Relaxation of a two-level system (TLS) into a resonant infinite-temperature reservoir with a Lorentzian spectrum is studied. The reservoir is described by a complex Gaussian-Markovian field coupled to the nondiagonal elements of the TLS Hamiltonian. The theory can be relevant for electromagnetic interactions in microwave high-Q cavities and muon spin depolarization. Analytical results are obtained for the strong-coupling regime, $\Omega_0 \gg \nu$, where $\Omega_0$ is the rms coupling amplitude (Rabi frequency) and $\nu$ is the width of the reservoir spectrum. In this regime, the population difference and half of the initial coherence decay with two characteristic rates: the most part of the decay occurs at $t \sim \Omega_0^{-1}$, the relaxation being reversible for $t \ll (\Omega_0^2 \nu)^{-1/3}$, whereas for $t \gg (\Omega_0^2 \nu)^{-1/3}$ the relaxation becomes irreversible and is practically over. The other half of the coherence decays with the rate on the order of $\nu$, which may be slower by orders of magnitude than the time scale of the population relaxation. The above features are explained by the fact that at $t \ll \nu^{-1}$ the reservoir temporal fluctuations are effectively one-dimensional (adiabatic). Moreover, we identify the pointer basis, in which the reduction of the state vector occurs. The pointer states are correlated with the reservoir, being dependent on the reservoir phase.

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I. INTRODUCTION

Decoherence of quantum systems is a subject of a significant current interest [1–4], since this phenomenon is of a great importance both for the fundamentals of the quantum theory [5] and for the field of quantum computation [6]. In most treatments of the decoherence and dissipation the thermal reservoir have been assumed to have a sufficiently broad spectrum (a short correlation time), so that the system-reservoir coupling can be considered weak, which allows one to use standard approaches, such as master equations [1,7,8].

For the generic case of a two-level system (TLS) this approach yields the familiar relation between the population (or longitudinal) and coherence (or transverse) relaxation times, $T_1$ and $T_2$ respectively: $T_1 \geq T_2/2$. The equality here is obtained in the case of a transverse reservoir, i.e., when the reservoir variables enter only the nondiagonal elements of the TLS Hamiltonian. Otherwise, when the reservoir is coupled also to diagonal elements of the Hamiltonian, the decoherence may occur much faster than the population relaxation.

Owing to the fast experimental progress in recent years, there appeared a number of novel materials and devices, such as photonic-band structures [9], high-Q cavities [10–12], and semiconductor heterostructures, many of which are of potential relevance for quantum information processing. By creating structures in the electromagnetic continuum, such devices often produce a strong system-reservoir coupling. In such cases the standard master-equation technique is inapplicable and one should resort to other approaches. Fortunately, for zero-temperature reservoirs, the TLS relaxation can be calculated for the general case in quadratures [13]. However, the finite-temperature case, which is important in the radio and microwave frequency ranges, is much less understood. Even the infinite-temperature limit, when the reservoir can be often modeled by a classical random field, is insufficiently studied.

In this paper we report an analytical solution for the dynamics of a TLS which is strongly coupled to a transverse reservoir with a Lorentzian spectrum in the infinite-temperature limit. In this case the reservoir can be described by a classical field, which is a complex Gaussian-Markovian random process [14,15]. This problem can be of relevance for microwave high-Q cavities [10,11] at temperatures of several degrees Kelvin and higher. It describes also a relaxation of a two-level atom in a resonant chaotic field [16–18].

Moreover, the above problem attracted a significant interest recently [15,19,20] in connection with NMR and muon spin depolarization experiments. In this case, numerical calculations [20] showed that the squared coherence may decay slower than the population difference, violating thus the above inequality. However, the above numerical studies did not elucidate the dependence of the relaxation on the parameters of the problem, not to mention a physical explanation of the phenomenon.

Below we obtain a comprehensive picture of the TLS relaxation and provide a physical interpretation of it. In particular, we show that in the strong-coupling regime about half of the coherence still survives after the population relaxation is practically over! The results obtained here are applied to the discussion of the pointer states, a concept of a significant current interest in the theory of quantum measurements [3,4,21,22].

The paper is organized as follows. In Sec. II we formulate the problem and introduce an analogy between the Liouville equation for a TLS and the Schrödinger equation for the spin 1. In Sec. III we discuss stochastic differential equations for partial averages. In Sec. IV
II. FORMULATION OF THE PROBLEM

We consider a TLS with the states $|1\rangle$ and $|2\rangle$ and the resonant frequency $\Delta_0$. The TLS Hamiltonian is

$$H(t) = \hbar \Delta_0 |2\rangle \langle 2| - \frac{\hbar}{2} [\Omega_c(t)|2\rangle \langle 1| + \text{H.c.}],$$

(1)

The interaction of the TLS with a high-temperature reservoir is described by a complex function $\Omega_c(t) = u(t) + iv(t)$, which is supposed to be a complex Gaussian-Markovian random process with the correlation function

$$\langle \Omega_c(t) \Omega_c^*(0) \rangle = \Omega_0^2 e^{-\nu t},$$

(2)

where $\Omega_0$ is the rms coupling amplitude and $\nu^{-1}$ is the correlation time of $\Omega_c(t)$. The reservoir spectrum has a Lorentzian shape with the half width at half maximum equal to $\nu$.

The Hamiltonian (1) has been used to describe muon spin depolarization experiments [19,20]. This Hamiltonian describes also [in the rotating wave approximation (RWA) and in the interaction representation] a TLS with the resonance frequency $\omega_0$ coupled to a Lorentzian reservoir with the spectrum centered at $\omega_c$ (which is the case, e.g., for microwave high-Q cavities [10,11]) or to a laser chaotic field with the nominal frequency $\omega_c$ [16–18]. The RWA validity conditions are $\Omega_0, \nu, |\Delta_0| \ll \omega_0$, where now $\Delta_0 = \omega_0 - \omega_c$.

It is convenient to write the TLS density matrix $\rho(t)$ as the column vector,

$$r \equiv (r_1, r_0, r_{-1})^T = (\rho_{12}, n, \rho_{21})^T,$$

(3)

where $n = \rho_{11} - \rho_{22}$ is the population difference and the superscript $T$ denotes the transpose. Then the Liouville equation for $\rho(t)$ can be written in the form

$$\dot{r} = A(t)r,$$

(4)

where

$$A(t) = i \begin{pmatrix} \Delta_0 & -\Omega_c^*(t)/2 & 0 \\ -\Omega_c(t) & 0 & \Omega_c^*(t) \\ 0 & \Omega_c(t)/2 & -\Delta_0 \end{pmatrix}.$$  

(5)

The solution of Eq. (4) is $r(t) = G(t)r(0)$. Here $G(t)$ is the Green function which obeys the equation

$$\dot{G} = A(t)G$$

(6)

with the initial condition $G(0) = I$, where $I$ is the unity matrix.

It is well known that the Liouville equation (4) can be cast as the equation [23]

$$\dot{s} = \bar{s} \times \vec{B}(t)$$

(7)

for the motion of a classical magnetic moment (pseudospin)

$$\bar{s} \equiv (s_x, s_y, s_z) = (2\text{Re}\rho_{21}, 2\text{Im}\rho_{21}, n)$$

(8)

in the effective magnetic field

$$\vec{B}(t) = (u(t), v(t), \Delta_0).$$

(9)

This analogy helps to obtain an insight into the TLS behavior.

