Neutrino Oscillations with Two $\Delta m^2$ Scales

Irina Mocioiu* and Robert Shrock†

C. N. Yang Institute for Theoretical Physics
State University of New York
Stony Brook, NY 11794-3840

An approximation that is often used in fits to reactor and atmospheric neutrino data and in some studies of future neutrino oscillation experiments is to assume one dominant scale, $\Delta m^2$, of neutrino mass squared differences, in particular, $\Delta m^2_{\text{atm}} \sim 3 \times 10^{-3}$ eV$^2$. Here we investigate the corrections to this approximation arising from the quantity $\Delta m^2_{\text{sol}}$ relevant for solar neutrino oscillations, assuming the large mixing angle solution. We show that for values of $\sin^2(2\theta_{13}) \sim 10^{-2}$ (in the range of interest for long-baseline neutrino oscillation experiments with either intense conventional neutrino beams such as JHF-SuperK or a possible future neutrino factory) and for $\Delta m^2_{\text{sol}} \sim 10^{-4}$ eV$^2$, the contributions to $\nu_\mu \rightarrow \nu_e$ oscillations from both CP-conserving and CP-violating terms involving $\sin^2(\Delta m^2_{\text{sol}} L/(4E))$ can be comparable to the terms involving $\sin^2(\Delta m^2_{\text{atm}} L/(4E))$ retained in the one-$\Delta m^2$ approximation. Accordingly, we emphasize the importance of performing a full three-flavor, two-$\Delta m^2$ analysis of the data on $\nu_\mu \rightarrow \nu_e$ oscillations in a conventional-beam experiment and $\nu_e \rightarrow \nu_\mu$, $\bar{\nu}_e \rightarrow \bar{\nu}_\mu$ oscillations at a neutrino factory. We also discuss a generalized analysis method for the KamLAND reactor experiment, and note how the information from this experiment can be used to facilitate the analysis of the subsequent data on $\nu_\mu \rightarrow \nu_e$ oscillations. Finally, we consider the analysis of atmospheric neutrino data and present calculations of matter effects in a three-flavor, two-$\Delta m^2$ framework relevant to this data and to neutrino factory measurements.

PACS: 13.15.+g, 14.60.Pq

*email: mocioiu@insti.physics.sunysb.edu

†email: robert.shrock@sunysb.edu
I. INTRODUCTION

There is increasingly strong evidence for neutrino oscillations, and thus neutrino masses and lepton mixing. All solar neutrino experiments that have reported results (Homestake, Kamiokande, SuperKamiokande, SAGE and GALLEX/GNO) show a significant deficit in the neutrino fluxes coming from the Sun [1,2]. This deficit can be explained by oscillations of the $\nu_e$’s into other weak eigenstate(s). The currently favored region of parameters to fit this data is the solution characterized by a neutrino mass squared difference $2 \times 10^{-5} \lesssim \Delta m^2_{s\text{ol}} \lesssim 2 \times 10^{-4} \text{ eV}^2$ and a large mixing angle (LMA), $\tan^2 \theta_{12} \sim 0.4$ [1]- [9]. Solutions yielding lower-likelihood fits to the data include the small-mixing angle (SMA) solution, with strong Mikheev-Smirnov-Wolfenstein (MSW) matter enhancement [10], and $\Delta m^2_{\text{sol}} \sim 0.6 \times 10^{-5} \text{ eV}^2$, $\tan^2 \theta_{12} \sim 10^{-3}$, the LOW solution with $\Delta m^2_{\text{sol}} \sim 10^{-7} \text{ eV}^2$ and essentially maximal mixing, and the vacuum oscillation solution, with $\Delta m^2_{\text{sol}} \sim 10^{-9} \text{ eV}^2$ and maximal mixing.

Another piece of evidence for neutrino oscillations is the atmospheric neutrino anomaly, observed by Kamiokande [11], IMB [12], Soudan [13], SuperKamiokande (also denoted SuperK, SK) with the highest statistics [14], and MACRO [15]. The SuperK experiment has fit its data by the hypothesis of $\nu_\mu \rightarrow \nu_\tau$ oscillations with $\Delta m^2_{\text{atm}} \sim 3 \times 10^{-3} \text{ eV}^2$ and maximal mixing, $\sin^2 2\theta_{\text{atm}} = 1$. The possibility of $\nu_\mu \rightarrow \nu_s$ oscillations involving light electroweak-singlet (“sterile”) neutrinos has been disfavored by SuperK, and the possibility that $\nu_\mu \rightarrow \nu_e$ oscillations might play a dominant role in the atmospheric neutrino data has been excluded both by SuperK and, for the above value of $\Delta m^2_{\text{atm}}$, by the Chooz and Palo Verde reactor antineutrino experiments [16,17]. The K2K long-baseline neutrino experiment between KEK and Kamioka has also reported results [18] which are consistent with the SuperK fit to its atmospheric neutrino data.

