}\) are in general contained in the boundaries of more than three \((k-1)\)-simplices which implies that the polygons \(F\) have in general more than three edges. The maps \(\partial_j, \varepsilon_j, \text{etc. of (2.34)}\) can be constructed inductively from the boundary relation of the simplicial complex.
Figure 5: The maps $\partial_j$ and $\varepsilon_j$ and the conditions (2.35). Here $N(f) = 3$, $\varepsilon_1 f = +1$, $\varepsilon_2 f = +1$ and $\varepsilon_3 f = (-1)$.

In the calculations of the next section, the following abbreviations are convenient: For a given edge $e \in E$, the sets

$$e_+ := \{ f \in F : e = \partial_j f, \; \varepsilon_j f = (+1) \; \text{for some} \; 1 \leq j \leq N(f) \}, \quad (2.36a)$$

$$e_- := \{ f \in F : e = \partial_j f, \; \varepsilon_j f = (-1) \; \text{for some} \; 1 \leq j \leq N(f) \}, \quad (2.36b)$$

contain all polygons that have the edge $e$ in their boundary with positive (+) or negative (−) orientation. For a given polygon $f \in F$, the set

$$f_0 := \{ v \in V : \; v = \delta_j f \; \text{for some} \; 1 \leq j \leq N(f) \}, \quad (2.37)$$

denotes all vertices that are contained in the boundary of the polygon $f$. Finally, the sets

$$f_+ := \{ e \in E : e = \partial_j f, \; \varepsilon_j f = (+1) \; \text{for some} \; 1 \leq j \leq N(f) \}, \quad (2.38a)$$

$$f_- := \{ e \in E : e = \partial_j f, \; \varepsilon_j f = (-1) \; \text{for some} \; 1 \leq j \leq N(f) \}, \quad (2.38b)$$

denote all edges in the boundary of the polygon $f$ with positive (+) or negative (−) orientation.

3 The duality transformation

In this section, we recall the duality transformation relating LGT for Lie groups on the dual generalized 2-complex $(V, E, F)$ with a spin foam model. This transformation was carried out in [17] on a hypercubic lattice and is formulated here for generic 2-complexes.

The calculation is presented entirely in terms of the 2-complex $(V, E, F)$ and does not refer to the simplicial complex $\Lambda^*(\ast)$. Its relation with $\Lambda^*(\ast)$ will be discussed in the following section. The calculation is furthermore valid in arbitrary dimension $d \geq 2$.

Definition 3.1. Let $G$ be a compact Lie group (or a finite group). The partition function of LGT on the finite generalized 2-complex $(V, E, F)$ with cyclic structure is defined by

$$Z = \left( \prod_{e \in E_G} d g_e \right) \prod_{f \in F} w(g_f), \quad g_f := g_{\partial_1 f}^{\varepsilon_1 f} \cdots g_{\partial_{N(f)} f}^{\varepsilon_{N(f)} f} \quad (3.1)$$

Here $\int_G$ denotes the normalized Haar measure on $G$, $w : G \rightarrow \mathbb{R}$ is the Boltzmann weight (1.1), and $g_f$ is the cyclicly ordered product of the group elements attached to the edges in the boundary of the polygon $f \in F$.

Remark 3.2. 1. Observe that even though this definition explicitly refers to the cyclic structure, the value of $Z$ is actually independent of it. The starting point for the cyclic numbering of edges in the boundary of a polygon does not matter because the Boltzmann
weight is given by a class function and thus invariant under cyclic permutation of the factors of \( g_f \). Reversal of the orientation is also a symmetry because it replaces \( g_f \) by \( g_f^{-1} \) which yields the complex conjugate of the class function, but this function is real.

2. Let \( h : V \to G, v \mapsto h_v \) associate a group element to each vertex. The weight \( w(g_f) \) in (3.1) is invariant under the local gauge transformations,

\[
g_e \mapsto h_{\partial + e} \cdot g_e \cdot h_{\partial - e}^{-1}. \tag{3.2}
\]

In order to prove this invariance, one has to make use of the conditions (2.35).

The first step of the duality transformation is to insert the character expansion (1.2) of the Boltzmann weight into (3.1),

\[
Z = \left( \prod_{e \in E} \int d g_e \right) \prod_{f \in F} \sum_{\rho_f \in \mathcal{R}} \hat{w}_{\rho_f} \sum_{n_f = 1}^{\dim V_{\rho_f}} t^{(\rho_f)}_{n_f}(g_f). \tag{3.3}
\]

The trace of the character is responsible for summations over one index \( n_f \) per polygon \( f \in F \). The application of coproduct and antipode (eq. (2.4a) and (2.4c)) to the product \( g_f \) (eq. (3.1)) yields further vector index summations. In total there is one summation per polygon and per vertex of that polygon. These summation variables are denoted by \( n(f, v) \) where \( f \in F \) and \( v \in f_0 \),

\[
Z = \left( \prod_{e \in E} \int d g_e \right) \prod_{f \in F} \sum_{\rho_f \in \mathcal{R}} \hat{w}_{\rho_f} \sum_{n(f, \partial - \partial f)} \cdots \sum_{n(f, \partial - \partial N(f), f)} \prod_{f \in F} t^{(\rho_f)}_{n(f, \partial + \partial f), n(f, \partial + \partial N(f), f)}(g_e). \tag{3.4}
\]

Recall that the conditions (2.35) of the 2-complex apply here. The above expression can now be reorganized, moving all summations to the left,

\[
Z = \left( \prod_{e \in E} \int d g_e \right) \left( \prod_{f \in F} \sum_{\rho_f \in \mathcal{R}} \prod_{\rho_f \in \mathcal{R}} \hat{w}_{\rho_f} \prod_{f \in F} \sum_{n(f, \partial + \partial e)} \cdots \sum_{n(f, \partial + \partial N(f), f)} \prod_{f \in F} t^{(\rho_f)}_{n(f, \partial + \partial e), n(f, \partial + \partial N(f), f)}(g_e) \right). \tag{3.5}
\]

Here the notation

\[
\left( \prod_{f \in F} \sum_{\rho_f \in \mathcal{R}} \right) := \sum_{\rho_f \in \mathcal{R}} \cdots \sum_{\rho_f \in \mathcal{R}} \tag{3.6}
\]

denotes one summation per polygon \( f \in F \). Sorting the product of representation functions by edge rather than by polygon amounts to just a slight change in the enumeration of polygons and edges,

\[
Z = \left( \prod_{f \in F} \sum_{\rho_f \in \mathcal{R}} \prod_{f \in F} \hat{w}_{\rho_f} \prod_{f \in F} \prod_{n(f, \partial + \partial e) = 1} \sum_{n(f, \partial + \partial N(f), f)} \prod_{e \in E} \left[ \prod_{e \in E} t^{(\rho_f)}_{n(f, \partial + \partial e), n(f, \partial + \partial N(f), f)}(g_e) \right] \right). \tag{3.7}
\]
The integrals can now be evaluated using the formula (2.12),

$$
\int_{G} dg_{e}[\cdots] = \sum_{P(e) \in \mathcal{P}_{e}} P(e) n(f, \partial^{+} e), \ldots n(f, \partial^{-} e), \ldots \quad \text{where } P_{e} \text{ denotes a basis of orthogonal } G\text{-invariant projectors onto the trivial components in the complete decomposition of (3.9).}
$$

(3.8)

The curly brackets in (3.8) indicate that there is one index \( n(f, \partial^{+} e) \) for each \( f \in e_{+} \) etc.. Finally, the sums over projectors are moved to the left of the expression,

$$
Z = \left( \prod_{f \in F} \sum_{\rho_{f} \in \mathcal{R}} \right) \left( \prod_{e \in E} \sum_{P(e) \in \mathcal{P}_{e}} \right) \left( \prod_{f \in F} \sum_{v \in V_{f}} \right) \left( \prod_{v \in V} C(v) \right)
$$

(3.11)

This formula can now be reorganized and yields the final result:

**Theorem 3.3.** The partition function (3.1) of LGT on the finite generalized 2-complex \((V, E, F)\) with cyclic structure is equal to the expression

$$
Z = \left( \prod_{f \in F} \sum_{\rho_{f} \in \mathcal{R}} \right) \left( \prod_{e \in E} \sum_{P(e) \in \mathcal{P}_{e}} \right) \left( \prod_{f \in F} \sum_{v \in V} \right) \left( \prod_{v \in V} C(v) \right)
$$

(3.11)

Here \( \mathcal{P}_{e} \) denotes a basis of orthogonal \( G\)-invariant projectors onto the trivial components in the complete decomposition of (3.9). The weights per polygon \( \hat{\omega}_{\rho_{f}} \) are the coefficients of the character expansion (1.2) of the original Boltzmann weight. The weights per vertex \( C(v) \) are given by a trace involving representations and projectors in the neighbourhood of the vertex \( v \in V \),

$$
C(v) = \left( \prod_{f \in F} \sum_{n_{f}} \right) \left( \prod_{e \in E} \sum_{v = \partial_{e} f_{v}} \right) \left( \prod_{e \in E} \sum_{v = \partial_{-} e} \right) \left( \prod_{f \in F} \sum_{v \in V} \right)
$$

(3.12)

Here the range \( f \in F: v \in f_{0} \) of the first product refers to all polygons \( f \in F \) that contain the vertex \( v \) in their boundary.