Moreover, it is of interest to consider another analogy, as follows. Under the linear transformation $\psi = S\chi$, where $S$ is a diagonal matrix with $S_{11} = -\sqrt{\Sigma}$, $S_{00} = 1$, and $S_{-1,-1} = \sqrt{2}$, Eq. (4) becomes the Schrödinger equation for a spin 1 in the magnetic field (9),

$$\dot{\chi} = i\tilde{G}(t) \cdot \tilde{S}\chi.$$  

(10)

Here $S_i$ ($i = x, y, z$) is the operator of the $i$th component of spin 1 in the representation of the eigenfunctions of $S_z$ (the cyclic-basis representation) [24]. Equations (7) and (10) imply that the effective gyromagnetic ratio equals 1. Note that the pseudospin $\bar{s}$ has the meaning of the polarization vector [24] for the above spin 1. The analogy (10) can simplify calculations, by allowing one to use the standard textbook techniques developed for the Schrödinger equation.

III. EQUATIONS FOR PARTIAL AVERAGES

Consider the TLS density matrix averaged over such realizations of the random process $\Omega_c(t)$ that assume the value

$$\Omega_c = u + iv \equiv \Omega e^{i\phi}$$

(11)

at $t$. This partially averaged density matrix $\rho(\bar{\Omega}, t)$ written as $r(\bar{\Omega}, t)$ [see Eq. (3)], where $\bar{\Omega} = (u, v)$, is given by

$$r(\bar{\Omega}, t) = G(\bar{\Omega}, t)r(0).$$

(12)

Here $G(\bar{\Omega}, t)$, the partial average of $G(t)$, obeys the equation [16]

$$\dot{G} = A(\bar{\Omega})G + LG,$$

(13)

where $A(\bar{\Omega})$ is given by Eq. (5) with a constant $\Omega_c$. The time dependence of the coupling is taken into account in Eq. (13) by the stochastic operator $L = L_u + L_v$, where

$$L_u = \nu \left(1 + u \frac{\partial}{\partial u} + \frac{\Omega_c^2}{2} \frac{\partial^2}{\partial u^2}\right).$$

(14)
The random vector $\vec{G}$ has a Gaussian stationary distribution $dW(\Omega) = f(\Omega) d\Omega$, where $d\Omega = du dv$ and

$$f(\Omega) = \exp(-\Omega^2/\Omega_0^2)/\pi \Omega_0^2.$$  

(15)

Assuming that the TLS does not interact with the reservoir at $t < 0$, the initial condition for Eq. (13) is $G(\vec{\Omega}, 0) = I f(\Omega)$, where $I$ is the unity matrix.

The fully averaged density matrix is given by

$$\bar{\rho}(t) = G(t) r(0),$$

(16)

where

$$G(t) = \int d\vec{\Omega} G(\vec{\Omega}, t).$$

(17)

Alongside with the forward partial averages $r(\vec{\Omega}, t)$ and $G(\vec{\Omega}, t)$, it may be expedient to consider the backward partial averages $\bar{\rho}(\vec{\Omega'}, t)$ and $\tilde{G}((\vec{\Omega'}, t)$ related by

$$\bar{\rho}(\vec{\Omega'}, t) = \tilde{G}((\vec{\Omega'}, t) r(0),$$

(18)

where the tilde denotes the average over $\vec{\Omega}(t)$ subject to the condition $\vec{\Omega}(0) = \vec{\Omega'}$. As shown in Appendix A, the forward and backward partial averages are related by

$$G(\vec{\Omega}, t) = S^{-2} \tilde{G}^T(\Omega, -\phi, t) S^2 f(\Omega).$$

(19)

Henceforth we shall focus on the exact-resonance case,

$$\Delta_0 = 0.$$  

(20)

Consider a matrix Fourier series

$$G(\vec{\Omega}, t) = \sum_{n = -\infty}^{n = \infty} T_n G_n(\Omega, t).$$

(21)

Here $T_n$ is a diagonal matrix,

$$T_n = \text{diag}(e^{i(n+1)\phi}, e^{in\phi}, e^{i(n-1)\phi}),$$

(22)

and $G_n(\Omega, t) = \langle T_n G(\vec{\Omega}, t) \rangle_\phi$, where the average over $\phi$ is denoted by $\langle \cdots \rangle_\phi = (2\pi)^{-1} \int_0^{2\pi} \cdots d\phi$. Multiplying Eq. (13) by $T_n$ from the left and integrating the both sides of the resulting equation with respect to $\phi$ from 0 to $2\pi$, one obtains the equations

$$\tilde{G}_n = A(\Omega) G_n + M_n G_n,$$

(23)

where $A(\Omega) = A(\Omega, \phi = 0)$ and $M_n = \text{diag}(L_{n+1}, L_n, L_{n-1})$. Here $L_k$ is defined so that for an arbitrary function $F(\vec{\Omega})$

$$\langle e^{ik\phi} L F(\vec{\Omega}) \rangle_\phi = L_k \langle e^{ik\phi} F(\vec{\Omega}) \rangle_\phi.$$  

(24)

On writing $L$ in the polar coordinates [29], one obtains that

$$L_k = L_0 - \kappa^2 \Omega_k^2/2\Omega^2,$$

(25a)

$$L_0 = \frac{\Omega_0^2 \nu}{2} \frac{\partial^2}{\partial \Omega^2} + \left( \nu \Omega + \frac{\Omega_0^2 \nu}{2\Omega} \right) \frac{\partial}{\partial \Omega} + 2\nu.$$  

(25b)

Taking into account that the only nonzero initial conditions for Eqs. (23) are $G_{n=0,n=0}(\Omega, 0) = f(\Omega)$ ($n = 0, \pm 1$), one obtains from Eqs. (23) and (21) that

$$G(\vec{\Omega}, t) = \left( \begin{array}{cc} R(\Omega, t) e^{-i\phi E(\Omega, t)/2} e^{-2i\phi P(\Omega, t)} & -e^{i\phi Q(\Omega, t)}/2 e^{-i\phi Q(\Omega, t)}/2 \\ e^{i\phi Q(\Omega, t)}/2 e^{i\phi Q(\Omega, t)}/2 & R(\Omega, t) \end{array} \right).$$

(26)

The functions entering Eq. (26) satisfy the following sets of equations,

$$\dot{\Phi} = -i\Omega E + L_0 N,$$

(27)

$$\dot{\tilde{P}} = - (i/2) \Omega Q + L_2 P,$$

(28a)

$$\dot{\tilde{R}} = (i/2) \Omega Q + L_0 R.$$  

(28b)

The only nonvanishing initial conditions for Eqs. (27) and (28) are

$$N(\Omega, 0) = f(\Omega), \quad R(\Omega, 0) = f(\Omega),$$

(29)

respectively. The functions $N(\Omega, t), R(\Omega, t)$ and $P(\Omega, t)$ are real, whereas $E(\Omega, t)$ and $Q(\Omega, t)$ are purely imaginary. This follows from the fact that, on changing the variables $E'(\Omega, t) = iE(\Omega, t)$ and $Q'(\Omega, t) = iQ(\Omega, t)$, Eqs. (27) and (28) become sets of equations with real coefficients and real initial conditions.