The LSND experiment has reported evidence for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ and $\nu_\mu \rightarrow \nu_e$ oscillations with $\Delta m^2_{\text{LSND}} \sim 0.1 - 1 \text{ eV}^2$ and a range of possible mixing angles [19]. This result is not confirmed, but also not completely ruled out, by a similar experiment, KARMEN [20]. The solar and atmospheric data can be fit in the context of three-flavor neutrino oscillations; global fits include [5-7]. We shall work within this context of three-flavor neutrino mixing [21].

The fact that these inferred values of neutrino mass squared differences satisfy the hierarchy $|\Delta m^2_{\text{sol}}| \ll |\Delta m^2_{\text{atm}}|$ has led to a commonly used approximation in fits to the reactor and atmospheric neutrino data, and in studies of CP-conserving effects in terrestrial neutrino oscillation experiments. In this approximation, which we shall denote the one-$\Delta m^2$ approximation (1DA), one neglects $\Delta m^2_{\text{sol}}$ compared with $\Delta m^2_{\text{atm}}$. For certain neutrino oscillation transitions, such as $\nu_\mu \rightarrow \nu_\tau$, this is an excellent approximation. It is worthwhile, however, to have a quantitative evaluation of the corrections to this approximation and a determination of the ranges of parameters where these corrections could become significant. Indeed, as we shall show, for sufficiently small values of the lepton mixing angle $\theta_{13}$ (defined below in eq. (2.1)), e.g., $\sin^2 2\theta_{13} \sim 10^{-2}$, and sufficiently large values of $\Delta m^2_{\text{sol}}$, e.g., $\Delta m^2_{\text{sol}} \sim 10^{-4} \text{ eV}^2$, this approximation is not reliable for certain oscillation channels such as $\nu_\mu \rightarrow \nu_e$. Here we are referring to CP-conserving quantities; the one-$\Delta m^2$ approximation is, of course, not used for calculating CP-violating quantities since neglecting $\Delta m^2_{\text{sol}}$ is equivalent to setting two of the neutrino masses equal (in the standard three-active-flavor context), which allows one to
rotate away the CP-violating phase that would appear in neutrino oscillation experiments and hence renders CP-violating quantities trivially zero. Since the values of \(\sin^2(2\theta_{13})\) and \(\Delta m_{sol}^2\) for which the one-\(\Delta m^2\) approximation breaks down are in the range of interest for future experimental searches for \(\nu_\mu \rightarrow \nu_\tau\) via both conventional neutrino beams generated by pion decay and via neutrino beams from neutrino “factories” based on muon storage rings, this complicates the analysis of the sensitivity and data analysis from these experiments.

There is another motivation for this study. There have been several fits to solar neutrino data that allow somewhat different weightings for different experiments. It has been argued that if, in particular, one assigns a somewhat larger uncertainty to the data from the Homestake chlorine experiment, then a considerably larger range of values of \(\Delta m_{sol}^2\) may be allowed [6], [7], [8], [22], enabling \(\Delta m_{sol}^2\) to extend up to values not much smaller than the Chooz limit \(0.7 \times 10^{-3}\) eV\(^2\). In turn, this leads to a breakdown of the one-\(\Delta m^2\) approximation at commensurately higher values of \(\sin^2(2\theta_{13})\). Moreover, in Ref. [23], the usual fit to the SuperK atmospheric neutrino data with a single \(\Delta m_{atm}^2 \simeq 3 \times 10^{-3}\) eV\(^2\) and \(\sin^2(2\theta_{atm}) = 1\) is compared with a very different fit with two equal mass squared differences, \(\Delta m_{23}^2 = \Delta m_{12}^2 = 0.7 \times 10^{-3}\) eV\(^2\), and it is argued that although the \(\chi^2\) for the latter is worse, it is still an acceptable fit (see also [24,25]). This strongly suggests that one should carefully assess corrections to the one-\(\Delta m^2\) approximation in studies of neutrino oscillations.

II. GENERALITIES ON NEUTRINO MIXING AND OSCILLATIONS

A. Mixing Matrix and Oscillation Probabilities

In this section we briefly record some standard formulas for neutrino oscillations that we shall use. In the framework of three active neutrinos, the unitary transformation relating the mass eigenstates \(\nu_i\), \(i = 1, 2, 3\), to the weak eigenstates \(\nu_a\) is given by \(\nu_a = \sum_{i=1}^{3} U_{ai} \nu_i\) where the lepton mixing matrix is

\[
U = R_{23} K R_{13} K^\ast R_{12} K' \begin{pmatrix}
    c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
    -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} e^{-i\delta} \\
    s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix} K' (2.1)
\]