**Remark 3.4.** 1. The projectors onto the trivial representations,

$$
P^{(e)}: \bigotimes_{f \in e_{+}} \rho_{f} \otimes \bigotimes_{f \in e_{-}} \rho_{f}^{*} \to \mathbb{C},
$$

(3.13)
Figure 6: The neighbourhood of a vertex $v \in V$ on the dual 2-complex in the three-dimensional case. The dotted lines denote the four edges attached to the vertex. Diagram (a) shows the weight $C(v)$ per vertex $v \in V$ occurring in the Spin Foam Model where the full dots denote projectors $P^{(e)}$, and the solid lines the representations $V_{\rho}$. Diagram (b) visualizes the weight $\tilde{C}(v)$ in the spin network expectation value. Here $Q^{(v)}$ is the morphism attached to $v$, and the dashed lines denote the representations $\tau_e$.

can be replaced via the isomorphisms $\text{Hom}(V \otimes W^*, \mathbb{C}) \cong \text{Hom}(V, W)$ by representation morphisms

$$\varphi^{(e)}: \bigotimes_{f \in e_+} \rho_f \rightarrow \bigotimes_{f \in e_-} \rho_f.$$ 

(3.14)

The partition function then contains a sum over a basis of the space of representation morphisms for each edge $e \in E$,

$$\text{Hom}(\bigotimes_{f \in e_+} \rho_f, \bigotimes_{f \in e_-} \rho_f).$$ 

(3.15)

2. The expression $C(v)$ is a trace in the category of finite dimensional representations $\text{Rep} G$, cf. Figure 6(a). Observe that all vector indices $n_f$ are contracted. The complexity of the $C(v)$ depends on the number of edges which contain $v \in V$ in their boundary. In order to generalize this Spin Foam Model to ribbon categories, $C(v)$ has to be replaced by a quantum trace. The main motivation for formulating LGT on the 2-complex dual to a triangulation is that it is now guaranteed that in dimension 4 there are always precisely 5 edges which contain $v$. Without this restriction, the generalization of $C(v)$ to the ribbon case would be much harder.

3. Observe that for $G = SU(2)$ in 3 dimensions, the $C(v)$ are essentially the $6j$-symbols of $SU(2)$.

The generic observables of LGT that have non-vanishing expectation values under the path integral are spin networks, the generalization of Wilson loops to the non-Abelian case.

**Definition 3.5.** Let $G$ be a compact Lie group (or a finite group), $(V, E, F)$ be a finite generalized 2-complex with cyclic structure and $Z$ denote the partition function of LGT of
Definition 3.1. Let $\tau: E \to \mathcal{R}$ assign a unitary finite-dimensional irreducible representation $\tau_e$ to each edge $e \in E$ and for each vertex $v \in V$, let

$$Q^{(v)}: \bigotimes_{e \in E: v = \partial_{+} e} \tau_e \to \bigotimes_{e \in E: v = \partial_{-} e} \tau_e$$

(3.16)

denote a representation morphism. The spin network labelled by $\tau_e$ and $Q^{(v)}$ associates to each configuration $E \to G, e \mapsto g_e$ the value

$$W(\tau, Q) := \frac{1}{Z} \left( \prod_{e \in E} \sum_{v} \dim V_e \right) \left( \prod_{e \in E} \tau_{k_e \ell_e} (g_e) \right) \left( \prod_{v \in V} \tau_e \right).$$

(3.17)

For more details on this definition and for the proof of the following result, we refer the reader to [17].

**Theorem 3.6.** Let $G$ be a compact Lie group (or a finite group) and $(V, E, F)$ be a finite generalized 2-complex with cyclic structure. Let $\tau_e$ and $Q^{(v)}$ define a spin network as in Definition 3.5. The expectation value of the spin network,

$$\langle W(\tau, Q) \rangle = \left( \prod_{e \in E} \int_G dg_e \right) \left[ W(\tau, Q) \prod_{f \in F} w(g_f) \right],$$

(3.18)

is equal to

$$\langle W(\tau, Q) \rangle = \frac{1}{Z} \left( \prod_{f \in F} \psi_f \right) \left( \prod_{e \in E} \psi_e \right) \left( \prod_{f \in F} \tilde{w}_{\rho_f} \right) \prod_{v \in V\tau_e} \left( \prod_{e \in E} \sum_{k_e=1}^{\dim V_e} \left( \prod_{v \in \partial_+ e} \dim V_e \right) \tilde{C}(v) \cdot Q^{(v)} \right).$$

(3.19)

Here $\overline{P}_e$ is a basis of orthogonal $G$-invariant projectors onto the trivial components in the complete decomposition of

$$\bigotimes_{f \in e_+} \rho_f \otimes \bigotimes_{f \in e_-} \rho_f^* \otimes \tau_e.$$

(3.20)

The weights per polygon $\tilde{w}_{\rho_f}$ are the coefficients of the character expansion (1.2) of the original Boltzmann weight. The weights per vertex $\tilde{C}(v)$ are given by the trace

$$\tilde{C}(v) = \left( \prod_{f \in F} \sum_{n_f=1}^{\dim V_{\rho_f}} \right) \left( \prod_{f \in e_+} \sum_{n_f=1}^{\dim V_{\rho_f}} \right) \left( \prod_{f \in e_-} \sum_{n_f=1}^{\dim V_{\rho_f}} \right) \left( \prod_{e \in E} \psi_e \right) \left( \prod_{f \in F} \psi_f \right).$$

(3.21)

**Remark 3.7.**

1. The above theorem is an example for an explicit calculation how the spin foams that are the configurations in the partition function couple to the spin network $W(\tau, Q)$. Figure 6(b) visualizes the trace which gives the weights per vertex $\tilde{C}(v)$.

2. If the set of edges for which the representations $\tau_e$ are non-trivial, forms a closed loop, then $W(\tau, Q)$ is non-zero only if the non-trivial $\tau_e$ are all isomorphic. The morphisms $Q^{(v)}$ are then unique up to normalization. In this case $W(\tau, Q)$ describes a Wilson loop.
4 The Spin Foam Model for Ribbon Categories

In Section 3, the Spin Foam Model dual to LGT was derived for the case in which the gauge group $G$ is a Lie group. If LGT is defined on the 2-complex dual to the triangulation, the partition function of the Spin Foam Model consists of a sum over all labellings of triangles with irreducible representations (simple objects) and of all tetrahedra with invariant projectors (representation morphisms), see Table 1.

This Spin Foam Model shall be generalized to a ribbon category $\mathcal{C}$ which replaces the category of representations $\text{Rep} G$ of the gauge group $G$. The partition function will contain the sum over all colourings of triangles with simple objects explicitly while the sum over all colourings of tetrahedra with morphisms will be implemented as a trace over suitable state spaces.

The definition of the Spin Foam Model is formulated in a first step for a given simplicial 4-complex. The definition thus refers explicitly to the linear order of vertices. In a second step we will prove that it does not depend on that order and that it is thus well-defined for any combinatorial complex that corresponds to the triangulation of a closed and oriented piecewise-linear 4-manifold.

4.1 Definition of the partition function

First we define the colourings which will be explicitly summed over in the partition function.

**Definition 4.1.** Let $\Lambda^{(*)}$ denote a simplicial complex, $\mathcal{C}$ be an admissible ribbon category (Definition 2.9) and $\mathcal{C}_0$ a set of colours.

1. A colouring $V : \Lambda^{(2)} \rightarrow \mathcal{C}_0$ associates an object $V(v_0, v_1, v_2) \in \mathcal{C}_0$ to each triangle $(v_0, v_1, v_2) \in \Lambda^{(2)}$ with standard vertex order $v_0 < v_1 < v_2$.

2. For any permutation $\sigma \in S_3$ (acting on $\{0, 1, 2\}$) define

$$V(v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}) := \begin{cases} V(v_0, v_1, v_2), & \text{if } \text{sgn} \sigma = 1, \\ V(v_0, v_1, v_2)^*, & \text{if } \text{sgn} \sigma = -1. \end{cases}$$

For given vertices $v_0, v_1, v_2 \in \Lambda$, we use the abbreviated notation

$$V_{012} := V(v_0, v_1, v_2), \quad V_{021} := V(v_0, v_2, v_1),$$

and so on, for example, $V_{021} = V_{012}^*$. 

