Non-factorization effects in heavy mesons and
determination of $|V_{ub}|$ from inclusive semileptonic $B$
decays.

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Abstract

The effects of spectator light quark in decays of heavy mesons are considered, which vanish in the limit of factorization of matrix elements of four-quark operators over the mesons. These effects include the difference of the total widths as well as of the semileptonic decay rates between the $D^0$ and $D_s$ mesons and also a contribution to inclusive semileptonic decay rates of $B^0$ and $B^{±}$ into the channel $X_u \ell \nu$ related to determination of the weak mixing parameter $V_{ub}$. If the observed difference of the lifetimes between $D_s$ and $D^0$ mesons is attributed to non-factorizable terms, such terms can naturally give rise to a significant difference in inclusive semileptonic decay rates of these mesons, and to a light-flavor dependent contribution to decays $B \to X_u \ell \nu$. The latter contribution affects mostly the upper part of the inclusive spectrum of the invariant mass of the lepton pair, and may significantly exceed the previously claimed in the literature theoretical uncertainty in determination of $|V_{ub}|^2$ from that part of the spectrum.
1 Introduction

The well known difficulty of determination of the weak mixing parameter $|V_{ub}|$ from the inclusive decay $B \to X_u \ell \nu$ is in discriminating this process from the much more abundant decays $B \to X_c \ell \nu$ (see e.g. in the review in Ref. [1] and references therein). The experimental method used so far is to impose a lower cutoff on the charged lepton energy at or above the kinematical boundary for the processes with charmed hadrons in the final state. Theoretically however a calculation of the lepton energy spectrum in the endpoint region is prone to a substantial uncertainty [1], which is unlikely to be resolved without knowledge of $B$ meson structure functions [2].

A somewhat more theoretically tractable scheme arises [3] if a cutoff is applied to the invariant mass squared $q^2$ of the lepton pair, $q^2 \geq q^2_0$, instead of the single-lepton energy. Clearly, the condition $q^2_0 \geq (M_B - M_D)^2$ makes a sample of semileptonic decay events free from the background from $B \to X_c \ell \nu$. The total decay rate into the part of the phase space limited by such constraint can be analysed, including nonperturbative contributions as well as the perturbative one, by applying the operator product expansion (OPE) for inclusive decay rates [4, 5]. Most recently an analysis along these lines was performed in Ref.[6].

The difficulty arising in this approach is that the “short distance” parameter in the OPE for the constrained inclusive decay rate is determined [6] by the typical momentum $\mu_c$ of the $u$ quark in the constrained kinematical region, where its maximal value is $(M^2_B - q^2_0)/2M_B$, which becomes slightly smaller than the mass of the charmed quark $m_c$, even if the cutoff parameter is chosen at exactly the kinematical limit for charmed final states, $q^2_0 \geq (M_B - M_D)^2$. This puts the theoretical status of the OPE for the constrained inclusive rate on par with or even somewhat worse than that of an OPE based analysis of total inclusive weak decay rates of charmed mesons and baryons. It is well known that in the latter analysis the nonperturbative terms in OPE, whose relative contribution is suppressed as $m_c^{-2}$ and $m_c^{-3}$, are essentially as significant as the “leading” perturbative (parton) term, which behavior prominently manifests itself in the large differences of the lifetimes both between the charmed mesons and baryons and within the meson and baryon light-flavor multiplets. This simple observation illustrates the acute necessity of considering the higher terms, formally suppressed by $\mu_c^{-2}$ and $\mu_c^{-3}$, in the OPE based analysis of the constrained decay rate of $B \to X_u \ell \nu$. The leading perturbative term as well as the nonperturbative contribution proportional to $\mu_c^{-2}$ were considered in Ref.[6], while the contribution relatively suppressed by $\mu_c^{-3}$ was relegated to a theoretical uncertainty. The magnitude of the uncertainty, arising from this source, was
estimated (essentially by guessing) to correspond to about 5% theoretical error in \( |V_{ub}| \). The purpose of this paper is to present the contribution of the third term in OPE in an explicit operator form, and to analyse, to an extent, the numerical significance of this term.

