Precision Determination of the Pion Form Factor and Calculation of the Muon $g - 2$

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Abstract

We perform a new calculation of the hadronic contributions, $a_{\text{Hadronic}}$, to the anomalous magnetic moment of the muon, $a_{\mu}$. For the low energy contributions of order $\alpha^2$ we carry over an analysis of the pion form factor $F_{\pi}(t)$ using recent data both on $e^+e^- \rightarrow \pi^+\pi^-$ and $\tau \rightarrow \bar{\nu}_\tau \pi^+\pi^0$. In this analysis we take into account that the phase of the form factor is equal to that of $\pi\pi$ scattering. This allows us to profit fully from analyticity properties so we can use also experimental information on $F_{\pi}(t)$ at spacelike $t$. At higher energy we use QCD to supplement experimental data, including the recent measurements of $e^+e^- \rightarrow \text{hadrons}$ both around 1 GeV and near the $\bar{c}c$ threshold. This yields a precise determination of the $O(\alpha^2)$ and $O(\alpha^2) + O(\alpha^3)$ hadronic part of the photon vacuum polarization pieces,

$$10^{11} \times a^{(2)}(\text{h.v.p.}) = 6909 \pm 64; \quad 10^{11} \times a^{(2+3)}(\text{h.v.p.}) = 7002 \pm 66$$

As byproducts we also get the masses and widths of the $\rho^0$, $\rho^+$, and very accurate values for the charge radius and second coefficient of the pion.

Adding the remaining order $\alpha^3$ hadronic contributions we find

$$10^{11} \times a^{\text{theory}}(\text{Hadronic}) = \begin{cases} 6993 \pm 69 & (e^+e^- + \tau + \text{spacel.; Q.c.m.}) \\ 6815 \pm 71 & (e^+e^- + \tau + \text{spacel.; Ch.m.}) \end{cases}$$

depending on whether the light-by-light scattering hadronic contributions are evaluated with the quark constituent model or a chiral model. The error above includes statistical, systematic and estimated theoretical errors. The figures given are obtained including $\tau$ decay data; if we restrict ourselves to $e^+e^-$ data, slightly lower values and somewhat higher errors are found.

This is to be compared with the figure obtained by subtracting pure electroweak contributions from the recent experimental value, obtained from measurements of the muon gyromagnetic ratio $(g - 2)$, which reads

$$10^{11} \times a^{\text{exp}}(\text{Hadronic}) = 7174 \pm 150.$$
1. Introduction

The appearance of a new, very precise measurement of the muon magnetic moment\(^1\) has triggered the interest in theoretical calculations of this quantity. Particularly, because the experimental figure (we give the result for the anomaly, averaged with older determinations\(^2\))

\[ 10^{11} \times a_\mu (\text{exp.}) = 116592030 \pm 150 \]  \hspace{1cm} (1.1)

lies slightly above theoretical evaluations based on the standard model, as much as 2.6\(\sigma\) in some cases.

It should be noted that all modern\(^1\) theoretical determinations\(^3\)–\(^7\) are compatible among themselves within errors (of order \(100 \times 10^{-11}\)) and that, with few exceptions, they are also compatible with the experimental result, (1.1), at the level of \(1.5\sigma\) or less. Because of this, it is our feeling that a new, complete evaluation would be welcome since, in fact, there exists as yet no calculation that takes fully into account all theoretical constraints and all the new experimental data. These experimental data allow an improved evaluation of the low energy hadronic contributions to \(a_\mu\), both directly from \(e^+e^-\) annihilations (in the \(\rho\) region\(^8\) and around the \(\phi\) resonance\(^9\)) and, indirectly, from \(\tau\) decays\(^10\) and, also indirectly, from measurements of the pion form factor in the spacelike region\(^11\). Moreover, the BES\(^12\) data, covering \(e^+e^-\) annihilations in the vicinity of \(\bar{c}c\) threshold, permit a reliable evaluation of the corresponding hadronic pieces. In fact, the main improvements of the present paper are the calculation of the two pion contribution to the hadronic part of \(a_\mu\), using all available experimental information and fulfilling compatibility with all our theoretical knowledge, and the pinning down of the multipion, \(KK\) and \(cc\) contributions. This we do in Sects. 3, 4 (in Sect. 2 we formulate the problem). In Sect. 5 we discuss other hadronic corrections, including one that, as far as we know, has been hitherto neglected, and which, though small (\(\sim 46 \times 10^{-11}\)) is relevant at the level of accuracy for which we are striving. Finally, in Sect. 6 we discuss our results and compare them with experiment.

The main outcome of our analysis is an accurate and reliable determination of the hadronic contributions to \(a_\mu\) at order \(\alpha^2\). In fact, in all regions where there are difficulties we perform at least two evaluations, and take into account their consistency (or lack thereof). Furthermore, we discuss in some detail (including ambiguities) the \(O(\alpha^3)\) hadronic contributions.

As a byproduct of the low energy calculations we can also give precise values for the \(\rho^0\), \(\rho^+\) masses and widths,

\[ m_{\rho^0} = 772.6 \pm 0.5 \text{ MeV}, \quad \Gamma_{\rho^0} = 147.4 \pm 0.8; \]
\[ m_{\rho^+} = 773.8 \pm 0.6 \text{ MeV}, \quad \Gamma_{\rho^+} = 147.3 \pm 0.9; \]  \hspace{1cm} (1.2)

for the P-wave \(\pi\pi\) scattering length,

\[ a^1_\pi = (41 \pm 2) \times 10^{-3} m_\pi^{-3}, \]  \hspace{1cm} (1.3)

and for the pion mean squared charge radius and second coefficient:

\[ \langle r_\pi^2 \rangle = 0.435 \pm 0.002 \text{ fm}^2, \quad c_\pi = 3.60 \pm 0.03 \text{ GeV}^{-4}, \quad (e^+e^- + \tau + \text{spacelike}); \]
\[ \langle r_\pi^2 \rangle = 0.433 \pm 0.002 \text{ fm}^2, \quad c_\pi = 3.58 \pm 0.04 \text{ GeV}^{-4}, \quad (e^+e^- + \text{spacelike}). \]  \hspace{1cm} (1.4)

We give results both using only direct results on \(F_\pi\), from \(e^+e^-\) annihilations, or involving also the decay \(\tau^+ \rightarrow \bar{\nu}_\tau \pi^+ \pi^0\), which last we consider to be our best estimates.

Our results for the hadronic part of the anomaly depend on which model one believes for the hadronic light-by-light contribution, to be discussed in Subsect. 5.1. So we write,

\[ 10^{11} \times a(\text{Hadronic}) = \begin{cases} 6993 \pm 69 & (e^+e^- + \tau + \text{spacel.: Q.c.m.}) \\ 6815 \pm 71 & (e^+e^- + \tau + \text{spacel.: Ch.m.}) \\ 6973 \pm 99 & (e^+e^- + \text{spacel.: Q.c.m.}) \\ 6795 \pm 100 & (e^+e^- + \text{spacel.: Ch.m.}) \end{cases} \]  \hspace{1cm} (1.5)

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\(^1\)By modern we here mean, somewhat arbitrarily, those obtained since 1985. A more complete list of references, including earlier work, may be found in ref. 7.
Note that in $a$(Hadronic) we include all hadronic contributions, $O(\alpha^3)$ as well as $O(\alpha^2)$. In (1.5) “Q.c.m.” means that the light by light hadronic contribution was calculated with the quark constituent model, and “Ch.m.” that a chiral model was used. The errors include the statistical errors, as well as the estimated systematic and theoretical ones. This is to be compared with the value deduced from (1.1) and electroweak corrections

$$10^{11} \times a_{\text{exp}} \text{(Hadronic)} = 7174 \pm 150.$$ 

Whether our results should be interpreted as providing further experimental validation of the standard model, or one can consider them as “harbingers of new physics”, we will leave for the reader to decide.

2. Contributions to $a_\mu$

We divide the various contributions to $a_\mu$ as follows:

$$a_\mu = a(\text{QED}) + a(\text{Weak}) + a(\text{Hadronic}). \quad (2.1)$$

Here $a(\text{QED})$ denote the pure quantum electrodynamics corrections, and $a(\text{Weak})$ are the ones due to $W$, $Z$ and Higgs exchange. The hadronic contributions can, in turn, be split as

$$a(\text{Hadronic}) = a^{(2)}(\text{h.v.p.}) + a(\text{Other hadronic}, O(\alpha^3)). \quad (2.2)$$

$a^{(2)}(\text{h.v.p.})$ are the corrections due to the hadronic photon vacuum polarization contributions (Fig. 1), nominally of order $\alpha^2$ (see Subsects. 3.3 and 5.2 for a qualification of this statement). We will discuss in detail the “Other hadronic, $O(\alpha^3)$” in Sect. 5.

According to the review of Hughes and Kinoshita\textsuperscript{[13]} one has

$$10^{11} \times a(\text{QED}) = 116584705 \pm 1.8$$
$$10^{11} \times a(\text{Weak}) = 151 \pm 4. \quad (2.3)$$

There is no dispute about these numbers. If we combine them with (1.1), we can convert this into a measurement of the hadronic part of the anomaly:

$$10^{11} \times a_{\text{exp}} \text{(Hadronic)} = 7174 \pm 150. \quad (2.4)$$

Our task in the present paper is the evaluation of this quantity.
We now say a few words about the piece $a^{(2)}(h.v.p.)$, which is the most important component of $a$(Hadronic). As Brodsky and de Rafael[14] have shown, it can be written as

$$a^{(2)}(h.v.p.) = 12\pi \int_{4m^2}^{\infty} dt \ K(t) \ Im \ II(t),$$

(2.4a)

$$K(t) = \frac{\alpha^2}{3\pi^2} \hat{K}(t); \ \hat{K}(t) = \int_{0}^{1} dx \ \frac{x^2(1-x)}{x^2 + (1-x)t/m^2}. $$

Here $II$ is the hadronic part of the photon vacuum polarization function. An alternate formula is obtained by expressing $Im \ II$ in terms of the ratio

$$R(t) = \frac{\sigma^{(0)}(e^+e^- \rightarrow \text{hadrons})}{\sigma^{(0)}(e^+e^- \rightarrow \mu^+\mu^-)} = \frac{4\pi\alpha^2}{3t};$$

$$a^{(2)}(h.v.p.) = \int_{4m^2}^{\infty} dt \ K(t)R(t).$$

(2.4b)

The superindex (0) here means ‘lowest order in the electromagnetic interactions’.

At low energy ($t \leq 0.8 \text{ GeV}^2$) we can separate the contribution from three pion states and that from two pions. The first will be discussed in Sect. 4. The two pion contribution in turn can be expressed in terms of the pion form factor, $F_\pi$,

$$\text{Im} \ II_{2\pi}(t) = \frac{1}{48\pi} \left(1 - \frac{4m^2}{t}\right)^{3/2} |F_\pi(t)|^2,$$

so that, for the two-pion contribution up to energy squared $t_0$, 

$$a_\mu(2\pi; t_0) = \frac{1}{t} \int_{4m^2}^{t_0} dt \ \left(1 - \frac{4m^2}{t}\right)^{3/2} K(t)|F_\pi(t)|^2.$$

(2.6)

3. The pion form factor

3.1. Theory

The evaluation of the pion form factor is slightly complicated by the phenomenon of $\omega - \rho$ interference. This can be solved by considering only the isospin $I = 1$ component, and adding later the $\omega \rightarrow 2\pi$ and interference in the standard Gounaris–Sakurai way. This is equivalent to neglecting, in a first approximation, the breaking of isospin invariance. We will also neglect for now electromagnetic corrections. In this approximation the properties of $F_\pi(t)$ are the following:

(i) $F_\pi(t)$ is an analytic function of $t$, with a cut from $4m^2$ to infinity.