---

Table 1: The partition function of the Spin Foam Model dual to LGT (3.11) is a sum over all colourings where the summands contain certain weights. Here colourings and weights are given for LGT living on the 2-complex dual to the triangulation.

| 4-simplex | vertex | — | $C(v)$ |
| tetrahedron | edge | morphism | — |
| triangle | polygon | simple object | $\tilde{w}_\rho$ |
| edge | — | — | — |
| vertex | — | — | — |
Figure 7: (a) The coupons denoting morphisms $\varphi_{0123} \in H_{0123}$ and their duals $\varphi^*_{0123}$ (4.4a) as well as (b) morphisms of the dual state spaces, $\overline{\varphi}_{0123} \in H^*_{0123}$ and their duals $\overline{\varphi}^*_{0123}$ (4.3b). Diagram (c) shows the pairing (4.5). All ribbons are drawn in blackboard framing.

Recall that $(V^*)^* \cong V$ is isomorphic in $\mathcal{C}$, but in general not equal. The above definition therefore describes an action of the symmetric group $S_3$ only up to isomorphism.

The state spaces are defined in the next step. A trace over these spaces will yield the summation over colourings of the tetrahedra with morphisms.

**Definition 4.2.** Let $V: \Lambda^{(2)} \rightarrow \mathcal{C}$ denote a colouring. The state space associated with a tetrahedron $(v_0, v_1, v_2, v_3)$ with arbitrary vertex order is defined by

$$H^{(V)}(v_0, v_1, v_2, v_3) := \text{Hom}(V(v_1, v_2, v_3) \otimes V(v_0, v_1, v_3), V(v_0, v_1, v_2) \otimes V(v_0, v_2, v_3)). \quad (4.3a)$$

The dual state space is then given up to isomorphism by the pairing (2.25),

$$H^{(V)}(v_0, v_1, v_2, v_3)^* := \text{Hom}(V(v_0, v_1, v_3)^* \otimes V(v_1, v_2, v_3)^*, V(v_0, v_2, v_3)^* \otimes V(v_0, v_1, v_2)^*). \quad (4.3b)$$

The following abbreviated notation is used,

$$H_{0123} = \text{Hom}(V_{123} \otimes V_{013}, V_{012} \otimes V_{023}), \quad (4.4a)$$

$$H^*_{0123} = \text{Hom}(V^*_{013} \otimes V^*_{123}, V^*_{023} \otimes V^*_{012}), \quad (4.4b)$$

so that the pairing (2.25) reads in this case

$$\langle \cdot, \cdot \rangle_{0123}: H^*_{0123} \otimes H_{0123} \rightarrow \mathbb{C}, (\overline{\varphi}_{0123}, \psi_{0123}) \mapsto \langle \overline{\varphi}_{0123}, \psi_{0123} \rangle_{0123}. \quad (4.5)$$

The ribbon diagrams corresponding to a morphism $\varphi_{0123} \in H_{0123}$ and its dual $\varphi^*_{0123}$ are depicted in Figure 7(a). The morphism of the dual state space $\overline{\varphi}_{0123} \in H^*_{0123}$ and its dual $\overline{\varphi}^*_{0123}$ are represented diagrammatically as in Figure 7(b). Dual morphisms are denoted by a star (*) whereas we indicate by a bar (−) that a morphism belongs to a dual state space. Figure 7(c) shows the pairing (4.5) for morphisms $\overline{\varphi}_{0123} \in H^*_{0123}$ and $\psi_{0123} \in H_{0123}$. In Figure 7 and in the following we use blackboard framing (see Section 2.2.2).

**Remark 4.3.**

1. Definition 4.2 applies to any order of vertices. In particular, the definition of $H_{0123}$ involves $V(v_0, v_1, v_2)$ etc. as given by (4.1).

2. The definition (4.3b) implements a special choice of isomorphism between $H_{0123}$ and $H^*_{0123}$ via the pairing (4.5). This choice is used consistently in the following.

The state spaces of Definition 4.2 for different vertex order are related by linear isomorphisms which are defined in the next step.
\[ \varphi_{1023} := \tau_{0}^{-1}(\varphi_{0123}) \in H_{1023} \quad \text{and} \quad \varphi_{1023} := \tau_{0}(\varphi_{0123}) \in H_{1023}^{*} \quad \text{for given } \varphi_{0123} \in H_{0123}^{*} \quad \text{and } \varphi_{0123} \in H_{0123}, \text{ see (4.6a).} \]

\[ \varphi_{0213} := \tau_{1}^{-1}(\varphi_{0123}) \in H_{0213} \quad \text{and} \quad \varphi_{0213} := \tau_{1}(\varphi_{0123}) \in H_{0213}^{*} \quad \text{for given } \varphi_{0123} \in H_{0123}^{*} \quad \text{and } \varphi_{0123} \in H_{0123}, \text{ see (4.6b).} \]

**Definition 4.4.** Let \( H_{0123} \) and \( H_{0123}^{*} \) denote the state space and its dual for a tetrahedron \( (v_0, v_1, v_2, v_3) \). The linear maps
\[
\tau_{0}: H_{0123} \rightarrow H_{0123}^{*}, \quad \tau_{1}^{*}-1: H_{0123}^{*} \rightarrow H_{1023}, \\
\tau_{1}: H_{0123} \rightarrow H_{0123}^{*}, \quad \tau_{1}^{-1}: H_{0123}^{*} \rightarrow H_{0123}, \\
\tau_{2}: H_{0123} \rightarrow H_{0132}^{*}, \quad \tau_{2}^{*}-1: H_{0123}^{*} \rightarrow H_{0132},
\]
are defined by the diagrams in Figure 8 to Figure 10.

Note that \( \tau_{j} \) exchanges the \( j \)-th and the \( (j + 1) \)-th vertex of the four arguments of \( H(v_0, v_1, v_2, v_3) \), counting from zero. This need not be the vertices with number \( j \) and \( j + 1 \), for example,
\[
\tau_{0}: H_{1234} \rightarrow H_{2134}^{*}, \quad \tau_{1}: H_{0214} \rightarrow H_{0124}^{*}.
\]

**Lemma 4.5.** Let \( \tau_{j} \), \( 0 \leq j \leq 2 \), denote the linear maps of Definition 4.4.

1. The \( \tau_{j} \) satisfy \( \tau_{j}^{-1} \circ \tau_{j} = \text{id} \) and \( \tau_{j} \circ \tau_{j}^{-1} = \text{id} \). In particular, the \( \tau_{j} \) and \( \tau_{j}^{-1} \) form linear isomorphisms.

2. The \( \tau_{j} \) satisfy \( \langle \tau_{j}^{-1}(\psi_{0123}), \tau_{j}(\varphi_{0123}) \rangle_{1023} = \langle \varphi_{0123}, \psi_{0123} \rangle_{0123} \) for all \( \varphi_{0123} \in H_{0123}^{*} \) and \( \psi_{0123} \in H_{0123} \) which motivates the notation \( \tau_{j}^{-1} \).

**Proof.**

1. The relations \( \tau_{j}^{-1} \circ \tau_{j} = \text{id} \) can be verified diagrammatically using the identities that hold in ribbon categories. Figure 11 shows the calculation for \( \tau_{1}^{-1} \circ \tau_{1} = \text{id}_{H_{0123}} \). The other cases are analogous.
Figure 10: The definition of the morphisms $\varphi_{0132} := \tau_2^{-1}(\varphi_{0123}) \in H_{0132}$ and $\varphi_{0132} := \tau_2(\varphi_{0123}) \in H^*_{0132}$ for given $\varphi_{0123} \in H_{0123}$ and $\varphi_{0123} \in H_{0123}$, see (4.6c).

Figure 11: Diagrammatic proof of the identity $\tau_1^{-1} \circ \tau_1 = \text{id}_{H_{0123}}$ in Lemma 4.5. Here $\varphi_{0123} \in H_{0123}$.

2. This claim can also be verified diagrammatically. It is essentially a consequence of the fact that the maps $\tau_j$ on the dual state spaces in Figure 8 to Figure 10 are given by the mirror images of the maps on the original state spaces.

Remark 4.6. In analogy with the three-dimensional case, one could conjecture that the $\tau_j$ generate an action of the symmetric group $S_4$ on some collection of state spaces. This is not the case. Only in the final step we will have an action of the symmetric group when it is proved that the partition function is well-defined.

At this point, the colourings $V_{jk\ell}$ and the spaces $H_{jk\ell m}$ are defined for a generic vertex order. The summation over all colourings of tetrahedra with morphisms which is part of the partition function, will be implemented as a trace. This trace is over the tensor product of maps $Z^{(V)}_{01234}$ for all 4-simplices $(v_0, \ldots, v_4) \in \Lambda^{(4)}$. These building blocks $Z^{(V)}_{01234}$ are defined first.