As will be discussed in Sec. 2 this contribution is described, as usual, by dimension 6 four-quark operators. The matrix element of the relevant combination of the operators over the \( B \) mesons vanishes in the limit of factorization. However, also as usual, the considered term is greatly enhanced numerically, so that a violation of the factorization relation by 10% makes this contribution about twice larger than the estimate of the corresponding uncertainty in Ref. [6]. Furthermore, the effect explicitly depends on the light quark flavor and thus is generally different in the decays of \( B^0 \) and \( B^\pm \) mesons. In Sec. 3 it is argued that a deviation from factorization of such magnitude would not be unusual, and in fact is possibly indicated by the observed difference of the lifetimes of the \( D_s \) and \( D^0 \) mesons, if that difference is analysed within the same OPE framework. Moreover, the discussed effect in the constrained decay rate of \( B \to X_u \ell \nu \) is very closely related to a similar effect in semileptonic charm decays, in particular to the yet unmeasured difference of total inclusive semileptonic decay rates of \( D_s \) and \( D^0 \). Thus a quantitative understanding of the discussed contribution would become possible, once this difference of the semileptonic decay rates is measured experimentally. The concluding Section 4 contains discussion and summary.

## 2 The third term in OPE for constrained inclusive decay rate.

The optical theorem of the scattering theory relates the total decay rate \( \Gamma_H \) of a hadron \( H_Q \) containing a heavy quark \( Q \) to the imaginary part of the ‘forward scattering amplitude’. For the case of inclusive decays generated by a particular term \( L_W \) in the weak interaction Lagrangian the latter amplitude is described by the following effective operator

\[
L_{\text{eff}} = 2 \Im \left[ i \int d^4x \, e^{ipx} T \left\{ L_W^\dagger(x), L_W(0) \right\} \right],
\]

in terms of which operator (at \( p^2 = m_Q^2 \)) the total decay rate is given by

\[
\Gamma_H = \langle H_Q | L_{\text{eff}} | H_Q \rangle.
\]

Using in eq. (1) the term

\[
L_{ub} = \frac{G_F V_{ub}}{\sqrt{2}} \left( \bar{u} \gamma_\mu \left( 1 - \gamma_5 \right) b \right) \ell_\mu
\]

\[\text{The non-relativistic normalization for the heavy quark states is used here: } \langle Q | Q^\dagger Q | Q \rangle = 1.\]
with $\ell_\mu = (\bar{\ell} \gamma_\mu (1 - \gamma_5) \nu$) in place of $L_W$, one would find the total inclusive decay rate of $B \to X_u \ell \nu$. The effective operator (1) is evaluated using short-distance OPE. The leading term in the expansion describes the perturbative decay rate, while subsequent terms containing operators of higher dimension describe the nonperturbative contributions. In particular, the leading term with no QCD radiative corrections applied, gives for the total inclusive rate of the decays $B \to X_u \ell \nu$ the familiar expression

$$\Gamma_0 = \frac{G_F^2 |V_{ub}|^2 m_b^5}{192 \pi^3}.$$  

(4)

For a constrained inclusive rate in the case of semileptonic decays and with selection in the invariant mass of the lepton pair, one should rather consider, instead of the full correlator (1), a correlator of the quark currents, and then perform integration over the lepton pair phase space with the weight function corresponding to the considered selection of events [6]. However for the third term in the OPE, suppressed by $m_Q^{-3}$ and expressed through four-quark operators of dimension 6, this procedure is not really necessary, since formally this contribution corresponds to an infinitesimally small momentum carried away by the final light quark, and all the momentum of the initial heavy quark flowing through the lepton pair. Thus this contribution formally comes only from $q^2 = m_Q^2$, and does not depend on the lower cutoff $q_0^2$. (This behavior of the discussed nonperturbative effect was also used in the estimate of Ref.[6].) Therefore the effect of this term constitutes a fixed fraction of the total rate of the decays $B \to X_u \ell \nu$.