(ii) On the cut, the phase of $F_\pi(t)$ is, because of unitarity, identical to that of the P-wave, $I = 1$, $\pi\pi$ scattering, $\delta^1_1(t)$, and this equality holds until the opening of the inelastic threshold at $t = t_0$ (Fermi–Watson final state interaction theorem).

(iii) For large $t$, $F_\pi(t) \simeq 1/t$. Actually, one knows the coefficient of this behaviour, but we will not need it here.

(iv) $F(0) = 1$.

The inelastic threshold occurs, rigorously speaking, at $t = 16m^2$. However, it is an experimental fact that inelasticity is negligible until the quasi-two body channels $\omega\pi, a_1\pi \ldots$ are open. In practice we will take 

$$t_0 \simeq 1 \text{ GeV}^2,$$

and fix the best value for $t_0$ empirically. It will be $t_0 = 1.1 \text{ GeV}^2$, and we will see that, if we keep close to this value, the dependence on $t_0$ is very slight.
The properties (i-iv) can be taken into account with the well-known Omnès-Muschelishvili method. We construct a function \( J(t) \) with the proper phase by defining

\[
J(t) = \exp \left\{ \frac{t}{\pi} \int_{4m_{\pi}^2}^{t_0} ds \frac{\delta_1^0(s)}{s(s-t)} + \frac{t}{\pi} \int_{t_0}^{\infty} ds \frac{\delta_1^0(s)}{s(s-t)} \right\}. \tag{3.1a}
\]

We have written the dispersion relation with one subtraction to ensure that \( J(0) = 1 \). The singular integrals are understood to be calculated replacing \( t \to t + \epsilon \), \( \epsilon > 0 \), and letting then \( \epsilon \to 0 \). In particular, we have

\[
|J(t)| = \exp \left\{ \frac{t}{\pi} \text{P.P.} \int_{4m_{\pi}^2}^{t_0} ds \frac{\delta_1^0(s)}{s(s-t)} + \frac{t}{\pi} \int_{t_0}^{\infty} ds \frac{\delta_1^0(s)}{s(s-t)} \right\}, \quad 4m_{\pi}^2 \leq t \leq t_0. \tag{3.1b}
\]

Defining then the function \( G \) by

\[
F_{\pi}(t) = G(t)J(t), \tag{3.2}
\]

it follows from properties i-ii that \( G(t) \) is analytic with only the exception of a cut from \( t_0 \) to infinity, as we have already extracted the correct phase below \( t = t_0 \).

We can, in Eq. (3.1a), take any value we like for the phase \( \delta_1^0(t) \), as a change of it only results in a redefinition of \( G \); but it is convenient to choose \( \delta_1^0(t) \) so that it joins smoothly \( \delta_1^0(t) \) at \( t = t_0 \) to avoid spurious singularities that would deteriorate the convergence, and so that \( J \) has the correct behaviour at infinity. Both properties are ensured if we take, simply,

\[
\delta_1^0(t) = \pi + \left[ \delta_1^0(t_0) - \pi \right] \frac{t_0}{t}
\]

so that \( \delta_1^0(t_0) = \delta_1^0(t) \) and, for large \( t \), \( \delta_1^0(t) \to \pi \) and we recover the behaviour \( 1/t \) of the form factor. Then we can rewrite more explicitly (3.1) by integrating the piece with \( \delta_1^0 \):

\[
J(t) = e^{-\delta_1^0(t_0)/\pi} \left( 1 - \frac{t}{t_0} \right)^{1-\delta_1^0(t_0)/\pi}[t_0/t]^{-1} \exp \left\{ \frac{t}{\pi} \int_{4m_{\pi}^2}^{t_0} ds \frac{\delta_1^0(s)}{s(s-t)} \right\}. \tag{3.3}
\]

3.1.1. The phase \( \delta_1^0 \)

We can apply the effective range theory to the phase \( \delta_1^0 \). According to this, the function

\[
\psi(t) \equiv \frac{2k^3}{\mu^{1/2}} \cot \delta_1^0(t), \quad k = \frac{\sqrt{t-4m_{\pi}^2}}{2} \tag{3.4a}
\]

is analytic in the variable \( t \) except for two cuts: a cut from \( -\infty \) to \( 0 \), and a cut from \( t = t_0 \) to \( +\infty \). To profit from the analyticity properties of \( \psi \) we will make a conformal transformation.\(^2\) We define

\[
w = \frac{\sqrt{t - \sqrt{t_0 - t}}}{\sqrt{t + \sqrt{t_0 - t}}} \tag{3.4b}
\]

When \( t \) runs the cuts, \( w \) goes around the unit circle. We may therefore expand \( \psi \) in a power series convergent inside the unit disc. However, the existence of the \( \rho \) resonance implies that we must have \( \cot \delta_1^0(m_{\rho}^2) = 0 \). It is therefore convenient to incorporate this piece of knowledge and expand not \( \psi \) itself but the ratio \( \psi(t)/(m_{\rho}^2 - t) \equiv \hat{\psi}(t) \): so we write,

\[
\hat{\psi}(t) = (m_{\rho}^2 - t) \hat{\psi}(t) = (m_{\rho}^2 - t) \left\{ b_0 + b_1w + b_2w^2 + \cdots \right\}. \tag{3.4c}
\]

The P-wave, \( I = 1 \pi \pi \) scattering length,\(^3\) \( a_1 \), is related to \( \psi \) by

\[
a_1^1 = \frac{1}{m_\pi \psi(4m_{\pi}^2)}. \tag{3.5a}
\]

\(^2\) The method of conformal transformations is rigorous, simpler and produces better results than that employed in ref. 4.

\(^3\) For details on \( \pi \pi \) scattering, including analyticity properties, see e.g. ref. 15.
Likewise, from the relation
\[
\frac{1}{\cot \delta_1(t) - i} \approx \frac{\text{const.}}{t^2 m_\rho^2 - t - 2k^3 \psi(t)}
\]
we find the expression for the rho width:
\[
\Gamma_\rho = \frac{2k^3}{m_\rho^2 \psi(m_\rho^2)}; \quad k_\rho = \frac{1}{2} \sqrt{m_\rho^2 - 4m_\pi^2}.
\]  
(3.5b)

Experimentally,\[^{15}\] \(a_1^+ \approx (0.038 \pm 0.003)m_\pi^{-3}\), and, according to the Particle Data Group Tables,\[^{16}\] \(m_\rho = 769.0 \pm 0.8; \quad \Gamma_\rho = 150.2 \pm 0.8\) MeV. Note, however, that we do not assume the values of \(m_\rho, \Gamma_\rho\). We only require that \(\psi\) has a zero, and will let the fits fix its location and residue.

It turns out that, to reproduce the width and scattering length, and to fit the pion form factor as well (see below), only two terms in the expansion are needed, so we approximate
\[
\delta_1(t) = \text{Arc cot} \left\{ \frac{t^{1/2}}{2k^3} \left[ b_0 + b_1 \sqrt{t - t_0} \right] \right\};
\]  
(3.6)

\(m_\rho, b_0, b_1\) are free parameters in our fits.

3.1.2. The function \(G(t)\)

Because we have already extracted the correct phase up to \(t = t_0\), it follows that the function \(G(t)\) is analytic except for a cut from \(t = t_0\) to \(+\infty\). The conformal transformation
\[
z = \frac{\frac{1}{2} \sqrt{t_0} - \sqrt{t_0 - t}}{\frac{1}{2} \sqrt{t_0} + \sqrt{t_0 - t}}
\]  
(3.7a)

maps this cut plane into the unit circle. So we may write the expansion,
\[
G(t) = 1 + A_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots
\]  
(3.7b)

that will be convergent for all \(t\) inside the cut plane. We can implement the condition \(G(0) = 1\), necessary to ensure \(F_\pi(0) = 1\) to each order, by writing
\[
A_0 = - \left[ c_1 z_0 + c_2 z_0^2 + c_3 z_0^3 + \cdots \right], \quad z_0 \equiv z|_{t=0} = -1/3.
\]

The expansion then reads,
\[
G(t) = 1 + c_1 (z + 1/3) + c_2 (z^2 - 1/9) + c_3 (z^3 + 1/27) + \cdots
\]  
(3.8)

We will need two-three terms in the expansion, so we will approximate
\[
G(t) = 1 + c_1 \left[ \frac{1}{2} \sqrt{t_0} - \sqrt{t_0 - t} \right] + c_2 \left[ \left( \frac{1}{2} \sqrt{t_0} - \sqrt{t_0 - t} \right)^2 - \frac{1}{8} \right],
\]
c1, c2 free parameters.

An interesting feature of our method is that, even if we only kept one term in each of the expansions (3.6, 8), that is to say, if we took \(b_1 = c_1 = c_2 = 0\), we could reproduce the experimental data with only a 15% error; so we expect (and this is the case) fast convergence of the series. It is important also that our expression for \(F_\pi(t)\) is valid in the spacelike as well as in the timelike region, provided only \(t < t_0\). What is more, (3.6, 8) represent the more general expressions compatible with analyticity, the Fermi–Watson theorem and the effective range theory, which follow only from the requirements of unitarity and causality. Therefore, by employing our expansions, we do not introduce uncontrolled biases in the analysis, and hence we minimize the model dependent errors.\(^4\)

\[^4\] The remaining approximations are neglect of the inelasticity between 16\(m_\pi^2\) and \(t_0\), experimentally known to be at the \(10^{-3}\) level or below, and we have the errors due to the truncation of the expansions; we will also check that they are small. By contrast, other functional forms used in in the literature are either incompatible with the phase of \(F_\pi\), or with its analyticity properties (or both), which causes biases in the fits.

The errors due to breaking of isospin and electromagnetic corrections will be discussed below.
3.2. Fits

In order to fit \( F_\pi \), and hence get the \( 2\pi \) low energy \((4m_\pi^2 \leq t \leq 0.8 \text{ GeV}^2)\) contribution to \( a^{(2)}(\text{h.v.p.}) \), we have available three sets of data:

- \( e^+e^- \to \pi^+\pi^- \), \( t \) timelike (Novosibirsk, ref. 8).
- \( F_\pi(t) \), \( t \) spacelike (NA7, ref. 11).
- In addition, one can use data from the decay \( \tau^+ \to \bar{\nu}_\tau \pi^+\pi^0 \) (Aleph and Opal, ref. 10).

For this last we have to assume isospin invariance, and neglect the isospin \( I = 2 \) component of \( \pi^+\pi^0 \), to write the form factor \( v_1 \) for \( \tau \) decay in terms of \( F_\pi \):

\[
v_1 = \frac{1}{12} \left( 1 - \frac{4m_\pi^2}{t} \right)^{3/2} |F_\pi(t)|^2,
\]

where, in terms of the weak vector current \( V_\mu = \bar{u}\gamma_\mu d \), and in the exact isospin approximation,

\[
\Pi^V_{\mu\nu} = (-p^2g_{\mu\nu} + p_\mu p_\nu) \Pi^V(t) = i \int d^4x \, e^{ip\cdot x} \langle 0 | TV^\pi_\mu(x) V_\nu(0)|0 \rangle; \quad v_1 = 2 \pi \text{ Im} \Pi^V. \quad (3.9b)
\]

Before presenting the results of the fits a few matters have to be discussed. A first point to clarify is that we will not include in the fits the old data on \( F_\pi \) in the spacelike or timelike regions, or on pion-pion phase shifts\(^\text{[17]}\). We have checked that, if we add the first two sets, the results of the fit vary very little; but they cause a bias. This is so because there is incompatibility\(^5\) between old spacelike and timelike data, and also with data on \( \pi\pi \) phase shifts, already noticed in CLY\(^\text{[4]}\). Doubtless, this is due to the fact that most old data for spacelike momentum were extracted from processes with one pion off its mass shell so that models were necessary to extrapolate to the physical form factor. In fact, a very important feature of the NA7\(^\text{[11]}\) data is that they are obtained from scattering of real pions off electrons, hence we do not require models to extract \( F_\pi \) from data.