Definition 4.7. Let $V : \Lambda^{(2)} \to C_0$ be a colouring and the state spaces for the tetrahedra be given by Definition 4.2.

1. For any 4-simplex $(v_0, \ldots, v_4)$ whose relative orientation in the manifold $M$ is positive, define the 4-simplex map

$$Z^{(V),(+)}_{01234} : H_{0234} \otimes H_{0124} \to H_{1234} \otimes H_{0134} \otimes H_{0123},$$

(4.8a)
to be the linear map that is related by the pairing (2.25) to the quantum trace
\[
Z_{01234}^{(V),(+)} : H_{1234}^* \otimes H_{0234} \otimes H_{0134}^* \otimes H_{0124} \otimes H_{0123}^* \to \mathbb{C},
\]
(4.8b)
which is depicted in Figure 12(a).

2. For any 4-simplex with negative relative orientation in \( M \), the 4-simplex map
\[
Z_{01234}^{(V),(-)} : H_{1234} \otimes H_{0134} \otimes H_{0123} \to H_{0234} \otimes H_{0124},
\]
(4.9a)
is defined by the quantum trace
\[
Z_{01234}^{(V),(-)} : H_{1234} \otimes H_{0234}^* \otimes H_{0134} \otimes H_{0124}^* \otimes H_{0123} \to \mathbb{C},
\]
(4.9b)
which is depicted in Figure 12(b).

**Remark 4.8.**

1. The assignment of the \( H_{jklm} \) to domain or codomain and the assignment of duality stars (*) in the above definitions is according to the orientation of the tetrahedra in the boundary of the 4-simplex,
\[
\partial(01234) = (1234) - (0234) + (0134) - (0124) + (0123).
\]
(4.10)

2. Observe that Figure 12(b) is the mirror image of (a) with all arrows reversed. This is different from the quantum trace of the dual morphism which would also replace the over-crossing by an under-crossing.
In order to obtain a summation over a basis of each state space $H_{jk\ell m}$, the partition function is defined as a trace over the tensor product of all 4-simplex maps.

**Definition 4.9.** Let $\Lambda^{(4)}$ be a finite combinatorial 4-complex corresponding to a triangulation of a closed oriented piecewise-linear 4-manifold $M$. Choose a fixed linear order of the vertices of $\Lambda$. Let $V: \Lambda^{(2)} \to C_0$ be a colouring and let the 4-simplex maps $Z_{jk\ell mn}^{(V), (\pm)}$ be given by Definition 4.7.

The **partition function per colouring** is defined as

$$Z^{(V)} := \text{tr}_H \left[ P \circ \left( \bigotimes_{(v_0, \ldots, v_4) \in \Lambda^{(4)}} Z_{01234}^{(V), (\pm)} \right) \right].$$  \hspace{1cm} (4.11)

Here $\varepsilon_{01234} \in \{+1, -1\}$ denotes the relative orientation of the 4-simplex $(v_0, \ldots, v_4) \in \Lambda^{(4)}$ in $M$. Since every tetrahedron occurs precisely twice in the boundary of a 4-simplex, once with positive and once with negative relative orientation, both domain and codomain of the tensor product over the $Z_{01234}^{(V), (\pm)}$ are permutations of the tensor factors of

$$H := \bigotimes_{(v_0, v_1, v_2, v_3) \in \Lambda^{(3)}} H^{(V)}(v_0, v_1, v_2, v_3).$$  \hspace{1cm} (4.12)

The permutation operator $P$ in (4.11) is the unique permutation which sorts the tensor factors of the codomain such that their ordering agrees with the ordering of factors in the domain.

**Remark 4.10.** The trace over the tensor product $H$ in the above definition essentially contains the quantum traces (4.8b) or (4.9b) for all 4-simplices plus an additional summation over bases of all state spaces. In the partition function the traces generalize the weights $C(v)$, cf. Table 1 and (3.12). Figure 12 is the four-dimensional analogue of Figure 6(a) with a particular choice of over- and under-crossings.

**Definition 4.11.** Let $\Lambda^{(4)}$ be a finite combinatorial 4-complex corresponding to a triangulation of a closed oriented piecewise-linear 4-manifold $M$. Choose a fixed linear order of the vertices of $\Lambda$. For each colouring $V: \Lambda^{(2)} \to C_0$, let the partition function per colouring, $Z^{(V)}$, be given by Definition 4.9. The **partition function** is defined as

$$Z := \sum_{V: \Lambda^{(2)} \to C_0} \left( \prod_{(v_0, v_1, v_2) \in \Lambda^{(2)}} \tilde{w}(v_0, v_1, v_2)(V_{012}) \right) Z^{(V)}. \hspace{1cm} (4.13)$$

The weights $\tilde{w}(v_0, v_1, v_2): C_0 \to \mathbb{R}$ assign a real number to the object associated with the triangle $(v_0, v_1, v_2) \in \Lambda^{(2)}$ and are required to satisfy the reality condition,

$$\tilde{w}(v_0, v_1, v_2)(V^*) = \tilde{w}(v_0, v_1, v_2)(V), \hspace{1cm} (4.14)$$

and to be functions on equivalence classes of isomorphic objects, i.e.

$$\tilde{w}(v_0, v_1, v_2)(V) = \tilde{w}(v_0, v_1, v_2)(\tilde{V}) \quad \text{if} \quad V \cong \tilde{V}. \hspace{1cm} (4.15)$$

Section 4.2 is devoted to proving that this definition is actually independent of the linear order of vertices and of the choice of colours $C_0$ up to isomorphism. The partition function is therefore well-defined for a combinatorial complex that corresponds to the triangulation of a closed and oriented manifold. In Section 5, we discuss some relevant special cases in more detail that are covered by (4.13), in particular the relation with the standard formulation of LGT for Lie groups and the Crane–Yetter state sum. There, we also comment on the convergence of (4.13) if $C_0$ is not a finite set.
4.2 Properties of the partition function

First we show that the partition function (4.13) does not depend on the choice of colours \( C \) up to isomorphism.

**Theorem 4.12.** Let \( V : \Lambda(2) \to C \) denote a colouring, and for each triangle \((v_0, v_1, v_2) \in \Lambda(2)\), \( v_0 < v_1 < v_2 \), let

\[
\Phi(v_0, v_1, v_2) : V(v_0, v_1, v_2) \to \tilde{V}(v_0, v_1, v_2)
\]  

(4.16)

be an isomorphism in \( C \) for some object \( \tilde{V}(v_0, v_1, v_2) \). Then the partition functions per colouring (4.11) for \( V \) and \( \tilde{V} \) agree,

\[
Z(V) = Z(\tilde{V}).
\]  

(4.17)

**Proof.** Using the standard abbreviations, the given isomorphisms are of the form \( \Phi_{012} : \tilde{V}_{012} \to \tilde{V}_{012} \) for all triangles \((v_0, v_1, v_2) \) with standard vertex order \( v_0 < v_1 < v_2 \). For any permutation \( \sigma \in S_3 \) we define isomorphisms \( \Phi_{\sigma(0)\sigma(1)\sigma(2)} : V_{\sigma(0)\sigma(1)\sigma(2)} \to \tilde{V}_{\sigma(0)\sigma(1)\sigma(2)} \) by

\[
\Phi_{\sigma(0)\sigma(1)\sigma(2)} := \begin{cases} 
\Phi_{012} : V_{012} \to \tilde{V}_{012}, & \text{if } \text{sgn} \sigma = 1, \\
\Phi_{012}^{-1} : V_{012} \to \tilde{V}_{012}, & \text{if } \text{sgn} \sigma = -1.
\end{cases}
\]  

(4.18)

Observe that this assignment is compatible with Definition 4.1. These definitions provide us with isomorphisms \( \Phi_{012} : \tilde{V}_{012} \to \tilde{V}_{012} \) and with their dual maps \( \Phi_{012}^* : \tilde{V}_{012}^* \to \tilde{V}_{012}^* \) for all triangles \((v_0, v_1, v_2) \) with arbitrary vertex order.