Here \(R_{ij}\) is the rotation matrix in the \(ij\) subspace, \(c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij}\), \(K = \text{diag}(e^{-i\delta}, 1, 1)\) and \(K' = \text{diag}(1, e^{i\delta}, e^{i\delta_2})\) involves further possible phases (due to Majorana mass terms) that do not contribute to neutrino oscillations (as can be seen from the invariance of the quantity \(K_{ab,ij}\) below under neutrino field rephasings). One can take \(\theta_{ij} \in [0, \pi/2]\) with \(\delta \in [0, 2\pi]\). The rephasing-invariant measure of CP violation relevant to neutrino oscillations is given by the Jarlskog invariant [26] determined via the product \(\text{Im}(U_{ij} U_{kn} U_{in}^* U_{kj}^*)\),

\[
J = \frac{1}{8} \sin(2\theta_{12}) \sin(2\theta_{23}) \sin(2\theta_{13}) \cos \theta_{13} \sin \delta \quad (2.2)
\]

In vacuum, the probability that a weak neutrino eigenstate \(\nu_a\) becomes \(\nu_i\) after propagating a distance \(L\) (assuming that \(E >> m(\nu_i)\) and the propagation of the mass eigenstates is coherent) is
\[ P(\nu_a \to \nu_b) = \delta_{ab} - 4 \sum_{i>j=1}^{3} \text{Re}(K_{ab,ij}) \sin^2 \phi_{ij} \]
\[ + 4 \sum_{i>j=1}^{3} \text{Im}(K_{ab,ij}) \sin \phi_{ij} \cos \phi_{ij} \] (2.3)

where
\[ K_{ab,ij} = U_{ai} U_{bj}^* U_{aj}^* U_{bi} \] (2.4)
\[ \Delta m_{ij}^2 = m(\nu_i)^2 - m(\nu_j)^2 \] (2.5)
and
\[ \phi_{ij} = \frac{\Delta m_{ij}^2 L}{4E} \] (2.6)

We recall that for the CP-transformed reaction \( \bar{\nu}_a \to \bar{\nu}_b \) and the time-reversed reaction \( \nu_b \to \nu_a \), the oscillation probabilities are given by eq. (2.3) with the sign of the \( \text{Im}(K_{ab,ij}) \) term reversed. Further, by CPT, \( P(\bar{\nu}_b \to \bar{\nu}_a) = P(\nu_a \to \nu_b) \) so that, in particular, \( P(\bar{\nu}_a \to \bar{\nu}_a) = P(\nu_a \to \nu_a) \). It is straightforward to substitute the elements of the lepton mixing matrix (2.1) and evaluate the general formula (2.3) for each of the relevant transitions.

For the special case \( P(\nu_a \to \nu_a) \), eq. (2.3) simplifies to
\[ P(\nu_a \to \nu_a) = 1 - 4 \sum_{i>j=1}^{3} |U_{ai}|^2 |U_{aj}|^2 \sin^2 \phi_{ij} \] (2.7)

We recall the elementary identity
\[ \Delta m_{32}^2 + \Delta m_{21}^2 + \Delta m_{13}^2 = 0 \] (2.8)
so that in general, three-flavor vacuum oscillations depend on the four angles \( \theta_{12}, \theta_{23}, \theta_{13}, \delta \) and two \( \Delta m^2 \)'s, which can be taken to be \( \Delta m_{32}^2 \) and \( \Delta m_{21}^2 \). The currently favored regions for \( \theta_{21} \) and \( \Delta m_{21}^2 \) are determined primarily by the (two-flavor oscillation fit to the) solar neutrino data. Here one can take \( \Delta m_{21}^2 > 0 \) with \( \theta_{12} \in [0, \pi/2] \) [27]. To distinguish between the first and second octants, the parameter regions allowed by these fits to the solar data can be expressed in terms of \( \Delta m_{21}^2 > 0 \) and \( \tan^2 \theta_{21} \).

The commonly used one-\( \Delta m^2 \) approximation is then based on the hierarchy
\[ \Delta m_{\text{sol}}^2 \equiv \Delta m_{21}^2 \ll |\Delta m_{31}^2| \approx |\Delta m_{32}^2| \equiv |\Delta m_{\text{atm}}^2| \] (2.9)

However, as mentioned above, the solar data itself or in combination with atmospheric and reactor data allows for rather large values of \( \Delta m_{\text{sol}}^2 \), up to \( \sim 2 \times 10^{-4} \text{ eV}^2 \) or, in the analyses of [6,8,23] even somewhat higher.

3
B. Two-flavor Oscillations

In the case of oscillations of two flavors, the oscillation probability in vacuum is, in an obvious notation,

\[ P(\nu_a \rightarrow \nu_b) = \sin^2(2\theta) \sin^2 \phi \]  \hspace{1cm} (2.10)