Averaging Eq. (12) over $\phi$ yields

$$r(\Omega, t) = G(\Omega, t) r(0),$$

(30)

where $r(\Omega, t) = \langle \Phi(\vec{\Omega}, t) \rangle_\phi$. As follows from Eq. (26),

$$G(\Omega, t) = \langle G(\vec{\Omega}, t) \rangle_\phi = \text{diag}(R(\Omega, t), N(\Omega, t), R(\Omega, t)).$$

This means that the population relaxation and the coherence relaxation proceed independently of each other,

$$n(\Omega, t) = N(\Omega, t) n(0),$$

(31)

$$\rho_{12(21)}(\Omega, t) = R(\Omega, t) \rho_{12(21)}(0).$$

(32)

Hence, finally,

$$\tilde{n}(t) = N(t) n(0), \quad \tilde{\rho}_{12(21)}(t) = R(t) \rho_{12(21)}(0),$$

(33)

where the population and coherence relaxation functions are obtained by

$$N(t) = 2\pi \int_0^\infty N(\Omega, t) \Omega d\Omega,$$

(34a)

$$R(t) = 2\pi \int_0^\infty R(\Omega, t) \Omega d\Omega.$$  

(34b)
IV. LIMITING CASES

a. Zero temperature. For the sake of comparison, we mention here the results obtained for the zero-temperature reservoir with a Lorentzian spectrum [28]. In this case, for an arbitrary initial state the TLS relaxation is determined by the equalities \( \rho_{22}(t) = N_0(t) \rho_{22}(0) \) and \( \rho_{12}(t) = R_0(t) \rho_{12}(0) \). The functions \( N_0(t) \) and \( R_0(t) \) can decay monotonously or in an oscillatory fashion, depending on the coupling strength, as described in Ref. [28]. However, for all values of the coupling strength and for sufficiently short times, i.e., as shown below, for the effective field \( \frac{\text{d}}{\text{d}t}\rho \), one can set \( N_0(t) = |R_0(t)|^2 \), which is the extension of the relation \( T_1 = T_2/2 \) for a nonexponential relaxation. The above equality is in sharp contrast with the present infinite-temperature case, as shown below.

b. Weak coupling. Returning to the infinite-temperature case, two limiting cases can be readily considered. For a weak coupling, \( \Omega_0 \ll \nu \), the average population difference and coherence decay exponentially with the decay times \( T_1 = \nu/\Omega_0^2 \) and \( T_2 = 2T_1 \) respectively [20].

c. The static limit. In the opposite case \( \nu \rightarrow 0 \) (the static limit) one can set \( L_u \approx 0 \) in Eqs. (27) and (28), yielding [16,18,25,26]

\[
\begin{align*}
N_{st}(t) & = 1 - \frac{1}{2} \Omega_0^2 t^2, \\
R_{st}(t) & = \frac{1}{2} \Omega_0^2 t^2 + 1/2,
\end{align*}
\]

(35a)

(35b)

where \( F(z) \) is Dawson’s integral [27],

\[
F(z) = e^{-z^2} \int_0^z dy e^{y^2}.
\]

The function (35a) (see Fig. 1, the dot-dashed line) simplifies in two limits,

\[
\begin{align*}
N_{st}(t) & \approx 1 - \frac{1}{2} \Omega_0^2 t^2 / |t| (t \ll \Omega_0^{-1}), \\
N_{st}(t) & \approx -2/\Omega_0^2 t^2 (t \gg \Omega_0^{-1}).
\end{align*}
\]

(37a)

(37b)

It vanishes at \( t = 1.85\Omega_0^{-1} \) and has one minimum equal to \(-0.285 \Omega_0^{-1} \) at \( t = 3.00\Omega_0^{-1} \).

As follows from Eq. (35b), \( \lim_{t \rightarrow \infty} R_{st}(t) = 0.5 \). This nonzero limit results from the fact that in the static limit the component of the TLS pseudospin parallel to the effective field \( \vec{B} = (u, v, \nu) \) is conserved. Equation (35) implies that for \( t \ll \Omega_0^{-1} \) one has \( N(t) \approx R^2(t) \), as in the weak-coupling case. In contrast, for \( t \gg \Omega_0^{-1} \) the ratio \( |N(t)|/R^2(t) \) tends to zero.

Note that in the static limit, the relaxation occurs due to the statistical spread of the coupling amplitude and, correspondingly, in principle it can be reversed by an echo technique. Irreversible relaxation is obtained only in the presence of temporal fluctuations (\( \nu \neq 0 \)).

d. Short times. Consider effects of the temporal fluctuations in the strong-coupling regime,

\[
\Omega_0 \gg \nu,
\]

(38)

for sufficiently short times, i.e., as shown below, for

\[
t^3 \ll D^{-1}.
\]

Here \( D = \Omega_0^2 \nu/2 \) is the diffusion coefficient for \( \dot{\Omega}(t) \) [cf. Eq. (14)]. For short times, one can use the time-dependent perturbation theory, as described in Appendix B, to calculate the backward partial average \( G(\Omega, t) \), which, in turn, yields \( G(\Omega, t) \) by Eq. (19). Comparing the resulting expression with Eq. (26) yields

\[
\begin{align*}
N(\Omega, t) & \approx f(\Omega) [a(\Omega, t) \cos \Omega t + \{D(1 - \Omega^2 t^2)/(2\Omega^3)\} \sin \Omega t], \\
R(\Omega, t) & \approx N(\Omega, t)/2 + f(\Omega)[1/2 - (Dt/2\Omega^2)(2 + \cos \Omega t) + (3D/2\Omega^3) \sin \Omega t],
\end{align*}
\]

(40)

(41)

where \( a(\Omega, t) = 1 - Dt^2/3 - Dt/2\Omega^2 \). Expressions for \( E(\Omega, t), P(\Omega, t) \), and \( Q(\Omega, t) \) are shown in Appendix B.

V. POPULATION RELAXATION

First, we discuss the population relaxation. Equations (27) are related to the Schrödinger equation for some TLS \( \{|a\}, |b\} \) coupled to a resonant chaotic field, as follows. Writing the Hamiltonian of the latter TLS in the form

\[
H_0 = \hbar V_c(t)|b\rangle\langle a| + \text{H.c.},
\]

(42)

where \( V_c(t) \) is a complex Gaussian-Markovian process, one obtains the Schrödinger equation

\[
\dot{U}_{aa} = -iV_c^*(t)u_{ba}, \quad \dot{U}_{ba} = -iV_c(t)u_{aa},
\]

(43)

where \( U_{aa}(t) \) and \( U_{ba}(t) \) are matrix elements of the evolution operator \( \dot{U}(t) \). If the Hamiltonians (1) and (42) describe the same interaction, then \( V_c(t) = -\Omega_c(t)/2 \) and hence

\[
V = \Omega/2, \quad V_0 = \Omega_0/2,
\]

(44)

where \( V = |V_c| \) and \( V_0^2 = \langle V^2 \rangle \).