Furthermore, there are induced linear isomorphisms of the state spaces,

\[
\Phi_{0123} : \text{Hom}(V_{123} \otimes V_{013}, V_{012} \otimes V_{023}) \to \text{Hom}(\tilde{V}_{123} \otimes \tilde{V}_{013}, \tilde{V}_{012} \otimes \tilde{V}_{023}),
\]  

(4.19a)

\[
\varphi_{0123} \mapsto (\Phi_{012} \otimes \Phi_{023}) \circ \varphi_{0123} \circ (\Phi_{123}^{-1} \otimes \Phi_{013}^{-1}),
\]

and

\[
\Phi_{0123}^* : \text{Hom}(\tilde{V}_{013} \otimes \tilde{V}_{123}, \tilde{V}_{023} \otimes \tilde{V}_{012}) \to \text{Hom}(V_{013} \otimes V_{123}, V_{023} \otimes V_{012}),
\]  

(4.19b)

\[
\varphi_{0123} \mapsto (\Phi_{023}^* \otimes \Phi_{012}^*) \circ \varphi_{0123} \circ (\Phi_{123}^{-1} \otimes \Phi_{013}^{-1}).
\]

A convenient abbreviated notation for these maps is \( \Phi_{0123} : H_{0123} \to \tilde{H}_{0123}, \Phi_{0123}^* : \tilde{H}_{0123}^* \to H_{0123} \) writing \( \tilde{H}_{0123} := \text{Hom}(\tilde{V}_{123} \otimes \tilde{V}_{013}, \tilde{V}_{012} \otimes \tilde{V}_{023}) \) etc.. Now the following diagram for the traces \( Z_{01234}(V, (+)) \) commutes:

\[
\begin{array}{ccc}
H_{1234} & \otimes & H_{0234} \\
\Phi_{1234}^{-1} & \Phi_{0234} & \Phi_{0123}^{-1} \\
\tilde{H}_{1234} & \otimes & \tilde{H}_{0234} \\
\Phi_{1234} & \Phi_{0234} & \Phi_{0123} \\
\tilde{H}_{1234} & \otimes & \tilde{H}_{0234} \\
\Phi_{1234}^{-1} & \Phi_{0234} & \Phi_{0123}^{-1} \\
\tilde{H}_{1234} & \otimes & \tilde{H}_{0234} \\
\end{array}
\]

To see this, imagine Figure 12(a) drawn for maps \( \varphi_{jk\ell m} \in \tilde{H}_{jk\ell m} \) etc. and insert the definitions of the linear isomorphisms \( \Phi_{jk\ell m} \) of (4.19). Then the isomorphisms in \( C \), \( \Phi_{jk\ell} : V_{jk\ell} \to \tilde{V}_{jk\ell} \), appear twice in each ribbon in a way such that they cancel.
Let $\langle \cdot, \cdot \rangle: \tilde{H}_{0123} \otimes \tilde{H}_{0123} \to \mathbb{C}$ denote the pairing (4.5) applied to the state spaces which use the colouring $\tilde{V}$. We find

$$\langle \tilde{\Phi}_{0123}^{-1}(\tilde{\psi}_{0123}), \tilde{\Phi}_{0123}(\psi_{0123}) \rangle = \langle \tilde{\Phi}_{0123}, \psi_{0123} \rangle,$$

for all $\tilde{\psi}_{0123} \in \tilde{H}_{0123}$ and $\psi_{0123} \in H_{0123}$. As a consequence the following diagram involving the 4-simplex maps themselves also commutes:

$\begin{array}{c}
H_{0234} \otimes H_{0124} \xrightarrow{Z_{0123}^{(V), (+)}} H_{1234} \otimes H_{0134} \otimes H_{0123} \\
\Phi_{0234} \quad \Phi_{0124} \\
\tilde{H}_{0234} \otimes \tilde{H}_{0124} \xrightarrow{Z_{0123}^{(\tilde{V}), (+)}} \tilde{H}_{1234} \otimes \tilde{H}_{0134} \otimes \tilde{H}_{0123} \\
\Phi_{0234} \quad \Phi_{0124}
\end{array}$

Analogous diagrams are available for $Z_{0123}^{(V), (-)}$ and Figure 12(b) in the case of opposite orientation.

Finally, each tetrahedron occurs precisely twice in the boundaries of 4-simplices, once with positive and once with negative relative orientation. Therefore the tensor product of all 4-simplex maps in (4.11) is conjugated by a linear isomorphism $\Phi$ which can be obtained from a tensor product of the $\Phi_{jk\ell m}$,

$$P \circ \left( \bigotimes_{\sigma \in \Lambda^{(4)}} Z_{\sigma, \epsilon_{\sigma}}^{(V)} \right) = \Phi \left[ P \circ \left( \bigotimes_{\sigma \in \Lambda^{(4)}} Z_{\sigma, \epsilon_{\sigma}}^{(\tilde{V})} \right) \right] \Phi^{-1}.$$  

(4.21)

Since $Z^{(V)}$ is the trace of (4.21), it agrees with $Z^{(\tilde{V})}$.

**Corollary 4.13.** The partition function (4.13) does not depend on the choice of colours $C_0$ up to isomorphism.

**Proof.** Consider another set of colours $\tilde{C}_0$ such that each colouring $V: \Lambda^{(2)} \to C_0$ induces a colouring $\tilde{V}: \Lambda^{(2)} \to \tilde{C}_0$ for which $V_{012} \cong \tilde{V}_{012}$ are isomorphic in $\tilde{C}$ for all triangles $(v_0, v_1, v_2) \in \Lambda^{(2)}$. The partition function (4.13) defined using $C_0$ agrees with that one defined using $\tilde{C}_0$ because the weights satisfy $\tilde{w}(v_0, v_1, v_2)(V_{012}) = \tilde{w}(v_0, v_1, v_2)(\tilde{V}_{012})$ and because $Z^{(V)} = Z^{(\tilde{V})}$ according to Theorem 4.12.

In order to prove the independence of the partition function (4.13) of the linear order of vertices, a generic 4-simplex $(01234)$ is considered. It is proved that any permutation of its vertices which results in different 4-simplex maps $H_{jk\ell m}$ according to Definition 4.2 and Definition 4.7, does not change the partition function (4.13).

This statement is verified for the four elementary transpositions of $S_5$ (acting on the vertices $\{0, 1, 2, 3, 4\}$). The following lemmas prepare the proof. They establish diagrammatical isotopies which permute the coupons in Figure 12(a) in order to reach a configuration similar to Figure 12(b). Recall that the orientation of the 4-simplex changes if an odd permutation is applied to its vertices.

**Lemma 4.14.** For any colouring $V: \Lambda^{(2)} \to C_0$ and morphisms $\varphi_{jk\ell m} \in H_{jk\ell m}$ and $\tilde{\varphi}_{jk\ell m} \in H_{jk\ell m}^{*}$, the quantum trace in Figure 12(a) is equal to the quantum trace in Figure 13.
Proof. The calculation is described in diagrammatic language and can be translated into equalities for morphisms of the ribbon category $C$ as described in Section 2.2. First, a number of coupons are moved around in the plane without twisting or braiding any ribbons: Move the coupon $\varphi_{0124}$ to the left and place it above $\varphi_{1234}$, then move $\varphi_{0134}$ down and to the right and place it below and right of $\varphi_{0234}$. Move $\varphi_{0123}$ to the right and place it below $\varphi_{0123}$ and below and left of $\varphi_{0134}$. Rotate the coupon $\varphi_{0123}$ by 360 degrees in order to place its ribbons as depicted in Figure 13. Finally, lift the ribbon labelled $V_{014}$ out of the plane, move it across the entire diagram, and place it as shown in Figure 13.

Lemma 4.15. For any colouring $V : \Lambda^{(2)} \rightarrow C_0$ and morphisms $\varphi_{jklm} \in H_{jklm}$ and $\varphi_{jklm}^* \in H^*_{jklm}$, the quantum trace in Figure 12(a) is equal to the quantum trace in Figure 14.

Proof. The proof is again explained diagrammatically: Start with Figure 12(a). Lift the ribbon $V_{012}$ out of the plane and move it across the coupon $\varphi_{1234}$ so that $V_{012}$ now overcrosses $V_{123}$, $V_{134}$ and $V_{124}$ rather than $V_{234}$. Then the coupons can be moved around in the plane without introducing twists or braidings such that the configuration in Figure 14 is obtained.

There exist two more lemmas that deal with the elementary transpositions (23) and (34) as well as four lemmas dealing with the case of opposite relative orientation. They are not stated explicitly here since they are very similar and completely analogous to prove.

The results of the preceding lemmas, Figure 13 and Figure 14, are furthermore related to the quantum trace of Figure 12(b) for a 4-simplex with a different order of vertices. This is stated in the following lemmas.

Lemma 4.16. Let $\tau = (01)$ and consider the 4-simplex (01234). Let $V : \Lambda^{(2)} \rightarrow C_0$ denote a colouring. Then there exists another colouring $\tilde{V}$ with isomorphic objects for each triangle, $\tilde{V}_{012} \cong V_{012}$, such that the following diagram commutes:

$$
\begin{array}{c}
H_{0234} \otimes H_{0124} \xrightarrow{Z_{01234}^{(V),(+)}} H_{1234} \otimes H_{0134} \otimes H_{0123} \\
\downarrow \text{id} \quad \tau_0 \quad \downarrow \text{id} \quad \tau_0 \quad \downarrow \tau_0 \\
H_{0234} \otimes H_{1024} \xrightarrow{Z_{10234}^{(\tilde{V})}} H_{1234} \otimes H_{1034} \otimes H_{1023}
\end{array}
$$

Here $Z_{01234}^{(V),(+)}$ is the 4-simplex map of Definition 4.7, the maps $\tau_0$ are given in Definition 4.4, and the bottom horizontal map is determined, using the pairing (4.5), by the 4-simplex map $Z_{10234}^{(\tilde{V})} : H_{0234} \otimes H_{1034} \otimes H_{1023} \rightarrow H_{1234} \otimes H_{1024}$.