C. Three-Flavor Oscillations with One-\(\Delta m^2\) Dominance

With the hierarchy (2.9), one has the following approximate formulas for vacuum oscillation probabilities relevant for experiments with reactor antineutrinos, atmospheric neutrinos and CP-conserving effects in terrestrial long-baseline oscillation studies:

\[ P(\nu_{\mu} \rightarrow \nu_{\tau}) = 4|U_{33}|^2|U_{23}|^2 \sin^2 \phi_{atm} \]
\[ = \sin^2(2\theta_{23}) \cos^4 \theta_{13} \sin^2 \phi_{atm} \]  \hspace{1cm} (2.11)

\[ P(\nu_{\mu} \rightarrow \nu_{e}) = 4|U_{13}|^2|U_{23}|^2 \sin^2 \phi_{atm} \]
\[ = \sin^2(2\theta_{13}) \sin^2 \theta_{23} \sin^2 \phi_{atm} \]  \hspace{1cm} (2.12)

\[ P(\nu_{e} \rightarrow \nu_{\tau}) = 4|U_{33}|^2|U_{13}|^2 \sin^2 \phi_{atm} \]
\[ = \sin^2(2\theta_{13}) \cos^2 \theta_{23} \sin^2 \phi_{atm} \]  \hspace{1cm} (2.13)

Since this one-\(\Delta m^2\) approximation removes CP-violating terms, it also implies that the oscillation probabilities for the CP-transformed and T-reversed transitions are equal to the probability for the original transition, \(P(\bar{\nu}_a \rightarrow \bar{\nu}_b)_{1DA} = P(\nu_b \rightarrow \nu_a)_{1DA} = P(\nu_a \rightarrow \nu_b)_{1DA}\).

For the analysis of data on reactor antineutrinos, i.e. tests of \(\bar{\nu}_e \rightarrow \bar{\nu}_x\), the two-flavor mixing expression is \(P(\nu_e \rightarrow \nu_e) = 1 - P(\nu_e \not\leftrightarrow \nu_e)\), where

\[ P(\nu_e \not\leftrightarrow \nu_e) = \sin^2(2\theta_{\text{reactor}}) \sin^2 \left(\frac{\Delta m^2_{\text{reactor}} L}{4E}\right) \]  \hspace{1cm} (2.14)

where \(\Delta m^2_{\text{reactor}}\) is the squared mass difference relevant for \(\bar{\nu}_e \rightarrow \bar{\nu}_x\). Combining (2.12) and (2.13), we have, in this approximation,

\[ \theta_{\text{reactor}} = \theta_{13}, \quad \Delta m^2_{\text{reactor}} = \Delta m^2_{32} \]  \hspace{1cm} (2.15)

For the analysis of atmospheric data with the transition favored by the current data, letting

\[ P(\nu_{\mu} \rightarrow \nu_{\tau}) = \sin^2(2\theta_{\text{atm}}) \sin^2 \left(\frac{\Delta m^2_{\text{atm}} L}{4E}\right) \]  \hspace{1cm} (2.16)
one has, using \((2.11)\),

\[
\sin^2(2\theta_{\text{atm}}) \equiv \sin^2(2\theta_{23}) \cos^4 \theta_{13}
\]  

and \(\Delta m^2_{\text{atm}}\) as given in \((2.9)\). Since the best fit value in the SuperK experiment is \(\sin^2(2\theta_{\text{atm}}) = 1\), it follows that

\[
\theta_{13} << 1
\]  

and hence

\[
\theta_{\text{atm}} \simeq \theta_{23}
\]

For the K2K experiment, using \(\theta_{13} << 1\), one has

\[
P(\nu_\mu \rightarrow \nu_\mu) \simeq 1 - \sin^2(2\theta_{23}) \sin^2 \phi_{\text{atm}}
\]  

All of these vacuum oscillation probabilities are independent of the sign of \(\Delta m^2_{\text{atm}}\), just as in the two-flavor vacuum case. However, the symmetry \(\theta \rightarrow \pi/2 - \theta\) of the two-flavor vacuum case is no longer present. For \(\theta_{13}\), one can immediately infer that this angle is near 0 rather than near to \(\pi/2\) from the fit to the atmospheric data, as noted above. For \(\theta_{23}\), the transformation \(\theta_{23} \rightarrow \pi/2 - \theta_{23}\) leaves the expression for \(P(\nu_\mu \rightarrow \nu_\tau)\) in \((2.11)\) invariant and interchanges the values of the oscillation probabilities \(P(\nu_e \rightarrow \nu_\mu)\) and \(P(\nu_e \rightarrow \nu_\tau)\). Because we know that the value of \(\theta_{23}\) is close to \(\pi/4\), this interchange does not make a large change in these probabilities \(P(\nu_e \rightarrow \nu_\mu)\) and \(P(\nu_e \rightarrow \nu_\tau)\). The atmospheric data places an upper bound on the transition \(P(\nu_\mu \rightarrow \nu_e)\), and the fact that this is small is implied by the fact that \(\theta_{13} << 1\), so that this upper bound does not determine how large the \(\sin^2 \theta_{23}\) factor is in \((2.12)\) is and hence whether \(\theta_{23}\) is slightly below or slightly above \(\pi/4\).