The averaging of the above quantity \( \dot{U}(t) \) was discussed in Ref. [29]. Writing equations for the partial average quantities \( U_{aa}(\vec{V}, t) \) and \( U_{ba}(\vec{V}, t) \) [29, Eqs. (3.3), (3.7)] and eliminating the field phase, one obtains the equations [30] which can be identified with Eqs. (27) under the substitutions

\[
\begin{align*}
U_{aa}(\vec{V}, t) & = U_{aa}(V, t) \rightarrow N(\Omega, t), \\
U_{ba}(\vec{V}, t) e^{-i\phi} & \rightarrow E(\Omega, t),
\end{align*}
\]

(45)

(46)

(a) \( V \rightarrow \Omega \), (b) \( V_0 \rightarrow \Omega_0 \).

(47)

For instance, for \( \Omega \ll \Omega_0 \), Eq. (40) can be obtained from Ref. [29, Eq. (5.25)], in view of Eqs. (39), (45), and (47). As follows from Eq. (45), the quantity \( \bar{U}_{aa}(t) = \int d\vec{V} U_{aa}(V, t) \) equals \( N(t) \) under the substitution (47b).
Thus, one can apply directly the results of the comprehensive study of $U_{aa}(t)$, performed in Ref. [29], to the population relaxation function $N(t)$. Plots of $N(t)$ for different values of $\nu/\Omega_0$ are shown in Fig. 1.

![Graph of population relaxation function](image)

**FIG. 1.** The population relaxation function $N(t)$ versus the dimensionless time $\Omega_0 t$ for the values of $\nu/\Omega_0$ shown in the plot.

Henceforth, we focus on the strong-coupling regime, Eq. (38). In this case, as follows from [29], the population relaxation is described by

$$N(t) = N_{st}(t)J(\alpha t),$$

(48)

where $\alpha = (2D)^{1/3} = (\Omega_0^2/\nu)^{1/3}$ and $J(\alpha t)$ is a dimensionless function of $\alpha t$, which describes the irreversible relaxation. For short times, the function $N(t)$ is [31]

$$J(\alpha t) \approx 1 - D t^3/3.$$  

(49)

One can show that the second-order correction to Eq. (49) is on the order of $D^2 t^6$, which implies the validity condition (38) for the above short-time results.

For $t^3 \gg D^{-1}$, $J(\alpha t)$ tends to zero, performing damped oscillations (see Fig. 2). Correspondingly, the characteristic rate of the irreversible relaxation is on the order of $D^{1/3} \sim \alpha$.

**VI. COHERENCE RELAXATION**

Consider the coherence relaxation. We shall focus on the strong-interaction regime, Eq. (38). At short times the coherence relaxation function $R(t)$ is obtained by inserting Eq. (41) into (34b) and performing the integration. For $t \ll \Omega_0^{-1}$ the $R(t)$ is very close to the static result, the discrepancy increasing with $t$. As discussed in Appendix B, for $\Omega_0^{-2} \ll t^2 \ll D^{-2/3}$

$$R(t) \approx 1/2 - 1/(\Omega_0^2 t^2) + \nu t (C_0 - \ln \Omega_0 t).$$

(50)

Here $C_0 = 5/3 - \gamma/2 = 1.38$, where $\gamma$ is the Euler constant [27].

![Graph of coherence relaxation function](image)

**FIG. 2.** The function $J(\alpha t)$ versus the scaled time $\alpha t$.

The coherence relaxation for $t^3 \gg D^{-1}$ can be found, as follows. Casting Eqs. (28) as an equation for the column vector

$$q = (q_1, q_0, q_{-1})^T = (P, Q, R)^T$$

(51)

and performing the linear transformation $\chi(\Omega, t) = S q(\Omega, t)$, Eqs. (28) become

$$\dot{\chi} = (i\Omega S_x + M)\chi.$$  

(52)

Here $M$ is a diagonal matrix with $M_{mm} = L_{1+m}$. Note that Eq. (52) can be obtained also from partially averaged Eq. (10), $\hat{\psi}(\Omega, t) = i\vec{B} \cdot \vec{S}\psi + L\psi$, with $\vec{B} = (u, v, 0)$, on defining

$$\chi_m(\Omega, t) = \langle e^{i(1+m)\phi}\psi_m(\Omega, t)\rangle/\psi_{-1}(0)$$

(53)

($m = 1, 0, -1$). Thus, $\chi(\Omega, t)$ has the meaning of a weighted partial average of the wave function for spin 1 in the stochastic magnetic field $\vec{B}(t) = (u(t), v(t), 0)$.

Next, we invoke the semiclassical dressed-state representation, by diagonalising the first term on the r.h.s. of Eq. (52). Since Eq. (52) has the form of a (partially averaged) Schrödinger equation, this can be done with the help of the standard finite-rotation operator. Namely, we rotate the $z$ axis around the $y$ axis by $\pi/2$, $\chi'(\Omega, t) = S_y\chi(\Omega, t)$, where $S_y = i(1/\pi)(\pi/2)$ is defined in [32]. As a result, one obtains

$$\dot{\chi}' = i[\Omega S_z - (D/2\Omega^2)C + L_0]\chi'.$$  

(54)

Here

$$C = \begin{pmatrix} 3 & -\sqrt{2} & 1 \\ -\sqrt{2} & 4 & -\sqrt{2} \\ 1 & -\sqrt{2} & 3 \end{pmatrix}.$$  

(55)

The vector $\chi'$ is related to $q$ by $\chi' = S_1 q$, where

$$S_1 = S_0 S = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 1 \\ -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$  

(56)
Correspondingly, the initial condition for Eq. (54) is
\[ \chi'(0, t) = f(\Omega)\chi(0), \]
where \( \chi(0) = (1/\sqrt{2}, 1/\sqrt{2})^T \).

The inverse linear transform \( q = S^{-1}\chi' \) yields
\[ R(\Omega, t) = \text{Re}\chi_1'(\Omega, t)/\sqrt{2} + \chi_0'(\Omega, t)/2. \]  
(57)

Here we took into account that \( \chi'_1(\Omega, t) = \chi''_{-1}(\Omega, t) \),
which follows from the form of Eq. (54) and the initial condition for it. On introducing new variables by
\[ \chi'_m(\Omega, t) = K_m(\Omega, t)\chi_0(\Omega, t), \]  
(58)
Eq. (57) becomes
\[ R(\Omega, t) = [\text{Re}K_1(\Omega, t) + K_0(\Omega, t)]/2. \]  
(59)

Note that \( K_{-1}(\Omega, t) = K_1(\Omega, t) \). As follows from Eq. (58)
\[ K_0(\Omega, 0) = K_1(\Omega, 0) = f(\Omega). \]  
(60)

For \( \Omega \gg \alpha \), the nondiagonal terms of the matrix coefficient in Eq. (54) are much less than the differences of the diagonal terms (which are on the order of \( 2\Omega \)) and can be neglected in the first (secular) approximation. This results, in view of Eqs. (58), in the following equations,
\[ K_1 = (i\Omega - 3D/2\Omega^2 + L_0)K_1, \]  
(61a)
\[ K_0 = (-2D/\Omega^2 + L_0)K_0, \]  
(61b)
with the initial conditions (60).