It is quite straightforward to find the expression for the third term in the expansion of the correlator (1) appropriate for the decays $B \to X_u \ell \nu$ [5, 7, 8] in terms of four-quark operators normalized at $\mu = m_b$:

$$L^{(3)}_{b \to u \ell \nu} = -\frac{2 G_F^2 |V_{ub}|^2 m_b^2}{3 \pi} \left( O_{V-A}^u - O_{S-P}^u \right) ,$$  

(5)

where the following notation [9] is used for the relevant four-quark operators:

$$O_{V-A}^q = (\bar{b}_L \gamma_\mu q_L)(\bar{q}_L \gamma_\mu b_L) , \quad O_{S-P}^q = (\bar{b}_R q_L)(\bar{q}_L b_R) ,$$

$$T_{V-A}^q = (\bar{b}_L t^a \gamma_\mu q_L)(\bar{q}_L t^a \gamma_\mu b_L) , \quad T_{S-P}^q = (\bar{b}_R t^a q_L)(\bar{q}_L t^a b_R) .$$

(6)

(The operators $T$, containing the color generators $t^a$, will appear in further discussion.)

The matrix elements of the four-quark operators over the $B$ mesons are parameterized [9] as

$$\langle B | O_{V-A}^u | B \rangle = \frac{f_B^2 m_B}{8} B_1 , \quad \langle B | O_{S-P}^u | B \rangle = \frac{f_B^2 m_B}{8} B_2 ,$$

(7)
where \( f_B \) is the meson annihilation constant, and \( B_1 \) and \( B_2 \) are generally phenomenological parameters, “bag constants”. In the limit of naive factorization, i.e. where the product of bilinear operators is saturated by only the vacuum insertion, these parameters are \( B_1 = B_2 = 1 \) for the \( B \) meson containing the same light quark as the operator (\( B^\pm \) mesons for the case of the operators \( O^a \)), and are zero for the other two mesons (\( B_d \) and \( B_s \) in this case). Clearly, in this limit the matrix element of the four-quark operator in eq.(5) over either of the \( B \) mesons is vanishing\(^2\). However, as useful as the factorization relation might be for other estimates, one does not expect this relation to be valid with better than about 10% accuracy (see e.g. in Ref.[10]), and it is conventionally implied that deviations of such order of magnitude are present. Moreover, it should be emphasized that these deviations refer not only to the relation between the constants: \( B_1 = B_2 \) for the meson with the same flavor as that of the quark \( q \) in the operator \( O^a \), but that these constants are generally non zero when the light quark flavors do not match. In other words, there are separate sets of the constants \( B \) describing the matrix elements \( B_{q'q} = \langle B_{q'}|O^a|B_q\rangle \) for \( q' = q \) and for \( q' \neq q \). Taking into account also the violation of the flavor SU(3) would further proliferate the constants \( B \).

In any case, a violation of the relation \( B_1 = B_2 \) by about 10% would not be surprising at all. On the other hand, a deviation from the exact factorization of such moderate magnitude would be quite significant for the discussed effect in the \( B \to X_u \ell \nu \) decays. Indeed, in terms of the “bag constants” the contribution \( \delta \Gamma \) to the decay rate of the term (5) is expressed according to equations (2) and (7) as

\[
\delta \Gamma \approx \frac{G_F^2 |V_{ub}|^2 f_B^2 m_b^3}{12 \pi} \left( B_2 - B_1 \right).
\]

Comparing this with the dominant semileptonic decay rate of \( B \to X_c \ell \nu \), and taking into account the known semileptonic branching ratio \( B_{sl}(B) \), one estimates that the discussed nonperturbative contribution corresponds in terms of the branching ratio to the correction

\[
\delta B(B \to X_u \ell \nu) \approx 3.9 \left( \frac{f_B}{0.2 \text{GeV}} \right)^2 \left( \frac{B_2 - B_1}{0.1} \right) |V_{ub}|^2,
\]

which easily can exceed by a substantial amount the previous estimate [6] of this correction as \( 2.0 |V_{ub}|^2 \) if the deviation from factorization is, as expected, at a 10% level.