Likewise, we also do not include in the fit the \( \pi\pi \) phase shifts, except for the scattering length, \( a_1^1 \), that we take as

\[
a_1^1 = (38 \pm 3) \times 10^{-3} m_\pi^{-3};
\]

we will see that the value our best fit returns for this quantity is satisfactorily close to this, as indeed we get \((41 \pm 2) \times 10^{-3} m_\pi^{-3}\).

The reason why the experimental \( \pi\pi \) phase shifts are not included is because they are obtained from scattering with one off-shell pion, hence their extraction is model dependent. However, we will use them as a very important \textit{a posteriori} check of our results.

A second remark concerns the matter of isospin breaking, due to electromagnetic interactions or the mass difference between \( u \), \( d \) quarks, that would spoil the equality (3.9a). It is not easy to estimate this. A large part of the breaking, the \( \omega \to 2\pi \) contribution and \( \omega \to \rho \) mixing, are taken into account by hand, but this does not exhaust the effects. For example, merely changing the quark masses from \( m_{\pi^+} + m_{\pi^-} \) to \( m_{\pi^+} + m_{\pi^0} \), in a Breit–Wigner model for the \( \rho \), shifts \( a^{(2)}(\text{h.v.p.}) \) by \(~50 \times 10^{-11}\), so a deviation of this order should not be surprising.

As stated above, Eqs. (3.9) were obtained neglecting the mass difference \( m_u - m_d \) end electromagnetic corrections, in particular the \( \pi^0 - \pi^+ \) mass difference. We can take the last partially into account by distinguishing between the pion masses in the phase space factor in (3.9a). To do so, write now (3.9b) as

\[
\Pi^V_{\mu\nu} = i \int d^4x \, e^{ip\cdot x} \langle 0 | TV^\pi_\mu(x) V_\nu(0)|0 \rangle = (-p^2g_{\mu\nu} + p_\mu p_\nu) \Pi^V(t) + p_\mu p_\nu \Pi^S; \quad v_1 = 2 \pi \text{ Im} \Pi^V. \quad (3.10a)
\]

We get

\[
v_1 = \frac{1}{12} \left\{ \left[ 1 - \frac{(m_{\pi^+} - m_{\pi^0})^2}{t} \right] \left[ 1 - \frac{(m_{\pi^+} + m_{\pi^0})^2}{t} \right] \right\}^{3/2} |F_\pi(t)|^2.
\]

To compare with the experimentally measured quantity, which involves all of \( \text{Im} \Pi^V_{\mu\nu} \), we have to neglect the scalar component \( \Pi^S \), which is proportional to \( (m_d - m_u)^2 \), and thus very small.

\(^5\) At the level of 1.5 to 2\(\sigma\).
To understand the situation we will proceed by steps. First of all, we start by fitting separately \( e^+e^- \) and \( \tau \) data, in the timelike region, using (3.9a) (we remark that although in \( a(2\pi; t \leq 0.8 \text{ GeV}^2) \) only enter the values of \( F_2 (t) \) for \( 4m_{\pi}^2 \) to 0.8 GeV\(^2\), we fit the whole range up to \( t = t_0 = 1.1 \text{ GeV}^2 \)). Then, we get quite different results:

\[
a(2\pi; t \leq 0.8 \text{ GeV}^2) = \begin{cases} 
4715 \pm 67 & (e^+e^-)/\chi^2/\text{d.o.f.} = 106/109 = 0.96 \\
4814 \pm 26 & (\tau)/\chi^2/\text{d.o.f.} = 52/48 = 1.09.
\end{cases} \tag{3.11a}
\]

This takes into account statistical errors only for \( e^+e^- \), but includes systematic ones for \( \tau \) decay as these are incorporated in the available data.

The slight advantage of the first figure in (3.11a) in what regards the \( \chi^2/\text{d.o.f.} \) makes one wonder that the difference is really caused by isospin breaking (in which case the value obtained from \( \tau \) decay should be rejected) or is due to random fluctuations of the data, as well as to the systematics of the experiments. The second explanation has in its favour that, if we include the spacelike data into the fit (but still use (3.9a)) the discrepancy is softened, and we get compatible results:

\[
a(2\pi; t \leq 0.8 \text{ GeV}^2) = \begin{cases} 
4754 \pm 55 & (e^+e^- + \text{spacelike})/\chi^2/\text{d.o.f.} = 179/154 \\
4826 \pm 23 & (\tau + \text{spacelike})/\chi^2/\text{d.o.f.} = 112/93.
\end{cases} \tag{3.11b}
\]

This last result allows us to draw the following conclusion: that part of the discrepancy between results obtained with \( e^+e^- \) and \( \tau \) decay is still of statistical origin, but also it would seem that part is genuine.

In an attempt to take into account at least some of the isospin breaking effects, we have fitted simultaneously \( e^+e^- \), \( \tau \) decay, both including spacelike data, allowing for different values of the mass and width of the rho (but keeping other parameters, in particular \( c_1 \), \( c_2 \), common for both \( e^+e^- \) and \( \tau \) fits). We, however, still use (3.9a). In this case we find convergence of the results; we have\(^6\)

\[
a(2\pi; t \leq 0.8 \text{ GeV}^2) = 4779 \pm 30, \quad \chi^2/\text{d.o.f.} = 248/204; \quad (e^+e^- + \tau + \text{spacelike}), \tag{3.12}
\]

which is compatible (within errors) with both numbers in (3.11b).

It is to be noted that, if we had not allowed for different masses and widths for the neutral and charged rho, we would have obtained, in this common fit,

\[
10^{11} \times a(2\pi; t \leq 0.8 \text{ GeV}^2) = 4822 \pm 30, \quad \chi^2/\text{d.o.f.} = 264/206; \quad (e^+e^- + \tau + \text{spacelike})
\]

i.e., a larger \( \chi^2/\text{d.o.f.} \) and a value quite different from that obtained with only \( e^+e^- \) and spacelike data. So it would appear that allowing for different parameters for the neutral and charged rho really takes into account a good part of the isospin breaking effects.

Finally, we take into account the kinematical effects of the \( \pi^\pm \), \( \pi^0 \) mass difference repeating the fit using (3.10b) now. The result of the fit with \( e^+e^- \) data only is of course unchanged, but we reproduce it to facilitate the comparison and for ease of reference. We find what we consider our best results:

\[
10^{11} \times a(2\pi; t \leq 0.8 \text{ GeV}^2) = 4774 \pm 31, \quad \chi^2/\text{d.o.f.} = 246/204; \quad (e^+e^- + \tau + \text{spacelike}). \tag{3.13}
\]

We remark that the results for the evaluation including \( \tau \) decays are rather insensitive to the use of (3.10b), but what change there is, it goes in the right direction: the \( \chi^2/\text{d.o.f.} \) has improved slightly, and the values for the anomaly including the \( \tau \) have become slightly more compatible with the figure obtained using \( e^+e^- \) data only. This makes us confident that most of the effects due to isospin breaking, both from \( u, d \) mass differences and from electromagnetic effects (about which we will say more in Subsects. 3.3 and 5.2) have already been incorporated in our calculation. The fit may be seen depicted in Fig. 2 for \( |F_\pi|^2 \), with timelike and spacelike data, and in Fig. 3 for the quantity \( v_2 \) in \( \tau \) decay.

The \( \chi^2/\text{d.o.f.} \) of the fits is slightly above unity; in next subsection we will see that including systematic errors cures the problem. For example, just adding the systematic normalization error for the spacelike

\(^6\) When evaluating \( a(2\pi; t \leq 0.8 \text{ GeV}^2) \) we of course use the parameters \( m_\mu, b_0, b_1 \) corresponding to \( \rho^0 \); see below.
Figure 2. Plot of the fit to $|F_\pi(t)|^2$, timelike (ref. 8) and spacelike (ref. 11) data. The theoretical curve actually drawn is that obtained by fitting also $\tau$ data, but the curve obtained fitting only $e^+e^-$ could not be distinguished from that drawn if we plotted it. A blowup of the fit in the spacelike region may be seen in Fig. 4.

Data\textsuperscript{[11]} gives a shift of the central value of $31 \times 10^{-11}$ and the $\chi^2$/d.o.f. decreases to 152/153 for the evaluation with $e^+e^-$ data only. The quality of the fit to the spacelike data is shown in Fig. 4, which is a blowup of the corresponding part of Fig. 2.

The parameters of the fits are also compatible. We have,

$$
c_1 = 0.23 \pm 0.02, \quad c_2 = -0.15 \pm 0.03; \quad b_0 = 1.062 \pm 0.005, \quad b_1 = 0.25 \pm 0.04 \quad (e^+e^- + \tau + \text{spacelike});
$$

$$
c_1 = 0.19 \pm 0.04, \quad c_2 = -0.15 \pm 0.10; \quad b_0 = 1.070 \pm 0.006, \quad b_1 = 0.28 \pm 0.06 \quad (e^+e^- + \text{spacelike}).
$$

In the second line the parameters $c_1$, $c_2$ are common for $\rho^0$, $\rho^\pm$. $b_0$ and $b_1$ vary very little; the ones shown correspond to the values of $m_{\rho^0}$, $\Gamma_{\rho^0}$ as given below in Eq. (3.15). The values $b_0 = 1$, $b_1 = 0$ would correspond to a perfect Breit–Wigner shape for the $\rho$. A fact to be mentioned is that including the corrected phase space factor (3.10b) helps a little to make compatible the parameters for both fits; if we had used (3.9a) we would have obtained

$$
c_1 = 0.23 \pm 0.01, \quad c_2 = -0.16 \pm 0.03; \quad b_0 = 1.060 \pm 0.005, \quad b_1 = 0.24 \pm 0.04 \quad (e^+e^- + \tau + \text{spacelike}).
$$
An important feature of our fit is that the coefficients decrease with increasing order. This, together with the fact that the conformal variables $w, z$ are of modulus well below unity in the regions of interest ($4m^2_{\pi} \leq t \leq 0.8 \text{ GeV}^2$ for $w$, $-0.25 \text{ GeV}^2 \leq t \leq 0.8 \text{ GeV}^2$ for $z$):

$$-0.57 \leq w \leq 0.24, \quad -0.38 \leq z \leq -0.02,$$
ensures good convergence of the expansions. We have also checked that including extra terms in the expansions does not improve the quality of the fits significantly.

Besides the results for the anomaly we obtain reliable determination of a set of parameters. We have those pertaining to the rho,

\[
\begin{align*}
\rho_m &= 772.6 \pm 0.5 \text{ MeV}, & \Gamma_{\rho} &= 147.4 \pm 0.8; \\
\rho_{m^+} &= 773.8 \pm 0.6 \text{ MeV}, & \Gamma_{\rho^+} &= 147.3 \pm 0.9.
\end{align*}
\]

(3.15)

The figures are in reasonable agreement with the Particle Data Group values\(^7\) given before.