Consider now Eq. (61a). Note first that in Eqs. (61a) and (25b) the second terms in the parentheses are small as compared to the first terms, respectively, and hence can be neglected. Furthermore, performing the change of variables \( K_1(\Omega, t) = e^{\nu t}K(\Omega, t) \), Eq. (61a) becomes approximately
\[ K = (i\Omega + L_\Omega)K, \]  
(62)
where \( L_\Omega \) is given by Eq. (14) with \( u \to \Omega \). Equation (62) describes the dephasing of a two-level system due to Gaussian-Markovian frequency fluctuations. The solution of Eq. (62) [33] implies that in the strong-interaction regime (38) considered
\[ K_1(\Omega, t) \approx K(\Omega, t) \approx f(\Omega)e^{i\Omega t - D^3/3}, \]  
(63)
i.e., \( K_1(\Omega, t) \) decays with the rate on the order of \( \alpha \).

On the other hand, for \( \Omega \lesssim \alpha \), the equations (54) for \( \chi'_m(\Omega, t) \) are strongly coupled. An analysis of Eq. (54) shows that for \( \Omega \lesssim \alpha \), \( \chi'(\Omega, t) \) and hence \( K_m(\Omega, t) \) [see Eq. (58)] disappear on the time scale \( \alpha^{-1} \). As a result, in view of Eqs. (59) and (63), for \( t \gg \alpha^{-1} \)
\[ R(\Omega, t) \approx \begin{cases} 0, & \Omega \lesssim \alpha \\ K_0(\Omega, t)/2, & \Omega \gg \alpha. \end{cases} \]  
(64)

Since \( \alpha \ll \Omega_0 \), the fact that \( K_0(\Omega, t) \approx 0 \) for \( \Omega \ll \alpha \) can be taken into account approximately by the boundary condition to Eq. (61b),
\[ K_0(0, t) = 0. \]  
(65)

Equation (61b) can be solved by the conjecture
\[ K_0(\Omega, t) = f(\Omega)g(h(t)\Omega), \]  
(66)
where the functions \( g(X) \) and \( h(t) \) are to be found, using the initial and boundary conditions (60) and (65). As shown in Appendix C, the solution yields
\[ K_0(\Omega, t) = C_2 f(\Omega)\chi K_n(\Omega, k, 2k + 1, -\zeta), \]  
(67)
where \( k = 1/\sqrt{2} \), \( \zeta = \Omega^2/\Omega_0^2(e^{2\nu t} - 1) \), \( M() \) is the degenerate hypergeometric function [27], and
\[ C_2 = \Gamma(1 + k)/\Gamma(1 + 2k), \]  
(68)
where \( \Gamma() \) is the \( \Gamma \)-function [27].

As follows from Eqs. (34b) and (64),
\[ R(t) \approx K_0(t)/2 \quad (t \gg \alpha^{-1}), \]  
(69)
where
\[ K_m(t) = 2\pi \int_0^\infty K_m(\Omega, t)\Omega d\Omega. \]  
(70)

Inserting here Eq. (67) and performing the integration with the help of Ref. [34, Eq. 7.621.4], one obtains
\[ K_0(t) = C_1 e^{-2k^2 t}F(k, k; 1 + 2k; e^{-2\nu t}). \]  
(71)

Here \( C_1 = \Gamma^2(1 + k)/\Gamma(1 + 2k) \approx 0.66 \) and \( F() \) is the hypergeometric function [27]. The function (71) monotonously decreases from 1 to 0 with the average rate on the order of \( \nu \). Equations (69) and (71) yield the following limits
\[ R(t) \approx 1/2 + (\nu t/2)(\ln t - C_3) \quad (\alpha^{-1} \ll t \ll \nu^{-1}), \]  
(72a)
\[ R(t) \approx C_1 e^{-2k^2 t} \quad (e^{2\nu t} \gg 1). \]  
(72b)

Here \( C_3 = 1 + 2k - 2\psi(1 + k) - 2\gamma - \ln 2 \approx 0.14, \) where \( \psi() \) is defined in [27]. Equation (72a) was obtained with the help of the expansion 15.3.11 in [27] for the hypergeometric function.

Equations (50) and (72a) describe the behavior of \( R(t) \) at \( t \ll \nu^{-1} \). The last term on the right-hand side (rhs) of Eq. (50) (which arises due to the temporal fluctuations) decreases with the time, increasing by the magnitude, in contrast to the second term (pertaining to the static limit). Note, however, that the last term on the rhs of Eq. (50) is much less than the second term, i.e., \( R(t) \approx R_m(t) \) for \( t \ll \alpha^{-1} \). The magnitudes of the above terms become of the same order \( (\nu/\Omega_0)^2/3 \) for \( t \sim \alpha^{-1} \), where \( R(t) \) has a maximum. Finally, for \( t \gg \alpha^{-1} \) \( R(t) \) decreases, the coherence dynamics being determined by the temporal fluctuations [see Eqs. (69), (71), (72)].
In the first approximation, $R(t)$ can be described for all times by the formula

$$R(t) = R\text{a}(t)K_0(t).$$

(73)

Indeed, as follows from Eq. (72a) and the above discussion, for $t \lesssim \alpha^{-1}$ the error in Eq. (73) increases with $t$ from 0 to the value of the order of $(\nu/\Omega_0)^{2/3} \ll 1$ at $t \sim \alpha^{-1}$, whereas for $t \gg \alpha^{-1}$ Eq. (73) is close to (69) with the relative error not exceeding by the order of magnitude $(\nu/\Omega_0)^{2/3} \ll 1$.

Note that the interpolation formula (73) is not unique. Instead of it, one can use with the same accuracy, e.g., the formula

$$R(t) = [N\text{a}(t) + K_0(t)]/2.$$

(74)

The both formulas provide similar results.

![Figure 3](image)

FIG. 3. The coherence relaxation function $R(t)$ versus $\log_{10}\Omega_0 t$ for the values of $\nu/\Omega_0$ shown in the figure. Solid lines, numerical solution; dashed lines, Eq. (73) for $\nu/\Omega_0 = 0.01$ and 0.0001.

Figure 3 demonstrates $R(t)$ for different values of $\Omega_0/\nu$. The solid lines are calculated numerically by inverting the continued fraction describing a Fourier transform of $R(t)$ [19], whereas the dashed lines are the plots of Eq. (73) for $\nu/\Omega_0 = 0.01$ and 0.0001. As follows from Fig. 3, Eq. (73) provides a good approximation to the exact result in the strong-coupling regime, the accuracy increasing with the decrease of $\nu/\Omega_0$ (in particular, at $\nu/\Omega_0 = 0.0001$ the solid and dashed curves coincide almost completely in Fig. 3).

VII. DISCUSSION

As shown above, in the regime of a strong coupling to the reservoir, $\Omega_0 \gg \nu$, the TLS behavior is rather complicated. The population relaxation is characterized by two time scales. On the time scale $\Omega_0^{-1}$, where most of the population relaxation occurs, the latter is of a reversible character, whereas on the time scale $\alpha^{-1}$ the population relaxation becomes irreversible and proceeds to completion.

The coherence relaxation is even more complicated. Roughly speaking, half of the coherence decays similarly to the population relaxation. However, for $t \gg \alpha^{-1}$, when the population relaxation is already completed, almost half of the initial coherence still survives. The decay time of the latter is on the order of $\nu^{-1}$. This time is greater than the largest population relaxation time $\alpha^{-1}$ by the large factor $(\Omega_0/\nu)^{2/3}$.