\(^2\)The naive factorization for the discussed effect actually correspond to the annihilation decay \( B^\pm \to \ell \nu \), which obviously is forbidden by chirality in the limit where the lepton mass is neglected.
3 Non-factorization effects in $D$ mesons

The deviation from naive factorization in mesons can in fact be probed more quantitatively from the data on charmed $D$ mesons. Indeed, in the limit of factorization, the OPE for the dominant Cabibbo unsuppressed nonleptonic decays due to $c \to s u \bar{d}$ predicts equal rates of decay for $D_s$ and $D^0$ mesons [5, 7]. Experimentally it is known by now [1] that there is a noticeable difference in lifetimes of these mesons: $	au(D_s)/\tau(D^0) = 1.20 \pm 0.025$, which cannot be described by spectator dependent effects in Cabibbo suppressed decay channels. Although this discrepancy can be merely attributed to the overall inaccuracy of the OPE in the inverse of the charmed quark mass\(^3\), a more constructive approach would be to attempt describing this difference in lifetimes as due to deviations from factorization. As will be discussed further in this section such approach also can be tested by measuring the difference of inclusive semileptonic decay rates of the $D_s$ and $D^0$ mesons, which difference is somewhat more directly related to the discussed matrix elements over the mesons in equations (8) and (9).

In the limit of flavor SU(3) symmetry the difference of the dominant inclusive nonleptonic decay rates of $D^0$ and $D_s$ mesons can be written [7] in terms of matrix elements of four-quark operators (normalized at $\mu = m_c$) as

$$\Gamma(D^0) - \Gamma(D_s) = \frac{2 G_F^2 \cos^4 \theta_c m_c^2 f_D^2 m_D}{9\pi} C_+ C_- \left( B_1^{ns} - B_2^{ns} - \frac{3}{4} \varepsilon_1^{ns} + \frac{3}{4} \varepsilon_2^{ns} \right), \quad (10)$$

where $\theta_c$ is the Cabibbo angle, $C_+$ and $C_-$ are the well known short-distance QCD renormalization coefficients for nonleptonic weak interaction: $C_- = C_+^2 = (\alpha_s(m_c)/\alpha_s(m_W))^{12/25}$, and the flavor non-singlet coefficients $B$ and $\varepsilon$ parameterize the following differences of the matrix elements:

\begin{align*}
\frac{1}{2} \langle O^{s}_{V-A} - O^{u}_{V-A} \rangle_{D_s - D^0} &= \frac{f_D^2 m_D}{8} B_1^{ns}, \\
\frac{1}{2} \langle O^{s}_{S-P} - O^{u}_{S-P} \rangle_{D_s - D^0} &= \frac{f_D^2 m_D}{8} B_2^{ns}, \\
\frac{1}{2} \langle T^{s}_{V-A} - T^{u}_{V-A} \rangle_{D_s - D^0} &= \frac{f_D^2 m_D}{8} \varepsilon_1^{ns}, \\
\frac{1}{2} \langle T^{s}_{S-P} - T^{u}_{S-P} \rangle_{D_s - D^0} &= \frac{f_D^2 m_D}{8} \varepsilon_2^{ns}, \quad (11)
\end{align*}

where the operators $O$ and $T$ are the same as in eq.(6) with the $b$ quark being replaced by $c$ and the notation $\langle X \rangle_{A-B} \equiv \langle A | X | A \rangle - \langle B | X | B \rangle$ is used. (The parameters $\varepsilon_1$ and $\varepsilon_2$ both

\(^3\)In this respect the situation is no better for the expansion of a constrained inclusive rate of the decays $B \to X_u \ell \nu$, where the expansion is governed by $\mu_c < m_c$ [6].
vanish in the limit of factorization.) It should be also mentioned that no attempt is being made here to allow for the breaking of the flavor SU(3) symmetry, thus no distinction is made between the annihilation constants or masses of the $D_s$ and $D_0$ mesons.