### III. GENERALIZED ANALYSIS OF REACTOR ANTI NEUTRINO DATA

The general three-flavor, two-\(\Delta m^2\) (i.e., two independent \(\Delta m^2\)) formula for antineutrino survival probability that is measured in reactor experiments such as G"osgen, Bugey, Chooz, Palo Verde, and KamLAND is

\[
P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - P(\bar{\nu}_e \not\rightarrow \bar{\nu}_e),
\]

where, using \((2.7)\), we have

\[
P(\bar{\nu}_e \not\rightarrow \bar{\nu}_e) = 4 \sum_{i>j=1}^3 |U_{ei}|^2|U_{ej}|^2 \sin^2 \phi_{ij}
\]

\[
= \sin^2(2\theta_{13}) \sin^2 \theta_{12} \sin^2 \phi_{32} + \sin^2(2\theta_{13}) \cos^2 \theta_{12} \sin^2 \phi_{31}
\]

\[
+ \sin^2(2\theta_{12}) \cos^4 \theta_{13} \sin^2 \phi_{21}
\]

Matter effects are negligible for these experiments. Let us consider the results from the Chooz experiment, since these place the most stringent constraints on the relevant parameters. This experiment obtained the result \([16]\)
\[ R = \frac{N_{\text{measured}}}{N_{\text{calculated}}} = 1.01 \pm 0.028(\text{stat.}) \pm 0.027(\text{sys.}) \]  

From this and the agreement between the measured and expected positron energy spectra, this experiment set the limit

\[ P(\bar{\nu}_e \not\rightarrow \bar{\nu}_e) < \epsilon_{Ch} \]  

where \( \epsilon_{Ch} \simeq 0.05 \). Within the usual context of two-flavor mixing described by (2.14), the Chooz experiment reported the 90\% confidence level (CL) limits

\[ \sin^2 2\theta_{\text{reactor}} < 0.1 \quad \text{for large} \quad \Delta m^2_{\text{reactor}} \]  

and

\[ |\Delta m^2_{\text{reactor}}| < 0.7 \times 10^{-3} \text{ eV}^2 \quad \text{for} \quad \sin^2 2\theta_{\text{reactor}} = 1 \]  

In the one-\( \Delta m^2 \) approximation, the first two terms of eq. (3.1) combine to make the term \( \sin^2 2\theta_{13} \sin^2 \phi_{\text{atm}} \) so that

\[ \sin^2 2\theta_{13} = \sin^2 2\theta_{\text{reactor}} < 0.1 \]  

Let us now work out the restrictions implied by the Chooz experiment in the context of general three-flavor neutrino oscillations with two independent \( \Delta m^2 \) quantities. Since each of the three terms \( T_i \) in the general equation (3.1) is positive-definite, we have, in an obvious notation, \( T_i < \epsilon_{Ch} \). Applying this to the third term and using the fact that \( \theta_{13} \ll 1 \), we obtain an upper bound on \( \sin^2 2\theta_{12} \sin^2 \phi_{21} \). From the plot of the Chooz excluded region [16], we infer the pair of bounds

\[ \Delta m^2_{21} < \begin{cases} 0.7 \times 10^{-3} \text{ eV}^2 & \text{for} \sin^2 2\theta_{12} = 1 \\ 0.9 \times 10^{-3} \text{ eV}^2 & \text{for} \sin^2 2\theta_{12} = 0.8 \end{cases} \]  

where the second upper bound applies for the central value of \( \sin^2 2\theta_{12} = 0.8 \) in the LMA solution,

\[ \text{LMA(central)} : \quad \tan^2 \theta_{12} = 0.4 , \quad \Delta m^2_{21} = 0.5 \times 10^{-4} \text{ eV}^2 \]  

For the central values of the LMA solution in (3.8), using \( L = 1 \text{ km} \) and a typical \( \bar{\nu}_e \) energy \( E \sim 4 \text{ MeV} \), the third term in (3.1) has a value of about \( 1.3 \times 10^{-5} \), which is negligibly small. This increases to about \( 2 \times 10^{-4} \) for \( \Delta m^2_{21} = 2 \times 10^{-4} \text{ eV}^2 \), which is again negligible compared with the range of \( \bar{\nu}_e \) disappearance, \(~ 0.05\), probed by Chooz.

Next, we consider the KamLAND long-baseline reactor experiment, which will use a liquid scintillator detector in the Kamioka mine to measure \( \bar{\nu}_e p \rightarrow e^+ n \) events initiated by \( \bar{\nu}_e \)'s from a number of power reactors and thereby test the LMA solution to the solar neutrino deficit and is expected to begin data-taking in 2001 [28,29]. The power reactors are located at various distances from 140 km to 200 km from Kamioka. It has been estimated that, in
the absence of oscillations, a total of 1075 $\bar{\nu}_e p \rightarrow e^+ n$ events per kton-yr will be recorded, and of these, 348 events per kton-yr will arise from the single most powerful reactor, the Kashiwazaki 24.6 GW (thermal) facility a distance $L = 160$ km away [28]. For the conditions of this experiment, $|\phi_{3j}| >> 1$ for $j = 1, 2$, so that the $\sin^2 \phi_{3j}$ factors average to 1/2 over the $\bar{\nu}_e$ energy spectra from the reactors, and hence (3.1) reduces to

$$P(\bar{\nu}_e \not\rightarrow \bar{\nu}_e)_{\text{Kam.}} = \frac{1}{2} \sin^2(2\theta_{13}) + \sin^2(2\theta_{12}) \cos^4 \theta_{13} \sin^2 \phi_{21}$$