To explain the above behavior, we write the pseudospin $\vec{s} = \vec{s}_\perp + \vec{s}_\parallel$, where $\vec{s}_\perp = (s_x^\perp, s_y^\perp, n)$ and $\vec{s}_\parallel = (s_x^\parallel, s_y^\parallel, 0)$ are respectively perpendicular and parallel to the initial effective field $\vec{B}(0) = (u', v', 0)$. Here we denoted $\Omega_c(0) = \Omega_c' = u' + iv' = \Omega' e^{i\phi'}$. The pseudospin component $\vec{s}_\perp$ describes the population difference $n$ and the ‘out-of-phase’ coherence $\rho_{21}^\perp = (s_x^\perp + is_y^\perp)/2 = ie^{i\phi'} \text{Im}(\rho_{21} e^{-i\phi'})$, whereas $\vec{s}_\parallel(t)$ describes the “in-phase” coherence $\rho_{21}^\parallel = (s_x^\parallel + is_y^\parallel)/2 = e^{i\phi'} \text{Re}(\rho_{21} e^{-i\phi'})$.

Except for the case $\Omega' \lesssim \alpha$, which has a negligibly small probability, during the time $\sim \alpha^{-1}$ the direction of $\Omega(t)$ almost does not change. As a result, the backward average of $\vec{s}_\perp(t)$ (i.e., the average with a given $\Omega'_c$) and hence the backward-averaged population and out-of-phase coherence decay at the rate $\alpha$, as in the case of collinear (adiabatic) field fluctuations [7,29,33]. By the same reason, the backward-averaged in-phase coherence almost does not decay at $t \lesssim \alpha^{-1}$. Actually, it can decay significantly only after the field rotates by an angle of the order of $\pi/2$, which, on the average, requires the time of the order of the correlation time $\nu^{-1}$. The averaging of the above quantities over $\Omega'_c$ (i.e., the full averaging) does not change the time scales, at which they vanish, whereas it provides equal contributions from the in- and out-of-phase coherences. This explains the above fact that the populations and half of the coherence relax faster than the other half of the coherence.

The above argument can be summarized as follows. The reservoir considered here is two-dimensional and symmetric with respect to the rotations in the $xy$ plane (of the pseudospin space). However, for $t \ll \nu^{-1}$ it behaves effectively as a one-dimensional one, since the direction of the effective field $\vec{B}(t)$ does not change significantly over short time intervals. In the strong-coupling regime, this fact results in a substantially asymmetric behavior of the pseudospin: the component $\vec{s}_\perp(t)$ decays much faster than $\vec{s}_\parallel(t)$, as discussed above. In contrast, in the weak-coupling regime, where the decay rates $T_1$ and $T_2$ are much longer than the correlation time $\nu^{-1}$ (see Sec. IV), the short-time behavior of the reservoir is not important, and the density matrix elements relax with similar rates.
The above results have a bearing on the the pointer states and related concepts [3, 4, 21, 22]. It is well known (Ref. [21] and references therein) that an important part of any quantum measurement is the reduction of the state vector, i.e., the diagonalization of the density matrix of the measuring apparatus in a special (pointer) basis, the diagonal elements remaining intact. The apparent contradiction of this process to quantum mechanics, where any evolution is unitary, has not obtained yet a completely satisfactory explanation. The resolution of the above contradiction suggested by Zurek [21] is that the reduction of the state vector occurs due to the coupling of the apparatus with the environment.

One can ask which conditions the coupling of the apparatus (called below the system) and the environment (reservoir) should satisfy to be capable to produce the state reduction. There are, at least, two such conditions, as follows. The first condition requires that the pointer basis be independent of the state of the reservoir [21]. Another condition stems from the apparent contradiction between the role, which the reservoir, according to Zurek, plays in obtaining the state reduction, and the second law of thermodynamics, which postulates that any evolution is unitary, has not obtained yet a completely satisfactory explanation. The resolution of the above contradiction suggested by Zurek [21] is that the reduction of the state vector occurs due to the coupling of the apparatus with the environment.

Consider the vector
\[ \tilde{\rho}(\phi', t) = 2\pi \int_0^\infty \tilde{\rho}(\Omega', t') f(\Omega') d\Omega'. \]  
(76)

The function \( \tilde{G}(\phi', t) = 2\pi \int_0^\infty \tilde{G}(\Omega', t') f(\Omega') d\Omega' \) is related to
\[ G(\phi, t) = 2\pi \int_0^\infty G(\Omega, t) d\Omega \]  
(77)
by \([\text{cf. Eq. (19)}]\)
\[ \tilde{G}(\phi', t) = S^{-2}G^T(-\phi', t)S^2. \]  
(78)

Combining Eqs. (26), (78) and (79) yields
\[ \tilde{G}(\phi', t) = \begin{pmatrix} R(t) & -e^{-i\phi'} Q(t)/2 & e^{-2i\phi'} P(t) \\ e^{i\phi'} E(t) & N(t) & -e^{-i\phi'} E(t) \\ e^{2i\phi'} P(t) & e^{i\phi'} Q(t)/2 & R(t) \end{pmatrix}. \]  
(79)

The functions entering Eq. (80) are obtained from the functions appearing in Eq. (26) by the integration, as in Eqs. (34). One can express \( q(t) = (P(t), Q(t), R(t))^T \) [cf. Eq. (51)] through \( K_m(t) \), Eq. (70), using \( q(t) = S^{-1}_1 \chi'(t) \), where \( \chi'_m(t) = K_m(t)\chi'^{(0)}_m \) [cf. Eq. (58)]. This yields
\[ P(t) = [2K_0(t) - K_1(t) - K_{-1}(t)]/4, \]  
(80a)
\[ Q(t) = [K_1(t) - K_{-1}(t)]/\sqrt{2}, \]  
(80b)
\[ R(t) = [2K_0(t) + K_1(t) + K_{-1}(t)]/4. \]  
(80c)

Inserting Eq. (80) into (77) yields \( \tilde{r}(\phi', t) \) and hence \( \tilde{\rho}(\phi', t) \) [cf. Eq. (3)]. The transformation of the latter to the basis (75) is performed by \( \tilde{S} = S^1\tilde{\rho}(\phi', t)\tilde{S} \), where
\[ \tilde{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ e^{i\phi'} & -e^{-i\phi'} \end{pmatrix}. \]  
(81)

Finally, one obtains the following expressions for the components \( \tilde{\rho}_{ij}(\phi', t) \) of \( \tilde{\rho}(\phi', t) \),
\[ \tilde{\rho}_{++}(\phi', t) = 1/2 \pm K_0(t) \text{Re}[\rho_{21}(0)e^{-i\phi'}], \]  
(82a)
\[ \tilde{\rho}_{+-}(\phi', t) = \tilde{\rho}^*_-(\phi', t) = [N(t) + Q(t)]n(0)/2 + i\text{Im} \left\{ [K_1(t) + K_{-1}(t) - 2E(t)]p_{21}(0)e^{-i\phi'} \right\} /2, \]  
(82b)

Equations (83) imply that \( \tilde{\rho}(\phi', t) \) becomes diagonal at \( t \gg \alpha^{-1} \), since the functions of time appearing in rhs of Eq. (83b) vanish at such times [see Secs. V and VI and Eq. (81b)]. Taking into account that \( K_0(t) \approx 1 \) for \( t \ll \nu^{-1} \), in the time interval
\[ \alpha^{-1} \ll t \ll \nu^{-1} \]  
(83)}
one obtains that
\[
\rho'(\phi', t) \approx \begin{pmatrix} \rho_{++}(0) & 0 \\ 0 & \rho_{--}(0) \end{pmatrix},
\]
(85)

where \( \rho_{++(-)}(0) = 1/2 \pm \text{Re}[\rho_{21}(0)e^{-i\phi}] \) are the diagonal elements of \( \rho'(\phi', 0) \).