The value for the scattering length the fit returns is comfortably close to the one obtained from \(\pi\pi\) phase shifts; we get

\[
a_1^\pi = (41 \pm 2) \times 10^{-3} \; m_\pi^{-3}.
\]

Also from our fits we obtain the low energy coefficients of the pion form factor,

\[
F_\pi^2(t) \approx 1 + \frac{1}{6} \langle r_\pi^2 \rangle t + c_\pi t^2 ;
\]

\[
\langle r_\pi^2 \rangle = 0.435 \pm 0.002 \; \text{fm}^2, \quad c_\pi = 3.60 \pm 0.03 \; \text{GeV}^{-4} \; (e^+e^- + \tau + \text{spacelike});
\]

\[
\langle r_\pi^2 \rangle = 0.433 \pm 0.002 \; \text{fm}^2, \quad c_\pi = 3.58 \pm 0.04 \; \text{GeV}^{-4} \; (e^+e^- + \text{spacelike}).
\]

(3.16)

These figures are compatible with, but much more precise than, the current estimates:\(^{18}\)

\[
\langle r_\pi^2 \rangle = 0.431 \pm 0.026 \; \text{fm}^2, \quad c_\pi = 3.2 \pm 1.0 \; \text{GeV}^{-4}.
\]

Another remark is that in all these fits we took \(t_0 = 1.1 \; \text{GeV}^2\). The dependence of the results on this parameter, \(t_0\), is very slight, provided we remain around this value. Thus, for example, if we take \(t_0 = 1.2 \; \text{GeV}^2\) the value of \(a(2\pi; t \leq 0.8 \; \text{GeV}^2)\) only increases by \(4 \times 10^{-11}\), and the global \(\chi^2\) only varies by one unit.

As a check of the quality and reliability of our results we may further mention the following. We have not used the experimental pion-pion scattering phase shifts as input (except for the value of the scattering length). So, the values that follow from our expression (3.6), with the values of the parameters given in (3.14), constitute really a prediction for \(a_1^\pi(t)\). This can be compared with the existing experimental values for this quantity,\(^{17}\) a comparison that may be found in Fig. 5. The agreement is remarkable.

Before finishing this section we have to clarify the matter of the \(\omega\) and \(\omega - \rho\) contribution to \(a(2\pi; t \leq 0.8 \; \text{GeV}^2)\). Our fits to \(e^+e^-\) data have actually been made including in the function \(F_\pi\) as given above, Eq. (3.2), a coefficient to take into account the \(\omega \rightarrow 2\pi\) contribution. To be precise, we have used the expression

\[
F_\pi^{\omega\pi}(t) = F_\pi(t) \times \frac{1 + \sigma \frac{m_\omega^2 - t - \text{imag} \Gamma_\omega}{1 + \sigma}}{m_\omega^2 - t - \text{imag} \Gamma_\omega},
\]

(3.17)

where the notation is obvious. We take from the PDG\(^{16}\) the values for the mass and width of the \(\omega\),

\[
m_\omega = 782.6 \pm 0.1 \; \text{MeV}, \quad \Gamma_\omega = 8.4 \pm 0.9 \; \text{MeV},
\]

and the fit gives a mixing parameter \(\sigma = (16 \pm 1) \times 10^{-4}\).

As is known, this Gounaris–Sakurai\(^{18}\) parameterization is only valid for \(t \simeq m_\omega^2\) and, in particular, its extrapolation to \(t \sim 0\) is not acceptable. This effect is very small, less than one part in a thousand. However, to play it safe, we have also adopted the following alternate procedure: we have obtained a first approximation to \(F_\pi\) by fitting the experimental data excluding the region \(0.55 \; \text{GeV}^2 \leq t \leq 0.65 \; \text{GeV}^2\). Then we have fitted only this region adding there also the \(\omega\) piece, as in (3.17). The resulting value for \(a(2\pi; t \leq 0.8 \; \text{GeV}^2)\) varies very little; it decreases by something between 2 and \(12 \times 10^{-11}\), depending on the fit. We may consider this as part of the theoretical error of our calculation, to be discussed in the next subsection.

\(^7\) It should be noted that the various determinations for \(m_\rho\) reported by the PDG\(^{16}\) actually cluster around two different values, with the more modern ones closer to our figures in Eq. (3.15) than the average value reported by the PDG.
To finish this subsection, we present in Table 1 a comparison both with old results that also use analyticity properties, and a recent one (which does not).

<p>| | | |</p>
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<tbody>
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<td>N1</td>
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<tr>
<td>N2</td>
<td>4730 ± 100</td>
<td>$e^+e^-$</td>
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<td>$e^+e^-$</td>
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<td>CLY-II</td>
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<td>$e^+e^- + \pi\pi$ ph. shifts</td>
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<td>4774 ± 31</td>
<td>$\tau + e^+e^-$</td>
</tr>
<tr>
<td>TY2</td>
<td>4754 ± 55</td>
<td>only $e^+e^-$</td>
</tr>
</tbody>
</table>

Table 1

Comparison of evaluations of $10^{11} \times a(2\pi; t \leq 0.8$ GeV$^2$).

N1, N2 are in ref. 7. TY denotes our result here (statistical errors only for the $e^+e^-$ and spacelike data).

The difference between the old CLY, AY and the new determinations is due to a large extent to the influence of the new Novosibirsk and NA7 data which allow us in particular to obtain a robust result: the CLY evaluation used only 18 data points!

The difference between the results of Narison (N), who does not take into account the Fermi–Watson theorem or the spacelike data and TY, who do, is due in good part to, precisely, the influence of the spacelike data which also help reduce the errors.
3.3. Systematic and theoretical errors for the pion form factor contribution

We now discuss the errors that stem from experimental systematics, as well as those originating from deficiencies of the theoretical analysis for the $2\pi$ contribution, in the low energy region $4m^2_t \leq t \leq 0.8 \text{ GeV}^2$.

We start with the systematic errors of the data. They are evaluated by taking them into account in a new fit. In this way we find, in units of $10^{-11}$, and neglecting the mass differences corrections (i.e., using (3.9a) for tau data)

$$
\text{Exp. Sys. } = \pm 40 \ (e^+e^- + \tau)
$$

$$
\text{Exp. Sys. } = \pm 66 \ (e^+e^-)
$$

To estimate the degree of correlation of the systematic errors pertaining to several experiments is a difficult task; we choose to consider the full range from 0 to 1. The error bars given cover all the possibilities. The $\chi^2$/d.o.f. of the fit improves when taking these systematic errors into account to

$$
\chi^2/\text{d.o.f. } = 214/204 \ (e^+e^- + \tau)
$$

$$
\chi^2/\text{d.o.f. } = 145/154 \ (e^+e^-)
$$

The error given for the case in which we include the decays of the tau would be smaller, and the $\chi^2$/d.o.f. would improve, if we used the correct kinematical formula for phase space, Eq. (3.10b); we would have obtained

$$
\text{Exp. Sys. } = \pm 30; \ \chi^2/\text{d.o.f. } = 213/204 \ (e^+e^- + \tau).
$$

In spite of this, we choose to accept the larger error ($\pm 40$) as we feel that it includes residual effects of isospin breaking and electromagnetic corrections, different for the tau and $e^+e^-$ cases, that we will discuss at the end of this subsection.

We discuss the systematic and theoretical errors in the higher energy regions in next section, but we mention here that the systematic error (4 $\times 10^{-11}$) for $2\pi$ between $t = 0.8$ and 1.2 GeV$^2$ is added coherently to the lower energy $2\pi$ piece.

In addition to this we have several theoretical sources of error. First, that originating in the approximate character of the Gounaris–Sakurai method for including the $\omega$ interactions. We will start with the analysis based on the pion form factor in the real world, where the $\pi^+\pi^-$ certainly interact electromagnetically. Not only this, but the initial particles ($e^+e^-$) also interact between themselves, and with the pions.

These last electromagnetic interactions, however, can be evaluated and they are indeed subtracted when presenting experimental data on $F_\pi$; the uncertainties this process generates are estimated and included in the errors provided with the data. We will thus only discuss the uncertainties associated to e.m. interactions of the $\pi^+\pi^-$ alone. These particles may exchange a photon, or radiate a soft photon that is not detected (see the corresponding figures in Subsect. 5.2). We may then define two quantities: $F_\pi^{(0)}$, which is the form factor we would have if there were no e.m. interactions; and $F_\pi^{(\text{real})}$, which is the measured quantity, even after removal of radiative corrections for initial states or mixed ones. Actually, $F_\pi^{(\text{real})}$ depends on the experimental setup through the cuts applied to ensure that no (hard) photon is emitted.

Our formalism, as developed in Subsect. 3.1, applies to $F_\pi^{(0)}$, but we fit $F_\pi^{(\text{real})}$. Therefore, we are introducing an ambiguity

$$
F_\pi^{(\text{real})} - F_\pi^{(0)}
$$

which is of order $\alpha$. 

---

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The error induced by this ambiguity should be small. In fact, what enters into \( a_u \) is the sum of the contribution of \( F^{(\text{real})}_\pi \), which is what we actually fit, and that of the state \( \pi^+\pi^- \gamma \), which can be obtained from the process

\[
e^+e^- \rightarrow (\gamma) \rightarrow \pi^+\pi^-\gamma.
\]

For this reason, we believe that the error due to the mismatch of \( F^{(\text{real})}_\pi \) and \( F^{(0)}_\pi \) is included in the errors to our fits here;\(^8\) the estimated error for the \( \pi^+\pi^-\gamma \) contribution, \( 9 \times 10^{-11} \), will be evaluated in Subsect. 5.2.

Tau decay presents the same difficulties, and we expect a similar uncertainty as for \( e^+e^- \) collisions. But apart from the effect \( F^{(\text{real})}_\pi \neq F^{(0)}_\pi \) discussed, it poses extra problems when relating it to \( F_\pi \). To discuss this, we take first \( m_u \neq m_d \); then we choose \( \alpha \neq 0 \), but \( m_u = m_d \). Higher effects, proportional to \( \alpha(m_u - m_d) \), \( \alpha^2 \) and \( (m_u - m_d)^2 \) shall be neglected.

For \( \alpha = 0 \), the masses of \( \pi^+ \) and \( \pi^- \) become equal, but isospin invariance is still broken. This means that, in particular, the quantity \( \Pi^8 \) in (3.10a) is nonzero and therefore the experimentally measured \( v_1 \) does not coincide with that in (3.10b). We expect this effect to be small, since it is of second order, \( (m_d - m_u)^2 \).

If the scale is the QCD parameter \( \Lambda \), then this will be of relative size \( 10^{-4} \); but other effects need not be so small. We have tried to take them into account by allowing for different parameters for \( \rho^\pm, \rho^0 \); this produced a substantial shift, of about \( 40 \times 10^{-11} \) for \( a_{\mu} \).

Then we set \( m_u = m_d \) and take e.m. interactions to be nonzero. Apart from the effects already discussed, this produces the mass difference between charged and neutral pions. This we took (partially) into account when using the modified phase space of (3.10b). The ensuing effect for \( a_{\mu} \) turned out to be small, \( \sim 4 \times 10^{-11} \).

Remnants of \( \alpha \neq 0 \) and \( m_u \neq m_d \) will likely still affect the determination of \( F_\pi \) from \( e^+e^- \) and \( \tau \) decay data; but we feel that accepting the systematic/theoretical error of \( 40 \times 10^{-11} \) covers the related uncertainty.

4. Contributions to \( a^{(2)}(\text{h.v.p.}) \) from \( t > 0.8 \text{ GeV}^2 \). The full \( a^{(2)}(\text{h.v.p.}) \)

4.1. The higher energy contributions, and the \( 3\pi \) contribution

At higher energies we will get a substantial improvement over determinations based on old data\(^{[20]} \) because of the existence of very precise measurements from Novosibirsk\(^{[9]} \) and Beijing,\(^{[12]} \) gathered in the last two–three years, which will help remove a large part of the existing errors. This is particularly true of the region up to \( t = 3 \text{ GeV}^2 \) which caused an important part of the total errors in pre-1998 calculations of \( a^{(2)}(\text{h.v.p.}) \). We turn to it next.