The expression (10) for the difference of the total decay rates corresponds numerically to

$$\Gamma(D^0) - \Gamma(D_s) \approx 3.3 \left( \frac{f_D}{0.22 \text{GeV}} \right)^2 \left( B_{1}^{ns} - B_{2}^{ns} - \frac{3}{4} \varepsilon_1^{ns} + \frac{3}{4} \varepsilon_2^{ns} \right) \text{ps}^{-1}. \quad (12)$$

Comparing this estimate with the experimental value for the difference of the total decay rates: $0.41 \pm 0.05 \text{ps}^{-1}$, one arrives at an estimate of corresponding combination of the non-singlet factorization parameters:

$$B_{1}^{ns} - B_{2}^{ns} - \frac{3}{4} \varepsilon_1^{ns} + \frac{3}{4} \varepsilon_2^{ns} \approx 0.12, \quad (13)$$

which perfectly complies with the understanding that non-factorizable contributions are at a level of about 10%.

The estimate (13) of the non-factorizable terms however can serve only as a semi-quantitative indicator of the magnitude of the spectator effects in the inclusive rate of the processes $B \rightarrow X_u \ell \nu$ described by a different combination of the factorization parameters in eq.(9) than in eq.(13). A somewhat more direct test of the relevant combination of the parameters would be possible from the difference of the total semileptonic decay rates of $D_s$ and $D^0$ mesons. Indeed, in the limit of flavor SU(3) symmetry this difference arises only in the decays due to $c \rightarrow s \ell \nu$ and is given in terms of the operators normalized at $\mu = m_c$ as [8]

$$\Gamma(sl(D^0)) - \Gamma(sl(D_s)) = \frac{G_F^2 \cos^2 \theta_c m_c^2 f_D^2 m_D}{12\pi} \left( B_{1}^{ns} - B_{2}^{ns} \right) \approx 1.1 \left( \frac{f_D}{0.22 \text{GeV}} \right)^2 \left( B_{1}^{ns} - B_{2}^{ns} \right) \text{ps}^{-1}. \quad (14)$$

Given that the total semileptonic decay rate of the $D^0$ meson is approximately $0.16 \text{ps}^{-1}$ [1], the discussed difference can easily amount to a quite sizeable fraction of the semileptonic rate, provided that $B_{1}^{ns} - B_{2}^{ns} \sim 0.1$. A more precise calculation of the relative effect of this difference in terms of the parameters $B_{1}^{ns}$ and $B_{2}^{ns}$ is somewhat difficult to achieve at present. The problem here is that the observed semileptonic decay rate of $D^0$ already includes, in comparison with the ‘bare’ parton rate, quite substantial negative QCD radiative corrections as well as the $O(m_c^{-2})$ corrections [11], while neither of these is included in eq.(14).
If compared with the ‘bare’ parton semileptonic decay rate of $c \to s \ell \nu$, the discussed effect is of a somewhat more moderate relative magnitude:

$$\frac{\Gamma_{sl}(D^0) - \Gamma_{sl}(D_s)}{\Gamma_0(c \to s \ell \nu)} = 3.4 \left( \frac{f_D}{0.22 \text{GeV}} \right)^2 (B_1^{ns} - B_2^{ns}) .$$

(15)

In either case however the difference should be quite conspicuous and can amount to several tens percent, provided that $|B_1^{ns} - B_2^{ns}| \sim 0.1$.

A measurement of the difference of the inclusive semileptonic decay rates of the $D^0$ and $D_s$ mesons would make it possible to more reliably predict the difference of the corresponding decay rates between $B^0$ and $B^\pm$ mesons: $\Gamma(B^0 \to X_u \ell \nu) - \Gamma(B^\pm \to X_u \ell \nu)$, which, according to the previous discussion is dominantly concentrated in the upper part of the spectrum of the invariant mass of the lepton pair. At the level of accuracy of the present discussion the only difference between the theoretical expressions for $B$ and for $D$ mesons arises through a different normalization point of the four-quark operators in the equations (8) and (14). Taking into account the ‘hybrid’ evolution of the operators containing $b$ quark down to $\mu = m_c$ gives the relation between the non-singlet factorization constants:

$$B_1^{ns}(m_b) - B_2^{ns}(m_b) = \frac{8 \kappa^{1/2} + 1}{9} [B_1^{ns}(m_c) - B_2^{ns}(m_c)] - \frac{2 (\kappa^{1/2} - 1)}{3} [\varepsilon_1^{ns}(m_c) - \varepsilon_2^{ns}(m_c)] ,$$