Thus we have proved that in the interval (84) the reservoir produces the reduction of the state vector in the basis (75). In contrast to the previous cases described in the literature, here the pointer states (75) depend on the state of the reservoir, due to their dependence on the reservoir phase \( \phi' \). Since this violates the above first condition, the strong coupling considered here is not of a type allowed for the apparatus-environment interaction.

IX. CONCLUSION

Above we presented a comprehensive analysis of the relaxation of a TLS coupled to an infinite-temperature reservoir. We showed that in the strong-coupling regime the decoherence can proceed much slower than the population relaxation and provided a physical interpretation of this effect. We identified the pointer states, which, in contrast to the previous findings, appeared to be correlated with the reservoir.

The present results can be checked, e.g., in experiments with high-\( Q \) microwave cavities [10,11]. Since the theory holds in the infinite-temperature limit, one should require that, at least, the average number of the photons \( n_{\text{ph}} \) in the resonance mode be large. Note that for a cavity mode with the frequency 21.5 GHz, used in experiments in Ref. [10], at the temperature 5 K \( n_{\text{ph}} = 4.4 \). An increase in the cavity temperature \( T \) and/or mode wavelength \( \lambda_c \) can significantly increase the above number, since for high temperatures \( n_{\text{ph}} > 1 \) \( n_{\text{ph}} \propto T \lambda_c \).

A more detailed estimation of the experimental conditions would require a consideration of corrections to the above results due to finite, though large, values of \( n_{\text{ph}} \), which is out of the scope of the present paper [37]. We believe, however, that the main results of this paper, in particular, those concerning the anomalously slow decoherence and the pointer states, will remain valid, at least, qualitatively, also for moderately large \( n_{\text{ph}} \).

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APPENDIX A: RELATION BETWEEN FORWARD AND BACKWARD PARTIAL AVERAGES

Let us derive the relation between \( \tilde{G}(\tilde{\Omega}', t) \) and \( G(\tilde{\Omega}, t) \). It is convenient to consider first the Green function \( U(t) \) of Eq. (10) defined by \( \psi(t) = U(t)\psi(0) \). The backward and forward partial averages of \( U(t) \) are denoted by \( \tilde{U}(\tilde{\Omega}', t) \) and \( U(\tilde{\Omega}, t) \). The latter function satisfies the equation
\[
\tilde{U} = [A_1(\tilde{\Omega}) + L]U
\]
(A1)

with the initial condition \( U(\tilde{\Omega}, 0) = I \). The Gaussian-Markovian random process satisfies the detailed-balance condition, which implies that Eq. (A2) can be recast as
\[
\frac{\partial \tilde{U}}{\partial t} = [A_1^T(\tilde{\Omega}) + L^T] \tilde{U},
\]
(A3)

where \( \tilde{U}(\tilde{\Omega}, t) = \tilde{U}(\tilde{\Omega}, t)f(\Omega) \). Note also that \( A_1(\tilde{\Omega}) \equiv A_1(\Omega, \phi) \), as defined above, obeys \( A_1^T(\Omega, \phi) = A_1(\Omega, -\phi) \). With the account of this, the comparison of Eqs. (A1) and (A3) shows that
\[
U(\tilde{\Omega}, t) = \tilde{U}^T(\Omega, -\phi, t)f(\Omega).
\]
(A4)

Taking into account that \( U(\tilde{\Omega}, t) = SG(\tilde{\Omega}, t)S^{-1} \) and \( \tilde{U}(\tilde{\Omega}, t) = SG(\tilde{\Omega}, t)S^{-1} \), Eq. (A4) yields finally Eq. (19).

APPENDIX B: PERTURBATION THEORY

Consider the perturbation theory in the strong-coupling regime, Eq. (38). One can write \( \tilde{\Omega}(t) = \tilde{\Omega}' + \tilde{W}(t) \), where \( \tilde{\Omega}' = (\nu', \nu') = \tilde{\Omega}(0) \) and \( \tilde{W}(t) = [W_1(t), W_2(t)] \) obeys \( \tilde{W}(0) = 0 \). Under the unitary transformation \( \psi' = S_2\psi \), where \( S_{2mm'} = D_{mm'}^{(1)}(\phi', \pi/2, 0) = i^{m-m'}(\pi/2)e^{im'\phi'} \) is the finite-rotation matrix [32], Eq. (10) becomes
\[
\dot{\psi}' = i[\Omega' S_z + A_2(t)]\psi'.
\]
(B1)

Here \( A_2(t) = -2[W_1(t)S_z + W_2(t)S_y] \), where \( W_1(t) \) and \( W_2(t) \) are defined by \( W_1(t) + iW_2(t) = [W_1(t) + iW_2(t)]e^{-i\phi} \).

Solving Eq. (B1) with the help of the time-dependent perturbation theory of the second-order in \( A_2(t) \) and averaging the result over \( \tilde{W}(t) \) yields
\[
\tilde{\psi}'(\Omega', t) = \tilde{U}^*(\Omega', t)\psi'(0),
\]
(B2)

where
\[ \tilde{U}'(\Omega', t) = \tilde{U}'_0(t) - i \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{U}_0(t - t_1) \langle A_2(t_1) \rangle \tilde{U}_0(t_2) \]
\[ \times \tilde{U}_0'(t_2). \]  
(B3)

Here the tilde and the angular brackets denote the average over \( \tilde{W}'(t) \), whereas \( \tilde{U}'_0(t) = e^{i\Omega'tS'}. \)

The functions \( \tilde{W}' + \tilde{W}'_1(t) \) and \( \tilde{W}'_2(t) \) are independent identical Gaussian-Markovian processes, which equal respectively \( \tilde{\Omega}' \) and \( 0 \) at \( t = 0 \). For \( t < \nu^{-1} \) \( \tilde{W}'_2(t) \) can be considered as a diffusion (Wiener) random process. The same holds for \( \tilde{\Omega}' + \tilde{W}'_1(t) \) unless \( \tilde{\Omega}' \) is too large in comparison with \( \Omega_0 \). As follows from Eq. (39), the approximation (B3) holds for \( t \ll \alpha^{-1} \ll \nu^{-1} \), which means that for \( \tilde{\Omega}' \ll (\tilde{\Omega}'_0 W'_1(t) \) and \( \tilde{W}'_2(t) \) in Eq. (B3) can be considered as independent identical diffusion processes with the initial conditions \( W'_{1,2}(0) = 0 \).