4.1.1. The region up to \( t = 3 \text{ GeV}^2 \)

We consider first the contribution of two, three, four pion, \ldots., and \( \text{KK} \) intermediate states for \( 0.8 \leq t \leq 1.2 \text{ GeV}^2 \). In what follows n.w.a. will mean narrow width approximation, r.d.a. resonance dominance approximation (but not narrow approximation) and s.o.i.c. sum over individual channels. For the n.w.a. we use the standard formula. Denoting by \( \Gamma_{ee}(V) \) to the width into \( e^+e^- \) of a vector resonance \( V \) with mass \( M \), its contribution to \( a^{(2)}(\text{h.v.p.}) \) is given in this approximation by

\[
a(V) = \frac{3\Gamma_{ee}(V)\hat{K}(M^2)}{\pi M}.
\]

\(^8\) In particular for the evaluation including \( \tau \) decay data, see below
3π states, $9m_{\pi}^2 \leq t \leq 1.2$ GeV$^2$

In the narrow width approximation one gets the $\omega$, $\phi$ contributions:

$$10^{11} \times a(3\pi;\omega) = 348 \pm 13$$

$$10^{11} \times a(3\pi;\phi) = 62 \pm 3,$$

but this misses the region between $\omega$ and $\phi$, and the interference effect just above the last. So we will use experimental data.\cite{ref9} This gives

$$10^{11} \times a(3\pi;t \leq 1.2 \text{ GeV}^2) = 438 \pm 4 \text{ (Stat.)} \pm 11 \text{ (Sys.)}.$$  \hspace{1cm} (4.2)

The fit to the 3π experimental cross section, with data from ref. 9, may be found in Fig. 6. The upper curve (continuous line in Fig. 6) is a fit to the CMD2 and SND data. We have used a Breit–Wigner parameterization for the $\omega$ and $\phi$ resonances, plus a constant term. The $\chi^2$/d.o.f. is 63/60; we consider this our central result here. The dashed curve fits instead the data from CMD2 and ND (Dolinsky et al, ref. 20); the quality of the fit is poorer ($\chi^2$/d.o.f. = 52/37). It fits better the region between the $\omega$ and $\phi$, but fails to reproduce the data beyond 1.06 GeV$^2$. In fact, we include the second fit to estimate the corresponding systematic uncertainty; the small difference in terms of the integrals between the two fits, $8 \times 10^{-11}$, is included into the systematic error.

2π states, $0.8$ GeV$^2 \leq t \leq 1.2$ GeV$^2$

This 2π state contribution is

$$10^{11} \times a(2\pi;t \leq 1.2 \text{ GeV}^2) = 230 \pm 3 \pm 4.$$  \hspace{1cm} (4.4)

The evaluation of the contribution of the 2π state has greatly improved (with respect to older calculations) because of the information from recent Novosibirsk\cite{ref8} data on $e^+e^- \to 2\pi$. We have fitted the experimental value of $|F_\pi|^2$ with an expression $1/(bt + a)$, $a$, $b$ completely free parameters; the result of this fit may be seen depicted in Fig. 7. (A similar result is obtained if we extended our earlier calculation of $F_\pi(t)$ to $t \sim 1.2$ GeV$^2$ by setting $t_0 = 1.2$; but we prefer the result based only on experimental data.) Of the two errors given for the 2π contribution the first is statistical and the second, systematic, has been added coherently to the systematic error on the low energy 2π contribution, as discussed in Subsect. 3.3.
The wiggle in Fig. 7 for $t \sim m_\phi^2$ is due to the interference of the decay $\phi \to 2\pi$. This is similar to the $\omega - \rho$ effect, and has been treated in a similar manner; we have incorporated it using the formulas and parameters given by Achasov et al.\cite{9} The influence of this effect on $a_\mu$ is minute.

**$KK$ states, 0.8 GeV$^2$ $\leq t \leq$ 1.2 GeV$^2$**

An important contribution is that of $KK$ states. In the n.w.a., this is given by the $\phi$:

$$10^{11} \times a(KK; \phi) = 332 \pm 9,$$

but this is a dangerous procedure here; the vicinity of the $KK$ threshold distorts the shape of the resonance. We thus have to calculate this $KK$ contribution directly from experiment. We have used two fitting procedures. In the first, we fit simultaneously the $K^+K^-$ and $KLKS$ data of Achasov et al.,\cite{9} with the same parameters for the $\phi$. We get,

$$10^{11} \times a(K^+K^-; t \leq 1.2 \text{ GeV}^2) = 185.5 \pm 1.5 \text{ (Stat.)} \pm 13 \text{ (Sys.)}$$

and

$$10^{11} \times a(KLKS; t \leq 1.2 \text{ GeV}^2) = 129.5 \pm 0.7.$$ 

The quality of the fit, shown in Fig. 8, is good ($\chi^2$/d.o.f. = 84/82).

In the second fitting procedure, we add the $KLKS$ data of Akhmetshin et al.,\cite{9} obtaining the result

$$10^{11} \times a(KLKS; t \leq 1.2 \text{ GeV}^2) = 128.4 \pm 0.5 \text{ (Stat.)} \pm 2.6 \text{ (Sys.)}.$$ 

The fit is now less good, but the integrals are essentially identical for both fits. Adding the $KK$ results together we find

$$10^{11} \times a(KK; t \leq 1.2 \text{ GeV}^2) = 314 \pm 2 \text{ (Stat.)} \pm 13 \text{ (Sys.)}.$$ 

The systematic errors have been obtained repeating the fits, including now the systematic errors given by the experiments.

We mention in passing that the ratio of contributions of $K^+K^-$ and $KLKS$, 1.44, agrees well with the ratio\cite{16}

$$\frac{\Gamma(\phi \to K^+K^-)}{\Gamma(\phi \to KLKS)} = 1.46 \pm 0.03.$$
The four pion contribution, including the quasi-two body state $\omega \pi$, may be obtained from recent Novosibirsk data,[9] or from the compilation of Dolinsky et al.[20] If we use the last we get

$$10^{11} \times a(4\pi; t \leq 1.2 \text{ GeV}^2) = 25 \pm 4;$$

if we fit the data of Akhmetshin et al.[9] we find

$$10^{11} \times a(4\pi; t \leq 1.2 \text{ GeV}^2) = 18 \pm 3.$$

Of the $4\pi$ contribution most is due to the $\omega \pi^0$ channel; only a small fraction ($2.4 \times 10^{-11}$) comes from the $\pi^+\pi^-\pi^+\pi^-$ states. We take, for this $4\pi$ contribution, the figure

$$10^{11} \times a(4\pi; t \leq 1.2 \text{ GeV}^2) = 20 \pm 5,$$

which covers all possibilities.

The five, six, . . . , pions as well as $\omega \to \eta + 2\pi^0$ contributions are very small,[16,20] Altogether, they give

$$10^{11} \times a(5\pi, 6\pi, \eta\pi^0, \pi^0, \cdots, t \leq 1.2 \text{ GeV}^2) = 4 \pm 2.$$  \hfill (4.8)

We present the summary of our results in the important region $0.8 \text{ GeV}^2 \leq t \leq 1.2$ plus the $3\pi$ contribution below $1.2 \text{ GeV}^2$ in the following Table 2:
<table>
<thead>
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<th>Channels</th>
<th>Comments</th>
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<td>(\pi^+\pi^-)</td>
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</tr>
<tr>
<td></td>
<td>(0.8 \leq t \leq 1.2) GeV(^2)</td>
</tr>
<tr>
<td>(3\pi)</td>
<td>(438 \pm 4 \pm 11)</td>
</tr>
<tr>
<td></td>
<td>(9m_\pi^2 \leq t \leq 1.2) GeV(^2)</td>
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<tr>
<td>(K^+K^-)</td>
<td>(186 \pm 2 \pm 13)</td>
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<tr>
<td>(K_LK_S)</td>
<td>(128 \pm 1 \pm 2)</td>
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<tr>
<td>(4\pi)</td>
<td>(20 \pm 5)</td>
</tr>
<tr>
<td></td>
<td>including (\omega\pi^0)</td>
</tr>
<tr>
<td>Multipion, (\eta + 2\pi, \cdots)</td>
<td>(4 \pm 2)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>(1006 \pm 19)</td>
</tr>
<tr>
<td></td>
<td>Syst. error for (2\pi) not included</td>
</tr>
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</table>

Table 2

Contribution to \(a^{(2)}\) of various channels up to \(t = 1.2\) GeV\(^2\) (\(2\pi\) below 0.8 GeV\(^2\) not included).

1.2 GeV\(^2\) \(\leq t \leq 2\) GeV\(^2\)

We consider three determinations:

\[
\begin{align*}
270 \pm 27 & \quad \text{(Here)} \\
278 \pm 25 & \quad \text{(s.o.i.c., CLY\(^[4]\))} \\
265 \pm 22 & \quad \text{(VMD+QCD; AY).} \\
\end{align*}
\]

The first is obtained from a numerical integration of the data,\(^{[20]}\) with a parabolic fit. The method referred to as “VMD+QCD; AY”, details of which can be found in the AY\(^[4]\) paper, consists in interpolating between a vector meson dominance (VMD) calculation for quasi-two body processes (\(\omega\pi, \rho\pi, \ldots\)), plus a Breit–Wigner expression for two-body channels (\(\pi\pi, KK, \ldots\)) at the lower end, and perturbative QCD at the upper end, the interpolation being obtained by fitting experimental data (see Fig. 9). Because we want to present a result as model-independent as possible, we will take as our preferred figure that obtained here from experimental data:

\[
10^{11} \times a(1.2\text{ GeV}^2 \leq t \leq 2\text{ GeV}^2) = 270 \pm 27. 
\]

2 GeV\(^2\) \(\leq t \leq 3\) GeV\(^2\)

\[
\begin{align*}
240 \pm 3 \text{ (J)} \pm 3 \text{ (Cond.)} & \quad \text{(QCD)} \\
250 \pm 19 & \quad \text{(N\(^[7]\), r.d.a.; only } e^+ e^-) \\
276 \pm 36 & \quad \text{(N\(^[7]\), r.d.a.; } e^+ e^- + \tau) \\
222 \pm 5 \text{ (St.)} \pm 15 \text{ (Sys.)} & \quad \text{(J, exp. data).} \\
\end{align*}
\]

\(\text{(J)}\) here denotes an evaluation, integrating with the trapezoidal rule, of a compilation of experimental data supplied by F. Jegerlehner.
For the QCD calculations we take the following approximation: for \( n_f \) massless quark flavours, with charges \( Q_f \), we write
\[
R^{(0)}(t) = 3 \sum_f Q_f^2 \left[ 1 + \frac{\alpha_s}{\pi} + (1.986 - 0.115n_f) \left( \frac{\alpha_s}{\pi} \right)^2 
+ \left[ -6.64 - 1.20n_f - 0.005n_f^2 - 1.240 \frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)} \left( \frac{\alpha_s}{\pi} \right)^3 \right] \right].
\]
To this one adds mass and nonperturbative corrections. We take into account the \( O(m^2) \) effect for quarks with running mass \( \bar{m}_i(t) \), which correct \( R^{(0)} \) by the amount\(^9\)
\[
-3 \sum_i Q_i^2 \bar{m}_i^2(t) \left[ 6 + 28 \frac{\alpha_s}{\pi} + (294.8 - 12.3n_f) \left( \frac{\alpha_s}{\pi} \right)^2 \right] t^{-1}.
\]
Finally, for the condensates we add
\[
\frac{2\pi}{3} \left( 1 - \frac{11}{12} \frac{\alpha_s}{\pi} \right) \langle \alpha_s G^2 \rangle \sum_f Q_f^2
\]
and
\[
24\pi^2 \left[ 1 - \frac{23}{24} \frac{\alpha_s}{\pi} \right] m_i \langle \bar{\psi}_i \psi_i \rangle.
\]
We neglect the condensates corresponding to heavy quarks \( c, b \) and express those for \( u, d, s \) in terms of \( f_0^2 m_c^2, f_K^2 m_b^2 \) using the well-known PCAC relations.
In the QCD calculation, the error labeled “Cond.” is found by inserting the variation obtained setting quark and gluon condensates to zero, and that labeled \( \Lambda \) by varying the QCD parameter. For this parameter we take the recent determinations\(^{21}\) that correspond to the value
\[
\alpha_s(M_Z^2) = 0.1172 \pm 0.003;
\]
to be precise, we have taken (in MeV, and to four loops),
\[
\Lambda = 373 \pm 80, \quad t \leq m_c^2; \quad \Lambda = 283 \pm 50, \quad m_c^2 \leq t \leq m_b^2; \quad \Lambda = 199 \pm 30, \quad t \geq m_b^2.
\]
\(^9\) The corrections of order \( m^4 \) may be found in the paper of Narison,\(^7\) together with references. We have checked that the effect of this correction is smaller than the errors of the leading terms.
For the gluon condensate we take $\langle \alpha_s G^2 \rangle = 0.07 \text{ GeV}^4$.