Proof. Consider Figure 13 whose quantum trace agrees with $Z_{01234}^{(V),(+)}$ of Figure 12(a) according to Lemma 4.14. The linear isomorphisms $\tau_0$ of Definition 4.4 can now be used to replace the dashed boxes of Figure 13 by morphisms of the state spaces $H_{jklm}$ with a different vertex order. The result is very similar to the trace $Z_{10234}^{(\tilde{V})}$ of Figure 12(b) for the 4-simplex (10234).

Observe, however, that in Figure 12(b) the arrows of the ribbons corresponding to the triangles (012), (013) and (014) are reversed compared with Figure 13. We can reverse these arrows in Figure 13 if we label them instead by $V^*_{012}$, $V^*_{013}$ and $V^*_{014}$, respectively.
Consider the triangle \((012)\). If \((v_0, v_1, v_2)\) is an even permutation of the standard vertex order, then Figure 13 contains \(V_{012}^* = V_{012}\) for some object \(V \in \mathcal{C}_0\), i.e. upon reversal of the arrows this label changes to \(V_{012}^* = (V^*)^\ast\). This is the same label as the label arising in \(Z_{\tau_0}^{(V)}(-)\).

If, however, \((v_0, v_1, v_2)\) is an odd permutation of the standard vertex order, then Figure 13 contains \(V_{012}^* = V_{012}^\ast\) for some object \(V \in \mathcal{C}_0\), i.e. upon arrow reversal this becomes \(V_{012}^\ast = V_{012}^\ast\). This is in general not identical, but still isomorphic to \(V\) which arises in \(Z_{\tau_0}^{(V)}(-)\) in this case. This is the reason why the present lemma holds only for a colouring \(\tilde{V}\) with isomorphic objects at all triangles.

Let \((w_0, w_1, w_2)\) denote any triangle in standard vertex order, \(w_0 < w_1 < w_2\). Define the colouring \(\tilde{V}\) by

\[
\tilde{V}_{\sigma(0)\sigma(1)\sigma(2)} := \begin{cases} 
(V_{012}^*)^\ast, & \text{if } \text{sgn } \sigma = 1, \\
V_{012}^\ast, & \text{if } \text{sgn } \sigma = -1,
\end{cases} \quad (4.23)
\]

if \(\{w_0, w_1, w_2\} \in \{(v_0, v_1, v_2), \{v_0, v_1, v_3\}, \{v_0, v_1, v_4\}\}\) and by \(\tilde{V}_{\sigma(0)\sigma(1)\sigma(2)} := V_{\sigma(0)\sigma(1)\sigma(2)}\) for the other triangles. Then the quantum trace of Figure 13 with arrows \((012), (013)\) and \((014)\) reversed, agrees with the trace of Figure 12(b) for the colouring \(\tilde{V}\). The following diagram therefore commutes:

Using Lemma 4.5, this implies the commutativity of the diagram claimed in the present lemma.

\[\square\]

**Lemma 4.17.** Let \(\tau = (12)\) and \(V: \Lambda^{(2)} \to \mathcal{C}_0\) be a colouring. Then there exists another colouring \(\tilde{V}\) with isomorphic objects for each triangle such that the following diagram commutes:

Here the \(\tau_j\) are the isomorphisms given in Definition 4.4, and the bottom horizontal map is determined, using the pairing \((4.5)\), by the 4-simplex map

\[
Z_{\tau_{01234}}^{(V),(-)}: H_{2134} \otimes H_{0234} \otimes H_{0213} \to H_{0134} \otimes H_{0214}.
\quad (4.24)
\]

**Proof.** Consider Figure 14 whose quantum trace agrees with \(Z_{\tau_{01234}}^{(V),(+)}\) of Figure 12(a) according to Lemma 4.15. The linear isomorphisms \(\tau_0\) and \(\tau_1\) of Definition 4.4 can now be used to replace the dashed boxes of Figure 14 by morphisms of the state spaces \(H_{jk\ell m}\) with a different
vertex order. The result is the trace $Z_{02134}(\tilde{V},(-))$ of Figure 12(b) for the 4-simplex (02134) up to the choice of isomorphic objects for the triangles (012), (123) and (124). These isomorphisms arise from double dualization as in Lemma 4.16.

The following diagram therefore commutes:

Employing Lemma 4.5, this proves the claim. □

There exist similar lemmas for the other elementary transpositions (23) and (34) as well as for the corresponding statements with opposite relative orientations, i.e. where $Z(V),(+)$ and $Z(V),(-)$ are exchanged. Their proofs are entirely analogous.

**Theorem 4.18.** The partition function (4.13) does not depend on the choice of the linear order of vertices.

**Proof.** Equip the set of vertices with a different linear order which is induced from the given one by a permutation $\tau$ of the vertices. The partition function using this new order can be expressed in terms of the definitions of Section 4.1 which use the original order, if $\tau$ is applied both to the vertices and to the colouring,

$$Z_\tau = \sum_{V: \Lambda(2) \rightarrow C_0} \left( \prod_{(v_0, v_1, v_2) \in \Lambda(2)} \hat{w}(v_{\tau^{-1}(0)}, v_{\tau^{-1}(1)}, v_{\tau^{-1}(2)})(\tau V)_{012} \right) Z_{\tau}(\tau V). \quad (4.25)$$

Here

$$Z_{\tau}(\tau V) = \text{tr}_H \left[ P \circ \left( \bigotimes_{s \in \Lambda(4)} Z_{\tau(s)}(\tau V,(\varepsilon_{\tau(s)})) \right) \right] \quad (4.26)$$

replaces the partition function per colouring in the case of the new vertex order, $\tau(s)$ denotes $(\tau^{-1}(0), \ldots, \tau^{-1}(4))$ for a given 4-simplex $s = (01234)$, and $\tau V$ is the colouring induced by $\tau$, i.e. $(\tau V)_{012} = V_{\tau^{-1}(0)}v_{\tau^{-1}(1)}v_{\tau^{-1}(2)}$ for all triangles $(v_0, v_1, v_2) \in \Lambda(2)$.

The permutation $\tau$ replaces triangles (012) by $(\tau^{-1}(0), \tau^{-1}(1), \tau^{-1}(2))$ and therefore just permutes the factors of the product in (4.25). This product can be reorganized so that we obtain

$$Z_\tau = \sum_{V: \Lambda(2) \rightarrow C_0} \left( \prod_{(w_0, w_1, w_2) \in \Lambda(2)} \hat{w}(w_0, w_1, v_2)(V_{012}) \right) Z_{\tau}(\tau V), \quad (4.27)$$

where the vertex order of the triangles does not matter because of the reality condition (4.14).

Any permutation $\tau$ which just permutes the 4-simplices but does not change the vertex order of these 4-simplices, permutes the tensor factors in (4.26) and therefore leaves the trace invariant. It is thus sufficient to prove invariance under permutations $\tau$ that change the vertex order for fixed 4-simplices.

Consider a 4-simplex $s = (01234)$ and and let $\tau$ be an elementary transposition, $\tau \in \{(01), (12), (23), (34)\}$. The colouring $\tau V$ associates with each triangle $(w_0, w_1, w_2) \in \Lambda(2)$
either the object \( V(w_3, w_4, w_5) \) assigned to some triangle \((w_3, w_4, w_5) \in \Lambda^2\) or the dual of that object.

Since the set of colours \( C_0 \) contains for each given object \( V \) exactly one object that is isomorphic to \( V^* \), there exists a unique colouring \( V : \Lambda^2 \to C_0 \) such that \( V_{012} \cong (\tau V)_{012} \) for all triangles. Moreover since \( \tau \) is a transposition, \((\tau V)_{012} \cong V_{012}\) so that \( \tau \) induces an involution on the set of colourings \( \Lambda^2 \to C_0 \). We can now sum over \( V \) rather than \( \overline{V} \) in (4.27) and obtain

\[
Z^V_\tau = \sum_{V : \Lambda^2 \to C_0} \prod_{(v_0, v_1, v_2) \in \Lambda^3} \tilde{w}(v_0, v_1, v_2)(V_{012}) Z^V_{\tau,012}, \quad (4.28)
\]

\[
Z^V_\tau = \text{tr}_H \left[ P \circ \left( \bigotimes_{s \in \Lambda^4} Z^V_{\tau,012} \right) \right], \quad (4.29)
\]

where we have used (4.14) and where we have written \( V \) instead of \( \overline{V} \) for simplicity.