Performing the averaging in Eq. (B3) with the help of the relations \( \langle \tilde{W}'(t_1) \rangle = 0 \) and \( \langle \tilde{W}'(t_1) \tilde{W}'(t_2) \rangle = 2D\tilde{\Omega}'_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol, yields

\[ \tilde{U}'(\Omega', t) = \tilde{U}'_0(t)[1 - (D/m^3) \tilde{S}'_x^2] - 2D \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{U}_0(t - t_1) \tilde{S}'_3 \tilde{U}_0(t_1 - t_2) \tilde{S}_3 \tilde{U}_0'(t_2). \]  
(B4)

Under the inverse linear transformation, \( \tilde{r}(\Omega', t) = S_3^{-1} \tilde{U}'(\Omega', t) \), where \( S_3 = S_2 S \), Eq. (B2) becomes (18) with

\[ \tilde{G}(\Omega, t) = S_3^{-1} \tilde{U}'(\Omega, t) S_3. \]  
(B5)

From Eqs. (B4), (B5), and (19) one obtains finally \( \tilde{G}(\Omega, t) \) and hence Eqs. (40), (42), and

\[ E(\Omega, t) \approx -2i f(\Omega) \left[ \frac{a(\Omega, t)}{2} \sin \Omega t + \frac{Dt^2}{4\Omega} \cos \Omega t \right], \]  
(B6)

\[ P(\Omega, t) \approx f(\Omega) \left[ \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{Dt^3}{3} - \frac{D^2t}{2\Omega^2} \right) \cos \Omega t \right. \]
\[ + \frac{D(1 + \Omega^2 t^2)}{4\Omega^3} \sin \Omega t - \frac{Dt^2}{\Omega^2} \right], \]  
(B7)

\[ Q(\Omega, t) \approx i f(\Omega) \left[ \left( 1 - \frac{Dt^3}{3} - \frac{D^2t}{2\Omega^2} \right) \sin \Omega t \right. \]
\[ + \frac{D(\Omega^2 t^2 - 4)}{2\Omega^3} \cos \Omega t + \frac{2D}{\Omega^3}. \]  
(B8)

Consider now the calculation of \( R(t) \) for \( \Omega_0^{-2} \ll t^2 \ll D^{-2/3} \). Inserting Eq. (41) into (34b) one obtains

\[ R(t) = 1/2 + N(t)/2 + \pi D t (I_1 - 2I_2). \]  
(B9)

Here \( N(t) \) is given by Eqs. (48) and (49),

\[ I_1 = \int_0^\infty \sin x - x \cos x \left( \frac{x}{t} \right) dx \]
\[ \approx f(0) \int_0^\infty \sin x - x \cos x \frac{x}{x^2} dx = f(0), \]  
(B10)

where the approximate equality holds since \( t \gg \Omega^{-2} \) and the latter equality results from Eq. 3.784.4 in Ref. [34], and

\[ I_2 = \int_0^\infty \frac{x - \sin x}{x^2} f(x) dx \equiv J_1 + J_2 + J_3. \]  
(B11)

In Eq. (B11) \( I_2 \) is splitted into three integrals, which are calculated for \( t \gg \Omega_0^{-1} \) as follows,

\[ J_1 = \int_0^1 \frac{x - \sin x}{x^2} f(x) dx \approx f(0) \int_0^1 \frac{x - \sin x}{x^2} dx \]
\[ = f(0)[\gamma + \sin 1 - 1 - Ci(1)], \]  
(B12)

\[ J_2 = -\int_1^\infty \frac{x - \sin x}{x^2} f(x) dx \approx f(0)[Ci(1) - \sin 1], \]  
(B13)

and

\[ J_3 = \int_1^\infty f(x) \frac{dx}{x^2} = f(0) \left( \frac{1}{2} \right) E_1 \left( \frac{1}{12} t^2 \right) \]
\[ \approx f(0)(\ln \Omega_0 t - \gamma/2). \]  
(B14)

Here \( E_1(\cdot) \) is the integral exponential function and \( Ci() \) is the integral cosine [27]. The integrals in Eqs. (B12) and (B13) were calculated with the help of the Mathematica software [38], whereas the approximate equality in Eq. (B14) results from Ref. [27, Eq. 5.1.11]. Combining Eqs. (B9)-(B14) yields Eq. (50).

**APPENDIX C: SOLUTION OF EQ. (61b)**

Inserting Eq. (66) into Eq. (61b) and taking into account Eq. (25b), one obtains the equation

\[ \hbar X \frac{h}{X} g' = Dh^2 g'' + \left( \frac{Dh^2}{X} - \nu X \right) g' - \frac{2Dh^2}{X} g, \]  
(C1)

where \( X = h(t) \Omega \) and the prime denotes the derivative of \( g(X) \) with respect to \( X \). By the assumption, \( g(X) \) depends on the time only through \( X \), which means that the coefficients in Eq. (C1) should not depend explicitly on the time. This is indeed so, if

\[ \hbar + \nu \hbar = -D \hbar^3. \]  
(C2)

Then Eq. (C1) becomes

\[ g'' + (1/X + X) g' - (2/X^2) g = 0. \]  
(C3)

Equations (61b) and (60) and the fact that \( L_0 f(\Omega) = 0 \) result in \( K_0(\Omega \to \infty) \to f(\Omega) \). The initial and boundary
where \( g(0) = 0 \), \( bh(0)\Omega = 1 \), \( g(\infty) = 1 \). (C4)

The comparison of conditions (C4b) and (C4c) shows that \( h(0) = \infty \).

Solving Eq. (C2) with the latter condition yields

\[
X = (\Omega/\Omega_0)\sqrt{2/(e^{2\nu t} - 1)}. \quad (C5)
\]

The solution of Eq. (C3) is obtained by the method in Ref. [39]. The substitution \( g(X) = (-Y)^k g_1(Y) \), where \( k = 1/\sqrt{2} \) and \( Y = -X^2/2 \), reduces Eq. (C3) to the degenerate hypergeometric equation [39, Sec. 2.113]

\[
Y g'' + (2k + 1 - Y)g' - kg_1 = 0, \quad \text{(C6)}
\]

where the prime denotes the derivative with respect to \( Y \). The solution of Eq. (C6) yields

\[
g(X) = C_2 \zeta^k M(k, 1 + 2k, -\zeta) + C_3 \zeta^{-k} M(-k, 1 - 2k, -\zeta), \quad \text{(C7)}
\]

where \( \zeta = X^2/2 \) and \( C_2, C_3 \) are constants. The boundary condition (C4a) yields \( C_2 = 0 \), whereas the boundary condition (C4c) and the asymptotic formula for \( M(\zeta) \) [27, Eq. 13.5.1] yield Eq. (68). As a result, Eqs. (66) and (C7) yield Eq. (67).

[30] Their Laplace transforms in a slightly modified form are given by Eqs. (B3) in Ref. [29].
[31] In principle, for short times the function \( N(t) \) is obtained by inserting Eq. (40) into (34a) and performing the integration. However, noting that for \( t >> \Omega_0^{-1} \), only small \( \Omega, \Omega \ll \Omega_0 \), contribute to the integral, one can use Eq. (5.26) in Ref. [29] to obtain Eq. (49), whereas for \( t \approx \Omega_0^{-1} \), \( J(\alpha t) \) is very close to 1 and one can still use Eq. (49), the error being of the order of or less than \( \nu/\Omega_0 \).
[34] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Academic, Orlando, 1965).