The four evaluations give comparable results, with the r.d.a. ones larger, and presenting also larger errors. As proved by the reliability of QCD calculations of semileptonic $\tau$ decays, a similar process in a similar energy range, we think perturbative QCD can be trusted here, so we select the corresponding value as our best result. We write thus

$$10^{11} \times a(2 \text{ GeV}^2 \leq t \leq 3 \text{ GeV}^2) = 240 \pm 6,$$  \hspace{1cm} (4.12)

where we have added linearly the errors due to $\Lambda$ and the condensates.

As a verification of the reliability of the calculation, as well as the improvement it presents when compared with earlier determinations, in the rather involved energy range $0.8 \leq t \leq 3 \text{ GeV}^2$ (including here the full $3\pi$ contribution), we compare our value here (adding, for the occasion, the channels $\omega, \phi \to \pi^0 \gamma, \eta \gamma$, see Subsect. 5.2) with that obtained by Narison\cite{7} who uses resonance dominance and s.o.i.c., and to the old CLY\cite{4} evaluation, with s.o.i.c. and QCD:

$$10^{11} \times a(0.8 \text{ GeV}^2 \leq t \leq 3 \text{ GeV}^2 + \omega \to 3\pi) = 1559 \pm 34 \text{ (Here)}$$
$$10^{11} \times a(0.8 \text{ GeV}^2 \leq t \leq 3 \text{ GeV}^2 + \omega \to 3\pi) = 1631 \pm 46 \text{ (Narison, } \tau + e^+ e^- \text{)}$$
$$10^{11} \times a(0.8 \text{ GeV}^2 \leq t \leq 3 \text{ GeV}^2 + \omega \to 3\pi) = 1675 \pm 65 \text{ (Narison, } e^+ e^- \text{)}$$
$$10^{11} \times a(0.8 \text{ GeV}^2 \leq t \leq 3 \text{ GeV}^2 + \omega \to 3\pi) = 1618 \pm 97 \text{ (CLY, } e^+ e^- \text{)}$$  \hspace{1cm} (4.13)

The compatibility between the results, using different methods of evaluation for many pieces, is reasonable.

4.1.2. The region $3 \text{ GeV}^2 \leq t \leq 4.6^2 \text{ GeV}^2$

This is another region where the availability of recent precise data\cite{12} in the neighbourhood of the $\bar{c}c$ threshold, previously poorly known, permit a reliable evaluation. As a byproduct, we get an experimental validation of QCD calculations.

$3 \text{ GeV}^2 \leq t \leq 2^2 \text{ GeV}^2$

We use perturbative QCD here and get

$$10^{11} \times a(3 - 2 \text{ GeV}^2) = 120 \pm 0.8 (A) \pm 0.8 \text{ (Cond.).}$$

$2^2 \text{ GeV}^2 \leq t \leq 3^2 \text{ GeV}^2$

We have now very good recent experimental data. So we present two determinations:

$$10^{11} \times a(2^2 - 3^2 \text{ GeV}^2) = 200 \pm 1 \text{ (A) \hspace{0.5cm} (QCD)}$$
$$10^{11} \times a(2^2 - 3^2 \text{ GeV}^2) = 210 \pm 3 \text{ (St.) \pm 14 (Sys.) \hspace{0.5cm} (Exp. BES)}$$

We only give the error due to $\Lambda$ here because that due to the condensates is negligible. When integrating the BES data we have used the trapezoidal rule. If instead we fitted a horizontal line, we would have obtained

$$10^{11} \times a(2^2 - 3^2 \text{ GeV}^2) = 207 \pm 2 \text{ (St.) \pm 13 (Sys.) \hspace{0.5cm} (Exp. BES)}$$

The BES\cite{12} purely experimental result and the QCD calculation are compatible, but one has to take into account the systematic errors of the first. This shows clearly the importance of systematic variations in $e^+ e^- \text{ annihilations}$ data. We take as our preferred value for the sum of the two intervals that obtained from the QCD calculations:

$$10^{11} \times a(3 \text{ GeV}^2 \leq t \leq 3^2 \text{ GeV}^2) = 320 \pm 2 \pm 1 = 320 \pm 3.$$  \hspace{1cm} (4.14)
Figure 10. Plot of BES experimental data and QCD for the $u, d, s$ quarks (lower $t$) and the same plus NRQCD for the $c$ quark contribution, from $t = 3.74^2$ GeV$^2$ to $4.6^2$ GeV$^2$ at the right. Only systematic errors shown for experimental data. Statistical ones are even smaller.

$3^2$ GeV$^2 \leq t \leq 4.6^2$ GeV$^2$

We give here the results in units of $10^{-11}$. We separate the contribution of the $J/\psi$, $\psi'$, that we calculate in the n.w.a., and the rest. For the first we have,

$$
\begin{align*}
62.0 \pm 4.0 & \quad J/\psi \\
14.8 \pm 1.3 & \quad \psi'
\end{align*}
$$

For the remainder we have the following possibilities:

$$
\begin{align*}
91 \pm 0.4 (\Lambda) & \quad uds; \quad \text{(QCD; } 3^2 \text{ GeV}^2 \leq t \leq 4.6^2 \text{ GeV}^2) \\
4.0 \pm 0.4 & \quad \psi'', \psi''', \psi^{IV} \quad \text{(N, r.d.a.)}
\end{align*}
$$

**Total:** $172 \pm 4$ (N; QCD+r.d.a.)

$$
\begin{align*}
91 \pm 0.4 (\Lambda) & \quad uds; \quad \text{(QCD; } 3^2 \text{ GeV}^2 \leq t \leq 4.6^2 \text{ GeV}^2) \\
46.8 - 28.6 & \rightarrow 38 \pm 10 \quad \bar{c}c. \quad \text{(AY, NRQCD)}
\end{align*}
$$

**Total:** $206 \pm 11$ (AY; QCD+NRQCD)

$$
\begin{align*}
54.7 \pm 0.3 (\Lambda) & \quad \text{(QCD): } 3.0^2 \leq t \leq 3.7^2 \\
56 \pm 0.3 & \quad \text{(Exp., BES): } 3.7^2 \leq t \leq 4.6^2
\end{align*}
$$

**Total:** $188 \pm 4 \pm 3$ (Exp., BES+QCD)

Here N refers to the paper of Narison\cite{7}, and AY is in ref. 4. BES are the experimental data from ref. 12. The first error for them is the statistical, the second the systematic one.

This region merits a somewhat detailed discussion, as there is a certain controversy about it. We have made the calculation in three different manners. First, we separate the $u, d, s$ quarks contribution, that can be evaluated using perturbative QCD. The contribution of the $\bar{c}c$ states is then evaluated saturating it by the $\psi$ resonances, in the r.d.a.; this is the result labeled (N, r.d.a.). This saturation procedure does not produce a good description.
In a second method one separates also the $u,d,s$ contribution; but the $\bar{c}c$ one is treated differently. If a resonance is below the channel for open charm production, which is set at $t = 4m_c^2$ (with $c$ the pole mass of the $c$ quark), then it is treated as a bound state, in the n.w.a. Above $\bar{c}c$ threshold, one uses nonrelativistic QCD (see AY, ref. 4 and ref. 22). The two values reported above for such a calculation (AY, NRQCD) are for two values of the $c$ quark mass: $m_c = 1.750$ GeV, in which case only the $J/\psi$ should be taken to be below threshold, and $m_c = 1.866$ GeV and then both $J/\psi$ and $\psi'$ are to be added below threshold. This last gives the smallest number (28.6). The result of the calculation is taken as the average of both numbers, with half the difference as the estimated error. In Fig. 10 one can see the BES$^{[12]}$ data and the predictions of QCD and NRQCD, the last for $m_c = 1.87$ GeV.

The third method, which is the one that yields our preferred number,

$$10^{11} \times a(3^2 \text{ GeV}^2 \leq t \leq 4.6^2 \text{ GeV}^2) = 188 \pm 4 \pm 3,$$

(4.15)
is obtained by using QCD for $u,d,s$ quarks plus $J/\psi$, $\psi'$ below $t = 3.7^2$, and experimental data (BES$^{[12]}$) above that energy.

All three methods give overlapping results, within errors, with the r.d.a. below experiment, and with an underestimated error, and with the NRQCD calculation reproducing better the data. This NRQCD calculation depends strongly on the mass of the $c$ quark and, in fact, one can turn the argument backwards and predict $m_c$ by requiring equality with the experimental figure. If we do so, we find

$$m_c \simeq 1.89 \text{ GeV},$$

a very reasonable estimate consistent with the two loop result$^{[23]}$ correct to $O(\alpha_s^4)$, which gives $m_c = 1.866 \pm 0.20$ GeV.

4.1.3. The region $t \geq 4.6^2$ GeV$^2$

The results from this region have not changed noticeably, but we give them for completeness.

$4.6^2$ GeV$^2 \leq t \leq 11.2^2$ GeV$^2$

For the first $\Upsilon$ resonances,

$$\begin{align*}
0.55 \pm 0.03 & \quad \Upsilon \\
0.18 \pm 0.01 & \quad \Upsilon'.
\end{align*}$$

Then,

$$88.8 \pm 1.0 (A) \quad udsc : \text{ (QCD)}, \quad 4.6^2 \text{ GeV}^2 \leq t \leq 11.2^2 \text{ GeV}^2$$

$$0.22 \pm 0.04 \quad \bar{b}b : (N, \text{n.w.a.}), \Upsilon'', \Upsilon''', \ldots$$

$$0.53 \pm 0.08 \quad \bar{b}b : (AY, \text{NRQCD})$$

Adding this, we get

**Total:** $90 \pm 1 \quad (N; \text{QCD+n.w.a.})$

**Total:** $90 \pm 1 \quad (AY; \text{QCD+NRQCD})$

The notation is like for the $c$ threshold region. The error in the (AY, NRQCD) evaluation is due to the error in the QCD parameter, $\Lambda$, and the $b$ quark pole mass, for which we have taken$^{[23]}$ $m_b = 5.00 \pm 0.10$ GeV. Both figures are essentially identical and we thus take

$$10^{11} \times a(4.6^2 \text{ GeV}^2 \leq t \leq 11.2^2 \text{ GeV}^2) = 90 \pm 1.$$

(4.16)

$11.2^2$ GeV$^2 \leq t \rightarrow \infty$ GeV$^2$

The use of QCD is mandatory here. The contribution above $t\bar{t}$ threshold is negligible, so we calculate with $n_f = 5$ and get,

$$10^{11} \times a(11.2^2 \text{ GeV}^2 \leq t \rightarrow \infty) = 21 \pm 0.1 (A).$$

(4.17)
4.2. The whole $a^{(2)}(h.v.p.)$

Our final result for the $O(\alpha^2)$ hadronic contribution to $a_\mu$ is then

$$10^{11} \times a^{(2)}(h.v.p.) = \begin{cases} 6909 \pm 64 & (e^+e^- + \tau + \text{spacel.}) \\ 6889 \pm 96 & (e^+e^- + \text{spacel.}) \end{cases}$$ (4.18)

To compare with other evaluations we have to add the contribution $(43 \pm 4) \times 10^{-11}$ of some of the radiative decays of the $\rho, \omega, \phi$ (see Sect. 5.2) that other authors include. This comparison is shown, for a few representative calculations,\footnote{A more complete list of evaluations, including some of the very earliest ones, may be found in the paper of Narison, ref. 7.} in Table 3.