In the preceding lemmas we have constructed linear isomorphisms which form the vertical maps in commutative diagrams of the following form:

\[
\begin{array}{ccc}
H_{0234} \otimes H_{0124} & \xrightarrow{Z^V_{01234}} & H_{1234} \otimes H_{0134} \otimes H_{0123} \\
\Phi_1 & \Phi_2 & \Phi_3 \quad \Phi_4 \quad \Phi_5 \\
H^{(1)} \otimes H^{(2)} & \longrightarrow & H^{(3)} \otimes H^{(4)} \otimes H^{(5)}
\end{array}
\]

Here the \( H^{(j)} \) are suitable state spaces such that the bottom horizontal map is related to the 4-simplex map \( Z^V_{\tau,01234} \) by the pairing (4.5). The colouring \( \tilde{V} \) is such that \( V_{012} \cong V_{012} \) for all triangles \( (012) \).

Since each tetrahedron occurs twice in the boundary of some 4-simplices, once with positive and once with negative relative orientation, each state space \( H^{(j)} \) occurs twice among the \( H^{(j)} \), once as \( H^{(j)} \) and once as the dual state space \( H^*(jklm) \). In both cases, the corresponding map \( \Phi_j \) is the same, either one of the \( \tau_i \) or the identity. Therefore the tensor product of all 4-simplex maps is conjugated by a linear isomorphism \( \Phi \) which can be obtained from a tensor product of these \( \Phi_j \),

\[
P \circ \left( \bigotimes_{s \in \Lambda^4} Z^V_{\tau,01234} \right) = \Phi \left[ P \circ \left( \bigotimes_{s \in \Lambda^4} Z^V_{\tau,01234} \right) \right] \Phi^{-1}. \quad (4.30)
\]

Observe that here \( \varepsilon_{\tau,01234} = -\varepsilon_s \) and that the colouring \( \tilde{V} \) can be replaced by \( V \) as a consequence of Theorem 4.12.

Since \( Z^V_\tau \) is the trace over (4.30), we find \( Z^V_\tau = Z(V) \) and therefore \( Z_\tau = Z \).

### 4.3 Gauge Symmetry

In order to understand the gauge symmetry of LGT in the picture of the Spin Foam Model, consider first the case in which \( C \) is the category of finite-dimensional representations of the gauge group \( G \). The group then acts on its representations via natural equivalences \( (t^V_g)_V \), \( g \in G \), i.e. natural isomorphisms \( t^V_g : V \to V \) for all objects \( V \).
In LGT on the 2-complex \((V, E, F)\) dual to the simplicial complex \(\Lambda^{(*)}\), consider a gauge transformation involving only one vertex \(v \in V\). This means that the group elements \(g_e\) attached to the edges \(e \in E\) are transformed as

\[
\begin{align*}
g_e &\mapsto g_e \cdot h_v^{-1}, & \text{if } v = \partial_- e, \\
g_e &\mapsto h_v \cdot g_e, & \text{if } v = \partial_+ e,
\end{align*}
\]

for \(h_v \in G\) while all other variables \(g_e\) remain unchanged. For each polygon containing the vertex \(v\) in its boundary, precisely two edges are affected in such a way that the effect of the transformation cancels for the polygon.

In the spin foam picture, only the 4-simplex dual to the vertex \(v \in V\) is affected. Let \((v_0, v_1, v_2)\) denote the triangle to which a polygon is dual. The gauge transformation then inserts natural isomorphisms \(t(V_{012})\) and \((t(V_{012}))^{-1}\) into the ribbon corresponding to that triangle, i.e. to the ribbon labelled by the object \(V_{012}\). These isomorphisms cancel.

In the categorical picture, however, this symmetry can be understood in other terms. Let now \(\mathcal{C}\) denote any admissible ribbon category and choose a colouring \(V: \Lambda^{(2)} \rightarrow \mathcal{C}_0\). Consider a morphism \(\varphi_{0123} \in H_{0123}\), i.e. \(\varphi_{0123}: V_{123} \otimes V_{013} \rightarrow V_{012} \otimes V_{023}\). Then for any natural equivalence \((t(V))_V\), naturality means

\[
\varphi_{0123} = (t(V_{013}) \otimes t(V_{012})) \circ \varphi_{0123} \circ (t(V_{123})^{-1} \otimes t(V_{012})^{-1}).
\]

If this transformation for the natural equivalence \((t(V))_V\) is applied simultaneously to all morphisms \(\varphi_{jk\ell m}\) in Figure 12(a) or (b), the isomorphisms \(t(V_{jk\ell})\) cancel pairwise in each ribbon, and the quantum trace remains unchanged.

In the categorical description of the Spin Foam Model, the gauge symmetry is therefore automatically implemented. It is just the naturality property of natural equivalences together with the fact that all ribbon diagrams used in the definition of the partition function are quantum traces.

### 4.4 Wilson loop and spin networks

Having generalized the partition function of LGT to ribbon categories, it is desirable to understand the corresponding generalization for the observables of LGT, namely for Wilson loops and spin networks (Definition 3.5).

In order to define the expectation value of a spin network, recall that the quantum traces in Figure 12 generalize the four-dimensional version of Figure 6(a). One should therefore extend Figure 12 and include five additional ribbons and one coupon for the spin network as Figure 6(b) suggests. However, it seems to be impossible to find a ribbon diagram which has the symmetries required in Section 4.2.

A possible explanation is the following argument. In the Lie group case, the spin network \((3.17)\) attaches representations to the edges and morphisms to the vertices of the 2-complex \((V, E, F)\). Its generalization to the ribbon case should therefore be given by a ribbon graph in four dimensions. In four dimensions, however, there is no canonical way of associating to each ribbon graph a morphism in the ribbon category \(\mathcal{C}\) because there is no four-dimensional analogue of the Reshetikhin–Turaev functor. It is conceivable that the notion of a spin network in four dimensions using ribbon categories is not a good definition.

For the construction of observables that generalize Definition 3.5 to the case of ribbon categories, one has therefore to choose a linear order of vertices on which the result will then depend in a crucial way. Spin network observables are thus defined for simplicial complexes, but not in general for combinatorial complexes.

These restrictions are important if one wants to construct particular physical models which are based on a Spin Foam Model using ribbon categories. It remains to be studied under which
conditions one can define at least a certain class of observables and how the dependence on
the vertex order can be interpreted. The reader is also referred to the diagrammatic approach
to observables in three dimensions in [3].

5 Special cases and generalizations

The partition function (4.13) of the Spin Foam Model using ribbon categories covers a number
of special cases which were already known in other contexts. In this section, we comment on
the relations between these models.

5.1 Lattice Gauge Theory

The category $C = \text{Rep}\, G$ of finite-dimensional representations of a compact Lie group $G$ forms
a semi-simple admissible ribbon category. The relation of the Spin Foam Model with LGT
holds if the set of colours $C_0$ is a set containing one representative of each equivalence class
of simple objects.

In this case one can use the generic Boltzmann weight (1.1) for any action which is local,
i.e. evaluated once for each polygon, and which is a real and bounded $L^2$-integrable class
function of $G$. In particular, the standard Wilson action and the heat kernel or generalized
Villain action are of this form. For details about these actions and about their character
expansion, we refer the reader to standard textbooks such as [15,16].

In general the set of representatives $C_0$ of the simple objects is countably infinite. However,
the partition function (4.13) is a convergent series because the Boltzmann weight is an $L^2$-
function, and its character expansion therefore forms a square summable series due to the
Peter–Weyl Theorem. For more details, see [17,25].

In this case both pictures, LGT and the Spin Foam Model, are well-defined and are dual
to each other in the sense of [17]. A comparison of the Spin Foam Model dual to LGT (3.11)
and the generalization (4.13) shows the following correspondences, cf. Table 1. The sum over
colourings of triangles/polygons and the weights $\hat{w}$ are explicitly contained in the partition
function. The sum over colourings of tetrahedra/edges with morphisms is explicit in (3.11)
and it is the result of the trace over the tensor product $\mathcal{H}$ in (4.11). The weights $C(v)$
per 4-simplex/vertex are given by the formula (3.12) and agree with the quantum traces of
Figure 12 which appear as a result of the trace over the 4-simplex maps in (4.11).

For standard actions of LGT such as Wilson’s action or the heat kernel action, the char-
acter expansion coefficients behave qualitatively like $\exp(\frac{1}{\beta}s^*(V_\rho))$ if the Boltzmann weight is
of the form $\exp(\beta s(g))$. Here $\beta$ is the inverse temperature, $s(g)$ denotes the action and $s^*(V_\rho)$
the dual action, a function assigning a real number to each finite-dimensional irreducible rep-
resentation of $G$. The transformation between LGT and the Spin Foam Model thus realizes
a low temperature — high temperature duality or a strong-weak duality in the bare coupling
$g_0$ if $\beta = 1/g_0^2$.