(3.9)

The first term has a maximum value of 0.05. For the central LMA values in (3.8), the second term has a value of approximately 0.1 (almost independently of $\theta_{13}$, given the bound (3.6)). Thus, if $\sin^2(2\theta_{12})$ and $\Delta m^2_{21}$ are characterized by the LMA solution and if $\sin^2(2\theta_{13})$ is near to its current upper bound, then the two terms in eq. (3.9) would make contributions that differ only by about a factor of 2. One can thus distinguish several possible outcomes for the KamLAND experiment:

- If this experiment sees a signal for oscillations of reactor $\bar{\nu}_e$’s with $P(\bar{\nu}_e \not\rightarrow \bar{\nu}_e) > 0.05$, this implies that there is at least some contribution to this signal from the second term.

- In general, if $P(\bar{\nu}_e \not\rightarrow \bar{\nu}_e)$ is nonzero, then, from the overall deficiency in the rate, one would not be able to determine the relative contributions of each term in (3.9). Instead, for one pathlength, $L$, one would have to perform a three-parameter fit involving the parameters $c_j$, $j = 1, 2, 3$, where $c_1$ is a constant, representing the first term in (3.9), and $c_j$, $j = 2, 3$ enter as $c_2 \sin^2(c_3E/(4L))$, representing the second term in (3.9). Since the KamLAND detector actually is sensitive to $\bar{\nu}_e$’s from a number of different reactors at different distances, the actual fit to the data would be more complicated than this, but our point here is that the oscillations would not, in general, be adequately described by a two-flavor, one-$\Delta m^2$ formula, and eq. (3.9) would apply for the three-flavor, two-$\Delta m^2$ analysis, given the size of $\Delta m^2_{atm}$ inferred from the atmospheric data.

- Finally, if the KamLAND experiment sees no signal for oscillations and sets the limit $P(\bar{\nu}_e \rightarrow \bar{\nu}_e) < \epsilon_{KL}$, then since each of the terms in (3.9) is positive-definite, one will have the bounds $\sin^2(2\theta_{13}) < 2\epsilon_{KL}$ and the usual excluded-region plot for the contribution of the second term. This will depend on the statistical uncertainties and the backgrounds and resultant systematic uncertainties, but, roughly speaking, it will have asymptotes $\sin^2(2\theta_{12}) < 2\epsilon_{KL}$ if $\phi_{21}$ is assumed to be large enough so that $\sin^2 \phi_{21}$ averages to 1/2 over the reactor $\bar{\nu}_e$ energy spectra, and a corresponding bound of $\sin^2 \phi_{21} < \epsilon_{KL}$ for maximal mixing, $\sin^2 2\theta_{12} = 1$ (given that one knows that the $\cos^4 \theta_{13}$ factor is very close to unity).

IV. GENERALIZED ANALYSIS OF A LONG-BASELINE EXPERIMENT TO MEASURE $\nu_\mu \rightarrow \nu_e$

In this section we shall discuss the general three-flavor, two-$\Delta m^2$ analysis of long-baseline accelerator experiments to measure $\nu_\mu \rightarrow \nu_e$ using conventional beams and $\nu_e \rightarrow \nu_\mu$ or its
conjugate using beams from a possible future neutrino factory based on a muon storage ring. There are several long-baseline accelerator experiments under construction to continue the study of neutrino oscillations after the pioneering work of the K2K experiment. These include the MINOS experiment from Fermilab to the Soudan mine, with $L = 730$ km, using a far detector of steel and scintillator and a neutrino flux peaked at $E \sim 3$ GeV [30]. In Europe, a program is underway to use a neutrino beam with $E \sim 20$ GeV from CERN a distance $L = 730$ km to the Gran Sasso deep underground laboratory, involving the OPERA experiment and also plans for a liquid argon detector [31]. Third, the JHF-SuperK neutrino oscillation experiment will use a $\nu_\mu$ beam from the 0.75 MW Japan Hadron Facility (JHF) High Intensity Proton Accelerator (HIPA) in Tokai, travelling a distance $L = 295$ km to Kamioka [32]. In a first stage, this would use SuperK as the far detector; a possibility that is discussed for a second stage involves an upgrade of JHF to 4 MW and the construction of a 1 Mton water Cherenkov far detector (denoted HyperKamiokande). The JHF-SuperK collaboration has stated that one of the goals of its first phase is to search for $\nu_\mu \rightarrow \nu_e$ oscillations down to the level $P(\nu_\mu \rightarrow \nu_e) \sim 0.003$ by taking advantage of the excellent particle identification ability and energy resolution of SuperK for electrons and muons [32]. This is the sensitivity for a narrow-band beam with $E = 0.7$ GeV, which, assuming that $|\Delta m^2_{32}| = 3 \times 10^{-3}$ eV$^2$, maximizes the factor $\sin^2 \phi_{32}$; the estimated sensitivity for a wide-band beam with $E$ peaked at about 1.1 GeV is $P(\nu_\mu \rightarrow \nu_e) \sim 10^{-2}$ [32]. This type of search will be pursued to some level also by the other long-baseline experiments. Other studies of possible long-baseline $\nu_\mu \rightarrow \nu_e$ oscillations using intense conventional neutrino beams have been carried out, including ones with a Fermilab-SLAC distance, 2900 km [33,34], one that might involve a possible future 600 kton water Cherenkov detector called UNO (Ultra Nucleon Decay and Neutrino Detector) located, e.g., in the Homestake mine or the Waste Isolation Pilot Plant (WIPP) in Carlsbad, NM [35], one for the distance between JHF and Beijing, 2100 km [36], and one for the 130 km distance from CERN to Frejus [37].