<table>
<thead>
<tr>
<th>Authors</th>
<th>$10^{11} \times a(h.v.p.)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>KNO</td>
<td>$7068 \pm 174$</td>
<td>$e^+e^-$ only</td>
</tr>
<tr>
<td>CLY</td>
<td>$7100 \pm 116$</td>
<td>$e^+e^- + \text{spacel.}$</td>
</tr>
<tr>
<td>CLY-II</td>
<td>$7070 \pm 116$</td>
<td>$e^+e^- + \text{spacel.}$</td>
</tr>
<tr>
<td>ADH</td>
<td>$7011 \pm 94$</td>
<td>$e^+e^- + \tau$</td>
</tr>
<tr>
<td>BW</td>
<td>$7026 \pm 160$</td>
<td>$e^+e^-$</td>
</tr>
<tr>
<td>AY</td>
<td>$7113 \pm 103$</td>
<td>$e^+e^- + \text{spacel.}$</td>
</tr>
<tr>
<td>DH</td>
<td>$6924 \pm 62$</td>
<td>$e^+e^- + \tau$</td>
</tr>
<tr>
<td>J</td>
<td>$6974 \pm 105$</td>
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</tr>
<tr>
<td>N1</td>
<td>$7031 \pm 77$</td>
<td>$e^+e^- + \tau$</td>
</tr>
<tr>
<td>N2</td>
<td>$7011 \pm 117$</td>
<td>$e^+e^- + \text{only}$</td>
</tr>
<tr>
<td>TY1</td>
<td>$6952 \pm 64$</td>
<td>$e^+e^- + \tau + \text{spacel.}$</td>
</tr>
<tr>
<td>TY2</td>
<td>$6932 \pm 96$</td>
<td>$e^+e^- + \text{spacel. only}$</td>
</tr>
</tbody>
</table>

Table 3

KNO: ref. 3; BW: ref. 3; J: ref. 6; CLY, CLY-II, AY: ref. 4; N: ref. 7; DH, ADH: ref. 5.

If we had added also the other radiative contributions ($\pi\pi\gamma$, and the continuum hadron + $\gamma$, cf. Subsect. 5.2) we would have found the hadronic vacuum polarization piece, correct to order $\alpha^2$ and $\alpha^3$,

$$10^{11} \times a^{(2+3)}(h.v.p.) = \begin{cases} 7002 \pm 66 & (e^+e^- + \tau + \text{spacel.}) \\ 6982 \pm 97 & (e^+e^- + \text{spacel.}) \end{cases}$$ (4.19)
5. Higher order hadronic contributions

5.1. Hadronic light-by-light contributions

A contribution in a class by itself is the hadronic light by light one. So we split

\[ a(\text{Other hadronic, } O(\alpha^3)) = a(\text{\textquoteleft One blob\textquoteright{} hadronic, } O(\alpha^3)) + a(\text{Hadronic light by light}). \]  

(5.1)

We will start by considering the last, given diagrammatically by the graph of Fig. 11. This can be evaluated only using models. One can make a chiral model calculation, in the Nambu–Jona-Lasinio version or the chiral perturbation theory variety, with a cut-off, or one can use a constituent quark model in which we replace the blob in Fig. 11 by a quark loop (Fig. 12). The result depends on the cut-off (for the chiral calculation) or on the constituent mass chosen for the quarks. The variation from one model to the other is, unfortunately, too large for comfort. We take as a typical value for the chiral calculation,\textsuperscript{24}

\[ 10^{11} \times a(\text{Hadronic light by light}) = -86 \pm 25 \text{ Chiral calculation; BPP, HKS.} \]  

(5.2a)

Earlier calculations with the chiral model, using VMD to cure its divergence, gave (HKS, ref. 24)

\[ 10^{11} \times a(\text{Hadronic light by light}) = -52 \pm 20 \text{ Chiral calculation (HKS).} \]  

(5.2b)

For the constituent quark model we use the results of Laporta and Remiddi.\textsuperscript{25} The contribution to \( a_\mu \) of light by light scattering, with a loop with a fermion of charge \( Q_i \), and mass \( m_i \) larger than the muon mass, is

\[ a_{lxl,i} = Q_i^4 \left( \frac{\alpha}{\pi} \right)^3 2c_{lxl,i}, \]

where, to order \( (m_\mu/m_i)^4 \),

\[ c_{lxl,i} = \left( \frac{m_\mu}{m_i} \right)^2 \left[ \frac{4}{9} \xi(3) - \frac{19}{9} \right] \]

+ \( \left( \frac{m_\mu}{m_i} \right)^4 \left[ -\frac{161}{810} \log^2 \frac{m_i}{m_\mu} - \frac{16189}{48600} \log \frac{m_i}{m_\mu} + \frac{13}{18} \xi(3) - \frac{161}{9720} \pi^2 - \frac{831931}{972000} \right] + \cdots. \]

Taking constituent masses,

\[ m_{u,d} = 0.33, \quad m_s = 0.50, \quad m_c = 1.87 \text{ GeV} \]

we find

\[ 10^{11} \times a(\text{Hadronic light by light}) = +92 \pm 20 \text{ (Quark const. model)} \]  

(5.2c)
and the error is estimated by varying $m_{u,d}$ by 10%. We note that, from general arguments, we can conclude that the bag model would also yield a positive figure.

One could also take the estimate of the $\pi^0$ pole from Hayakawa and Kinoshita\cite{24} and add the constituent quark loop, in which case we get

$$10^{11} \times a_{\text{Hadronic light by light}} \sim 10 \pm 30 \quad \text{(Quark const. model+ pion pole)} \quad (5.2d)$$

This procedure could be justified in models where the strong coupling “freezes” at low energy, but we will not discuss this here.

These variations are unfortunate; one expects the chiral calculation to be valid for small values of the virtual photon momenta, and the constituent model to hold for large values of the same.\footnote{Strictly speaking, one would also need large momentum of the external photon to get a really trustworthy evaluation with the constituent model.} So, neither evaluation is fully believable: we will give results with the extreme choices, (5.1a,c).

### 5.2 Photon radiation corrections to the hadronic vacuum polarization

The $a^{('\text{One blob}'} \text{ hadronic, } O(\alpha^3))$ corrections are obtained by attaching a photon or fermion loop to the various lines in Fig. 1. They can be further split into two pieces: the piece where both ends of the photon line are attached to the hadron blob, $a^{(h.v.p., \gamma)}$, shown in Fig. 13, and the rest. So we write,

$$a^{('\text{One blob}'} \text{ hadronic, } O(\alpha^3)) = a^{(h.v.p., \gamma)} + a^{('\text{One blob}'} \text{ hadronic, rest).} \quad (5.3)$$

The last can be evaluated\cite{26} in terms of the hadronic contributions to the photon vacuum polarization, finding

$$10^{11} \times a^{('\text{One blob}'} \text{ hadronic, rest}) = -101 \pm 6. \quad (5.4)$$

(Note, however, that this result has not, as far as we know, been checked by an independent calculation).

The only contribution that requires further discussion is that depicted in Fig. 13, $a^{(h.v.p., \gamma)}$. In principle, this contribution can be evaluated straightforwardly by a generalization of the Brodsky–de Rafael method. We can write

$$a^{(2)}(h.v.p.) + a^{(h.v.p., \gamma)} = \int_{4m^2}^{\infty} dt \ K(t) R^{(2)}(t), \quad (5.5)$$

where

$$R^{(2)}(t) = \frac{\sigma^{(0)}(e^+e^- \rightarrow \text{hadrons}) + \sigma^{(2)}(e^+e^- \rightarrow \text{hadrons}) + \sigma^{(0)}(e^+e^- \rightarrow \mu^+\mu^-)}{\sigma^{(0)}(e^+e^- \rightarrow \mu^+\mu^-)}. $$

The notation means that we evaluate the hadron annihilation cross section to second order in $\alpha$, and we add to it the first order annihilation into hadrons plus a photon.
For energy \((t)\) large enough, this can be calculated with the parton model, and leads to a correction \(\frac{3}{2} \alpha/\pi\) times the h.v.p. contribution. Taking then \(t \geq 1.2\) GeV\(^2\), this is \(2.0 \times 10^{-11}\). Since in many pieces of the evaluation of the cross section \(\sigma(e^+e^- \rightarrow \text{hadrons})\) we took experimental data, where it is not clear whether photons are included or not, we think it safest to take half this figure, \(1.0 \times 10^{-11}\), as an error. So we write \((2 \pm 1) \times 10^{-11}\) for this contribution.

Then comes the contribution of small momenta, \(t \leq 1.2\) GeV\(^2\). We start by discussing the process involving two pions. In our determination in Sects. 3, 4 of \(a^{(2)}(\text{h.v.p.})\), we made calculations by fitting the experimental cross section \(e^+e^- \rightarrow \pi^+\pi^-\), which specifically excludes radiation of \textit{hard} photons (hard photons defined as those that are identified experimentally). Diagrammatically, this means that our evaluations of Sects. 3, 4 included the diagrams of Fig. 14A (where a soft photon is one that is not detected), but not
those of Fig. 14B (radiation of a hard photon). So, we have to include this radiation into \( a(\text{h.v.p.}, \gamma) \). This can be easily done if we consider this region to be dominated by the rho, hence we approximate

\[
\sigma^{(0)}(e^+e^- \rightarrow \text{hadrons}; \gamma) \simeq \sigma^{(0)}(e^+e^- \rightarrow (\rho) \rightarrow \pi^+\pi^-\gamma).
\]

The last can be evaluated in terms of the branching ratio for the decay \( \rho \rightarrow \pi^+\pi^-\gamma \), which is indeed measured experimentally (see the review of Dolinsky et al., ref. 20) from the reaction \( e^+e^- \rightarrow \rho \rightarrow \pi^+\pi^-\gamma \). In the narrow width approximation for the rho, the contribution to \( a/\mu \) is

\[
\frac{\Gamma(\rho \rightarrow \pi^+\pi^-\gamma)}{\Gamma_\rho} \cdot \frac{3G_F(\rho)\hat{K}(m_\rho^2)}{\pi m_\rho}.
\]  

(5.6a)

In this way, we find

\[
a^{10^{11}}(\text{h.v.p.}, \pi^+\pi^-\gamma) = 45 \pm 7 \quad \text{(n.w.a.)},
\]

(5.6b)

and the error is that induced by the experimental error in the width \( \Gamma(\rho \rightarrow \pi^+\pi^-\gamma) \).