For the heat kernel action, the dual action $s^*(V_\rho) \sim C^{(2)}_\rho$ is essentially given by the second
order Casimir operator $C^{(2)}_\rho$ of the representation. One can thus sort the configurations of
the Spin Foam Model by the sum of the Casimir eigenvalues over all triangles, and recovers
the full strong coupling expansion of non-Abelian LGT.

Observe finally that LGT was formulated here on the 2-complex dual to a generic triang-
gulation. In order to obtain the usual continuum limit, the Boltzmann weight $\hat{w}(v_0,v_1,v_2)$
should now depend on the geometry of the triangle $(v_0,v_1,v_2)$ in a suitable way.
5.2 Gauge theory with finite groups

If \( \mathcal{C} = \text{Rep } G \) is the category of representations of a finite group \( G \), all comments of Section 5.1 still apply. In this case, there are only finitely many simple objects up to isomorphism and the partition functions in both pictures, in LGT and in the Spin Foam Model, are well-defined.

It is now also possible to study the ‘topological’ Boltzmann weight

\[
\hat{w}(g) = \delta(g) := \begin{cases} 
|G|, & \text{if } g = 1, \\
0, & \text{else}
\end{cases} \quad \text{i.e.} \quad \hat{w}_\rho = \text{dim } V_\rho. \tag{5.1}
\]

With suitable prefactors, the partition function is then independent of the triangulation and thus forms a topological invariant which is well known and depends only on the gauge group and on the first fundamental group of the manifold. See, for example, the comments in Section 2.2 of [28].

5.3 The Crane–Yetter state sum

Let \( \mathcal{C} \) be a finitely semi-simple and admissible ribbon category satisfying the conditions of Corollary 2.11 and \( \mathcal{C}_0 \) be a set containing one representative for each equivalence class of simple objects. This case is beyond the standard formulation of LGT, and only the Spin Foam Model (4.13) makes sense. The partition function is a finite sum. It is again possible to choose ‘topological’ Boltzmann weights which here means the quantum dimension of the simple objects,

\[
\hat{w}(V) = q\text{dim } V. \tag{5.2}
\]

With suitable prefactors the partition function (4.13) agrees with the Crane–Yetter invariant [10]. For a comparison of Figure 12(a) with the main diagram in [10], observe that the state spaces \( H_{0123} \) used in the present paper can be further decomposed employing semi-simplicity (2.26), for example,

\[
H_{0123} = \operatorname{Hom}(V_{123} \otimes V_{013}, V_{012} \otimes V_{023}) \cong \bigoplus_{J \in \mathcal{C}_0} \operatorname{Hom}(V_{123} \otimes V_{013}, J) \otimes \operatorname{Hom}(J, V_{012} \otimes V_{023}). \tag{5.3}
\]

If this decomposition is applied to the state spaces associated with all tetrahedra, one has to colour in addition the tetrahedra with simple objects (the \( J \) in (5.3)) and the tetrahedra (0123) with two types of morphisms, \( \operatorname{Hom}(V_{123} \otimes V_{013}, J) \) and \( \operatorname{Hom}(J, V_{012} \otimes V_{023}) \). These colourings are used in the standard formulation of the Crane–Yetter state sum in [10]. Note that the additional weight \( 1/q\text{dim } J \) per tetrahedron in [10] is a consequence of the choice of bases of \( \operatorname{Hom}(V_{123} \otimes V_{013}, J) \) and \( \operatorname{Hom}(J, V_{012} \otimes V_{023}) \).

5.4 The generic case

The construction presented generalizes LGT and the Crane–Yetter state sum, but also contains the generic case. Here \( \mathcal{C} \) is any admissible ribbon category, in particularly not required to be semi-simple, and the weights \( \hat{w}(V) \) for given simple objects \( V \) can be quite freely chosen. If the set of colours \( \mathcal{C}_0 \) is finite, the partition function is a finite sum and thus well-defined for any choice of weights. If \( \mathcal{C}_0 \) is a countable set, similar convergence issues arise as for Lie groups [17]. Note that here it is also necessary to examine the quantum traces of Figure 12 in order to prove convergence of the partition function.
5.5 Generalizations and the Barrett–Crane model

If $\mathcal{C} = \text{Rep} G$ for a compact Lie group, for example, $G = SU(2)$, and if the Boltzmann weight is chosen to be ‘topological’,

$$w(g) = \delta(g), \quad \hat{w}(V_\rho) = \dim V_\rho, \quad (5.4)$$

the partition function is just a (divergent) formal expression. This is the case for the Ooguri model [13] which can be formulated in the LGT or in the spin foam picture.

The simplest version of a Spin Foam Model of Barrett–Crane type [14] is obtained from the Ooguri model for $SO(4)$ in the spin foam picture by restricting the representations in all sums to the simple representations of $SO(4)$. Simple here means that the representation is of the form $V \otimes V$ as a representation of $SU(2) \times SU(2)$ for some irreducible representation $V$ of $SU(2)$.

In order to implement this restriction one can choose the set of colours $\mathcal{C}_0$ to contain one representative per isomorphism class of simple representations of $SO(4)$. However, in addition one has to restrict the sum over $J$ in (5.3) to simple representations. As a consequence the state spaces $H_{0123}$ are certain subspaces of $\text{Hom}(V_{123} \otimes V_{013}, V_{012} \otimes V_{023})$.

The results of the present paper can be generalized to state spaces that are subspaces of $\text{Hom}(V_{123} \otimes V_{013}, V_{012} \otimes V_{023})$ as long as the pairing (4.5) and the maps $\tau_j$ of Definition 4.4 can be consistently restricted to these subspaces. The correspondence with LGT with a partition function (3.1) is, however, lost as a consequence of this generalization.

6 Conclusion and Outlook

The Spin Foam Model for ribbon categories defined in the present paper generalizes the Spin Foam Model dual to Lattice Gauge Theory (LGT) and can be used as a definition of LGT for gauge groups which are quantum groups rather than Lie groups. Furthermore the definition presented here encompasses state sum models that are of interest both in topology and in quantum gravity. The definition presented provides a bridge between the standard (Lie group) formulation of LGT and the Crane–Yetter invariant which uses ribbon categories. It can also be used to construct other Spin Foam Models that do not correspond to Topological Quantum Field Theories and provides proofs that they are well-defined. This work might finally help to make the relation of LGT and the Spin Foam Models used in other areas more transparent and the common concepts and open questions more accessible.

If one seeks to construct even more general Spin Foam Models than defined here, it is worth pointing out that consistency of the definition restricts the quantum traces of Figure 12 very tightly. The introduction of further weights, however, seems to be much easier to achieve.

The definition of LGT with ribbon categories presented here is restricted to 4 dimensions. Technically, this is due to the fact that the key diagrams in Figure 12 are hand made for this construction. Due to the generality of ribbon categories it involves choices of over- or under-crossings, and only with a good choice is the partition function well-defined. While the corresponding approaches in $d = 3$ [2,3] are canonical in the sense that their construction is well-defined due to general principles, the $d = 4$ construction presented here involves choices and one has to verify a posteriori that it is consistent. It is not obvious whether the result of [29] in arbitrary dimension in the Lie group case can be generalized to ribbon categories. It is, in any case, a striking observation that there exist constructions in $d = 3$ [3] and in $d = 4$ (presented here) which both generalize the Spin Foam Model dual to LGT. Notice, however, that the $d = 3$ case can be handled with spherical categories [7] which are more general than ribbon categories. For the construction in $d = 4$, we make explicit use of ribbon categories because the basic diagrams in Figure 12 always contain a crossing. If ribbon categories are
used in the construction in $d = 3$, all spin network observables can be defined [3] whereas in $d = 4$ there are no canonical expressions for the spin network observables anymore. It therefore seems that one has to use more and more restrictive structures if one wishes to increase the dimension.

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**References**


Figure 13: This diagram is isotopic to the quantum trace $Z_{01234}(V)$ in Figure 12(a), cf. Lemma 4.14 and Lemma 4.16. The morphisms in the dashed boxes are by definition of $\tau_0$ (Definition 4.4) just morphisms $\varphi_{1024} \in H_{1024}$, $\varphi_{1034} \in H_{1034}$ and $\varphi_{1023} \in H_{1023}$. With these replacements this quantum trace is similar to Figure 12(b) defining $Z_{10234}(V)$ for opposite relative orientation with a non-standard order (10234) of vertices. Note that the permutation $(01) \in S_5$ also replaces $V_{012}$, $V_{013}$ and $V_{014}$ by their duals according to Definition 4.1.
Figure 14: This diagram is isotopic to the quantum trace defining $Z_{01234}^{(V),(+)}$ in Figure 12(a), cf. Lemma 4.15 and Lemma 4.17.