A different approach that has been considered in detail is that of a neutrino “factory”, in which one would obtain a very intense beam of $\nu_\mu$ and $\bar{\nu}_e$’s from the decays of $\mu^-$’s in a muon storage ring in the form of a racetrack or bowtie, and similarly a beam of $\bar{\nu}_\mu$ and $\nu_e$’s from stored $\mu^+$’s. These beams would have a definitive and precisely understood flavor composition and would make possible neutrino oscillation searches using very long-baselines of order 3000 km [38]- [43]. Typical design parameters are $E = 20$ GeV for the stored $\mu^\pm$ energy and $L = 10^{20}$ $\mu$ decays per Snowmass year ($10^7$ sec). With a stored $\mu^-$ beam, say, one would carry out a measurement of the $\nu_\mu \rightarrow \nu_\mu$ survival probability via the charged current reaction yielding a final state $\mu^-$ and an appearance experiment with $\bar{\nu}_e \rightarrow \bar{\nu}_\mu$ yielding a final state $\mu^+$, a so-called wrong-sign muon signature. It has been estimated that with a moderate-level neutrino factory, one could search for $\nu_e \rightarrow \nu_\mu$ or its conjugate reaction down to the level $\sin^2(2\theta_{13}) \sim 3 \times 10^{-4}$ [40]. For such long pathlengths, matter effects are important [43] and can be used to get information on the sign of $\Delta m^2_{32}$. It may also be possible to measure leptonic CP violation using either a conventional beam or a beam from a neutrino factory.

In view of this intense effort, it is important to realize that at the levels of $\sin^2 2\theta_{13}$ that will be probed, the one-$\Delta m^2$ approximation used in many planning studies may well be
inadequate, and one should use a more general theoretical framework. The full expression
for the $\nu_\mu \to \nu_e$ oscillation probability in vacuum (matter effects are discussed below) is
obtained in a straightforward manner from the formulas (2.3) and (2.1) and has the form, in
a compact notation,

$$P(\nu_\mu \to \nu_e) = 2 \sin(2\theta_{13}) s_{23} c_{13} s_{12} (s_{12} s_{23} s_{13} - c_{12} c_{23} c_\delta) \sin^2 \phi_{32} +$$

$$+ 2 \sin(2\theta_{13}) s_{23} c_{13} c_{12} (c_{12} s_{23} s_{13} + s_{12} c_{23} c_\delta) \sin^2 \phi_{31} -$$

$$- 2 \sin(2\theta_{12}) c_{13}^2 \left[ s_{12} c_{12} (s_{13}^2 s_{23}^2 - c_{23}^2) + s_{13} s_{23} c_{23} (s_{12}^2 - c_{12}^2) c_\delta \right] \sin^2 \phi_{21}$$

$$+ \frac{1}{2} \sin(2\theta_{12}) \sin(2\theta_{13}) \sin(2\theta_{23}) c_{13} s_\delta \left[ \sin \phi_{32} \cos \phi_{32} -$$

$$- \sin \phi_{31} \cos \phi_{31} \right] \left[ \sin \phi_{21} \cos \phi_{21} \right]$$

(4.1)

(As in (2.3), $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_e \to \nu_\mu)$ are given by (4.1) with the sign of the $\sin \delta$ term reversed.) Now for sufficiently small $\Delta m_{21}^2$, as would be true in the solar neutrino fits
with SMA, LOW, or $\Delta m_{21}^2 \sim 10^{-9}$ eV$^2$ vacuum oscillations, and sufficiently large $\sin^2(2\theta_{13})$, subject to the constraint (3.6), the full eq. (4.1) reduces to (2.12), in which the oscillation
is driven by the terms involving $\sin^2 \phi_{\text{atm}} = \sin^2 \phi_{32} \simeq \sin^2 \phi_{31}$. However, if $\Delta m_{21}^2$ and $\sin^2 2\theta_{21}$
are at the upper end of the LMA region, then the one-$\Delta m^2$ approximation can break down.