We will elaborate a bit more on this contribution. The final state interaction of the \( \pi^+\pi^- \) in the state \( \pi^+\pi^- \gamma \) is very strong. The pions are produced in an S-wave, which presents a wide enhancement in the energy region \( E_{\pi\pi\gamma} \simeq 0.6 \pm 0.2 \text{ GeV} \). However, this is only a small part of the contribution to the rate for \( \pi^+\pi^- \gamma \). According to Dolinsky et al., ref. 20, pp. 126 ff., most of the effect would be due to Bremsstrahlung by the pions. Above procedure to estimate this, in terms of the \( \pi^+\pi^- \gamma \) decay of the \( \rho \) would be exact only if the experimental cuts made for identifying this decay, and to measure the pion form factor were the same. A more accurate procedure is as follows. We write

\[
a(\pi^+\pi^-\gamma, t \leq 1.2) = \int_{4m_\pi^2}^{1.2} dt K(t)R_{\pi^+\pi^-\gamma}(t)
\]

(5.7a)

where

\[
R_{\pi^+\pi^-\gamma}(t) = B(t, E_\gamma)R_{\pi^+\pi^-\gamma}(t)
\]

and the Bremsstrahlung factor \( B \) is given by \( \text{[27]} \)

\[
B(t, E_\gamma) = \frac{8t^{1/2}a}{\pi(t - 4m_\pi^2)^{3/2}} \int_{E_\gamma}^{k_m} \frac{dk}{k} I(k),
\]

(5.7b)

\[
I(k) = k_m \left( \frac{t - 2m_\pi^2}{2t^{1/2}} - k \right) \log \frac{1 + \xi}{1 - \xi} - \left[ k_m \left( \frac{t^{1/2} - k}{2} \right) \right]^2 \xi.
\]

(5.7b)

Here \( \xi = [(k_m - k)/(\frac{t}{2}^{1/2} + k)]^{1/2}, k_m = (t - 4m_\pi^2)/2t^{1/2} \) is the maximum energy of the photon and, finally, \( E_\gamma \) is the minimum energy for photon detection.

To evaluate \( E_\gamma \), we have to look at the setup of experiments measuring the pion form factor. Typically, one takes that no (hard) photon has been emitted when the angle between the pion momenta differs from \( \pi \) by less than a small given amount, \( \eta_0 \). The energy cut is, in this case,

\[
E_\gamma = \frac{\sqrt{1 - 4m_\pi^2}}{2} \eta_0.
\]

The effective \( \eta_0 \) depends on the specific cuts made in each experiment; those in ref. 8 are covered if we take \( \eta_0 = 0.15 \pm 0.05 \). Using this we find the result, for this range of \( \eta_0 \), of

\[
10^{11} a(\pi^+\pi^-\gamma, t \leq 1.2) = 46 \pm 0.5 \pm 9.
\]  

(5.8)

The first error corresponds to the error in the integral of \( |F_\pi|^2 \) and the second is induced by the dispersion in the values of \( \eta_0 \) of the various experiments. (5.8) is practically identical to the n.w.a. result, Eq. (5.6b) (which of course is a satisfactory result). The reason for this agreement is that, when detecting \( \pi^+\pi^-\gamma \), the energy cut made, \( E_\gamma = 50 \text{ MeV} \) (Dolinsky et al., ref. 20), turns out to be very similar to the average energy cut made when measuring the pion form factor.

A similar analysis ought to be made, in principle, for other radiative intermediate states like \( 3\pi + \gamma \) and \( K\bar{K} + \gamma \), which can be estimated in terms of the corresponding decays of the \( \omega \) and \( \phi \), but they give a contribution below the \( 10^{-11} \) level and we neglect them.
The lowest energy contributions to \( \sigma^{(0)}(e^+e^- \rightarrow \text{hadrons}; \gamma) \) are those of the intermediate states \( \pi^0\gamma \) and \( \eta\gamma \), Fig. 15. At energies below the rho mass, one can evaluate the first (the only one that gives a sizable contribution) by relating the process to the decay \( \pi^0 \rightarrow 2\gamma \). We write an effective interaction, corresponding to the vertex factor in the Feynman rules of
\[
\frac{iG_\pi}{2m_\pi} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta;
\]
with it
\[
\Gamma(\pi^0 \rightarrow 2\gamma) = \frac{G_\pi^2 m_\pi}{256\pi}
\]
so that \( G_\pi^2 = 4.6 \times 10^{-5} \). Then, with \( e \) the electron charge,
\[
\text{Im } \Pi^{\gamma\gamma}(t) = \frac{G_\pi^2 t}{384\pi e^2 m_\pi^2} \left( 1 - \frac{m_\pi^2}{t} \right)^3.
\]
This gives a very small contribution to \( a_\mu \), about \( 0.76 \times 10^{-11} \) if we integrate up to \( t^{1/2} \approx 0.7 \) GeV, and \( 0.96 \times 10^{-11} \) if we go to \( t^{1/2} \approx 0.84 \) GeV (the integral only grows logarithmically). We only integrate up to the rho, i.e., to \( t^{1/2} \approx 0.7 \) GeV with this pointlike model.

Around the rho region we have to take into account the excitation of this resonance, which produces the corresponding enhancement. This piece can be obtained in terms of the radiative width \( \rho \rightarrow \pi^0\gamma \). More important is the \( \omega \rightarrow \pi^0\gamma \) process which gives \( (33 \pm 2) \times 10^{-11} \). Likewise, the contribution of the \( \eta\gamma \) state is evaluated in terms of the decay \( \phi \rightarrow \eta\gamma \). Finally, the contribution from \( \pi^0\pi^0\gamma \) is taken from ref. 28. Collecting all of this, we get
\[
10^{11} \times a(\text{h.v.p., } \rho \rightarrow \pi^0\gamma) = 4 \pm 1
\]
\[
10^{11} \times a(\text{h.v.p., } \rho \rightarrow \eta\gamma) = 1.1 \pm 0.4
\]
\[
10^{11} \times a(\text{h.v.p., } \omega \rightarrow \pi^0\gamma) = 33 \pm 2
\]
\[
10^{11} \times a(\text{h.v.p., } \phi \rightarrow \eta\gamma) = 5 \pm 1;
\]
\[
\text{TOTAL} : 43 \pm 4
\]
we have included the lower energy contribution of \( \pi^0\gamma \) into \( a(\text{h.v.p., } \rho \rightarrow \pi^0\gamma) \) and, because we are relying on models, we added the errors linearly. For the \( \pi\pi\gamma \) states,
\[
10^{11} \times a(\text{h.v.p., } \pi^+\pi^-\gamma) = 46 \pm 9
\]
(5.9a)
\[
10^{11} \times a(\text{h.v.p., } \pi^0\pi^0\gamma) = 2 \pm 0.3;
\]
(5.9b)
and, for the high energy piece,
\[
10^{11} \times a(\text{hadrons } + \gamma, \ t \geq 1.2 \text{ GeV}^2) = 2 \pm 1.
\]
(5.9c)
Other contributions are negligible. The total effect of the states hadrons + γ is thus,

\[ 10^{11} \times a(\text{hadrons} + \gamma) = 93 \pm 11. \]  

(5.10)

6. Conclusions

Taking into account all contributions, and errors, we complete the best values for the h.v.p. piece, and the whole hadronic part of the anomaly:

\[ 10^{11} \times a^{(2+3)}(\text{h.v.p.}) = \begin{cases} 
7002 \pm 66 & (e^+e^- + \tau + \text{spacel.}) \\
6982 \pm 97 & (e^+e^- + \text{spacel.}), 
\end{cases} \]  

(6.2)

and adding the other radiative and light-by-light corrections,

\[ 10^{11} \times a(\text{Hadronic}) = \begin{cases} 
6993 \pm 69 & (e^+e^- + \tau + \text{spacel.}; \text{Q.c.m.}) \\
6815 \pm 71 & (e^+e^- + \tau + \text{spacel.}; \text{Ch.m.}) \\
6973 \pm 99 & (e^+e^- + \text{spacel.}; \text{Q.c.m.}) \\
6795 \pm 100 & (e^+e^- + \text{spacel.}; \text{Ch.m.}) 
\end{cases} \]  

(6.3)

Eq. (6.3) is of course the main outcome of the present paper. Because, even after adding systematic and theoretical errors the evaluation including τ decay data is more precise, we may consider it to provide the best result for \( a(\text{Hadronic}) \) available at present.

We can add to (6.3) the pure electroweak contributions and present the result as the standard model prediction for \( a_\mu \):

\[ 10^{11} \times a_\mu = \begin{cases} 
116591849 \pm 69 & (e^+e^- + \tau + \text{spacel.}; \text{Q.c.m.}) \\
116591671 \pm 71 & (e^+e^- + \tau + \text{spacel.}; \text{Ch.m.}) 
\end{cases} \]  

(6.4)

We will next compare our results with other recent evaluations of the same quantities. In Figs. 16 we show a number of last century’s theoretical evaluations (open circles), and recent ones (this century, black dots), together with the experimental band. Older evaluations are presented as given by the authors; the more recent ones are shown incorporating the contribution of the \( \pi^+\pi^-\gamma \) and \( \pi^0\pi^0\gamma \) channels, and with the chiral model result for the hadronic light-by-light piece. For the recent evaluations we present, in Fig. 16B, also the results when using the quark constituent model for the hadronic light-by-light contribution.

A further point to emphasize is the importance of using, in the low energy region, parametrizations of \( F_\pi(t) \) compatible with unitarity and analyticity. Only in this way we can incorporate data on \( F_\pi(t) \) for spacelike \( t \) into the fits. As discussed in Subsect. 3.2, these data not only provide a substantial shift for \( a_\mu \) (of \( 39 \times 10^{-11} \) in the evaluation with \( e^+e^- \) data only) but, by so doing, allow compatibility of these with the results from τ decay, hence allowing a combination of the two in a meaningful way: this permits an important reduction of the errors of the calculation.\(^{12}\)

To finish this section, we can add a few words on prospects for improvements. In our view they are rather dim in the sense that it is not easy to see how one could get an error estimate below the \( 70 \times 10^{-11} \) mark, when taking into account systematic and theoretical errors. In fact, as Fig. 16 shows, the central values have moved little, and the errors have not improved much, since 1985. It is true that experiments planned or in progress can clear further the region between 1.2 and 3 GeV\(^2\). However, a serious improvement of the very important low energy region for \( \pi\pi \) is unlikely: as our evaluations show, one can get a fit to all data relevant for the hadronic component of \( a_\mu \), with a \( \chi^2 / \text{d.o.f.} \) of unity and verifying all theoretical constraints, with an error of at least \( 51 \times 10^{-11} \), already for \( t \leq 0.8 \) GeV\(^2\) (see Eq. (3.19)). In this respect the improvement obtained by adding τ decay data, although not negligible, is minor: statistical errors are smaller, but theoretical ones are increased.

All this is not too bad; it is possible that one could get a theoretical estimate of \( a^{(2+3)}(\text{h.v.p.}) \) with an error comparable to the best experimental error attainable in the near future, of the order of \( 60 \times 10^{-11} \). However, it is difficult to see how one can get a reliable handle on the evaluation of the light-by-light

\(^{12}\)Or, put conversely, not using data at spacelike \( t \) for \( F_\pi \) implies a hidden error of about \( 40 \times 10^{-11} \).
Figure 16A. Theoretical results on \( a_{\text{Hadronic}} \times 10^{-11} \), and experiment. The chiral calculation result used for the hadronic light-by-light contribution.

Figure 16B. The same, but with the quark model result for the hadronic light-by-light contribution.

CLY: ref. 4 (no \( \pi\pi \) phase shifts); KNO: ref. 3; ADH, DH: ref. 5; N1: ref. 7, data from \( e^+e^- + \tau \). N2: id., \( e^+e^- \) only. T1, T2: this paper with data from \( e^+e^- + \tau \) or data from \( e^+e^- \) only, respectively (including syst. and th. errors).

scattering: a contribution which shifts the results for \( a_\mu \) by as much as \( 178 \times 10^{-11} \), and certainly at least \( 40 \times 10^{-11} \), depending on the model used. To get a reliable value for this quantity is a nontrivial challenge for theorists.
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References