(16)

where $\kappa = (\alpha_s(m_c)/\alpha_s(m_b))$. However, modulo the unlikely case that the difference of the constants $\varepsilon$ in this relation is much bigger than the difference between the constants $B$, the renormalization effect is quite small, and most likely is at the level of other uncertainties in the considered approach (such as the accuracy of the flavor SU(3) symmetry, higher QCD corrections, contribution of higher terms in $m_c^{-1}$, etc.). Thus with certain reservations, one can use the approximate relation $B_1^{ns}(m_b) - B_2^{ns}(m_b) \approx B_1^{ns}(m_c) - B_2^{ns}(m_c)$ to relate directly the differences in the inclusive semileptonic decay rates:

$$\Gamma(B^0 \to X_u \ell \nu) - \Gamma(B^\pm \to X_u \ell \nu) \approx \left| V_{ub} \right|^2 \left| V_{cs} \right|^2 \frac{f_B^2}{f_D^2} \frac{m_b^3}{m_c^3} \left( \Gamma_{sl}(D^0) - \Gamma_{sl}(D_s) \right) .$$

(17)

4 Discussion and summary

Naturally, the largest uncertainty of the approach pursued in the present paper is perceived to be coming from applying the operator product expansion for the amplitude in eq.(1) in Minkowski space at energy equal to mass of the charmed quark. This raises both the issue of the contribution of higher terms as well as that of the applicability of quark-hadron
duality at such energy. The situation is further complicated by the poor knowledge of the hadronic matrix elements of four-quark operators, including the deviation from the naive factorization in mesons. Same uncertainties are pertinent in the same, or greater, measure to a calculation of the constrained inclusive rate of the decays $B \to X_u \ell \nu$ at \( q^2 \geq (M_B - M_D)^2 \). At present these issues are unlikely to be resolved purely theoretically, and an experimental input is acutely needed. The accuracy of the approach can be probed by testing at a quantitative level its theoretical predictions, most notably that of a large enhancement \cite{8} of the semileptonic decay rates of the strange charmed hyperons $\Xi_c^+; \Xi_c^0$ as compared to that of $\Lambda_c$. As discussed in this paper, an experimental measurement of the difference of inclusive semileptonic decay rates between $D^0$ and $D_s$, that looks quite feasible, could as well shed light on the problem of the deviations from factorization in heavy mesons. Unless a better understanding of the non-factorizable terms is achieved, they will stand in the way of determining the mixing parameter $|V_{ub}|^2$ from inclusive semileptonic decay spectra with accuracy better than $O(15\%)$, including the method discussed in Refs.\cite{3, 6}.

In summary. It is shown that non-factorizable terms can provide a quite noticeable contribution to inclusive rates of the decays $B \to X_u \ell \nu$, which depends on the flavor of the light spectator quark, and is thus different for $B^0$ and $B^\pm$ mesons. This yet unknown contribution can limit the theoretical accuracy of determining $|V_{ub}|^2$ from the lepton spectra beyond the kinematical limit for decays with charm in the final state. The effect of such contribution can be substantially larger than the previously estimated \cite{6} uncertainty, if the non-factorizable terms are evaluated from the experimentally observed difference of the lifetimes of $D_s$ and $D_0$ mesons. A more direct conclusion about the non-factorization effects in charmless semileptonic decays of the $B$ mesons can be drawn once the difference of the total semileptonic decay rates of $D^0$ and $D_s$ is measured experimentally. If the current understanding is correct and the non-factorizable terms amount to about $10\%$, the latter difference can amount to a significant fraction of the total semileptonic decay rate of charmed mesons.

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References