As a numerical example, one can consider the parameter set $\sin^2 2\theta_{12} = 0.8$, $\Delta m_{21}^2 = 2 \times 10^{-4}$
eV$^2$, $\sin^2 2\theta_{13} = 0.01$, $\delta = \pi/6$, with the usual central SuperK values $\Delta m_{32}^2 = 3 \times 10^{-3}$ eV$^2$
and $\sin^2 2\theta_{23} = 1$. Further, take the JHF-SuperK pathlength $L = 295$ km and narrow-bandbeam energy $E = 0.7$ GeV, and label this total set of parameters as set (a). Then, if one
were to evaluate the $\nu_\mu \to \nu_e$ oscillation probability using the one-$\Delta m^2$ approximation, again
denoted 1DA, eq. (2.12), one would obtain

$$P(\nu_\mu \to \nu_e) = 5.0 \times 10^{-3} \text{ for set (a) with 1DA}$$

(4.2)

However, correctly including the contribution from the term involving $\sin^2 \phi_{21}$, using the full
expression (4.1), one gets an oscillation probability that is more than twice as large as the
one predicted by the one-$\Delta m^2$ approximation:

$$P(\nu_\mu \to \nu_e) = 1.4 \times 10^{-2} \text{ for set (a)}$$

(4.3)

This clearly shows that for experimentally allowed input parameters involving the LMA sol-
lar fit, and in particular, for a value of $\sin^2 2\theta_{13}$ that can be probed by the JHF-SuperK
experiment and others that could achieve comparable sensitivity, the one-$\Delta m^2$ approximation
may not be valid. Thus, it is important that the KamLAND experiment will test the
LMA and anticipates that, after about three years of running, it will be sensitive to the level
$\Delta m_{\text{sol}}^2 \lesssim 10^{-5}$ eV$^2$ [28]. This information should therefore be available by the commissioning
of JHF in 2007. The adequacy of the three-flavor theoretical framework will also be tested
by the miniBOONE experiment within this period. If, indeed, the LMA parameter set is
confirmed by KamLAND, then it may well be necessary to take into account three-flavor
oscillations involving two independent $\Delta m^2$ values in the data analysis for the JHF-SuperK
experiment and other $\nu_\mu \to \nu_e$ neutrino oscillation experiments that will achieve similar sensitivity. This point is thus certainly also true for long-baseline experiments with a neutrino factory measuring $\nu_e \to \nu_\mu$, $\bar{\nu}_e \to \bar{\nu}_\mu$ oscillations, since they anticipate sensitivity to values of $\sin^2 2\theta_{13}$ that are substantially smaller than the level to which the JHF-SuperK collaboration will be sensitive, and as one decreases $\theta_{13}$ with other parameters held fixed, the $\sin^2 \phi_{21}$ corrections to the one-$\Delta m^2$ approximation become relatively more important.

In passing, we observe that in the limit $\theta_{13} \to 0$, eq. (4.1) reduces to

$$P(\nu_\mu \to \nu_e) = \sin^2(2\theta_{12}) \cos^2 \theta_{23} \sin^2 \phi_{21} \quad \text{for} \quad \theta_{13} = 0$$

(4.4)

In this limit, the term involving $\sin^2 \phi_{21}$, rather than the terms involving $\sin^2 \phi_{32}$ or $\sin^2 \phi_{31}$, are driving the $\nu_\mu \to \nu_e$ oscillations.

V. $\nu_\mu \to \nu_\mu$ DISAPPEARANCE EXPERIMENTS

All long-baseline accelerator neutrino experiments, including K2K, MINOS, CNGS, JHF-SuperK, and other possible ones such as CERN-Frejus and those that might involve UNO and/or a neutrino factory, will perform a measurement of the $\nu_\mu \to \nu_\mu$ survival probability. The one-$\Delta m^2$ approximation yields the result

$$P(\nu_\mu \not\to \nu_\mu) = 4(1 - |U_{\mu 3}|^2) |U_{\mu 3}|^2 \sin^2 \phi_{32}$$

$$= \left[ \sin^2(2\theta_{23}) \cos^2 \theta_{13} + \sin^2(2\theta_{13}) \sin^4 \theta_{23} \right] \sin^2 \phi_{32}$$

(5.1)

Since SuperK infers a maximal $\nu_\mu \to \nu_\tau$ oscillation to fit its atmospheric neutrino data, and since this implies that $\theta_{13} << 1$, the second term in (5.1) is quite small compared to the first. As a numerical example, for $\sin^2 2\theta_{13} = 0.01$ and $\theta_{23} = \pi/4$, the ratio of the second to the first term in (5.1) is $2.5 \times 10^{-3}$. The one-$\Delta m^2$ approximation is a very good one for this transition; for experiments such as MINOS and JHF-SuperK, the relative corrections are typically of order $\lesssim O(10^{-2})$.

VI. $\nu_e \to \nu_\tau$

This transition is more difficult to measure than $\nu_\mu \to \nu_e$ since (a) the optimal neutrino energy to maximize the oscillation factor is below $\tau$ threshold, and (b) even if this were not the case, the $\tau$ is not observed directly. For completeness, however, it should be noted that again the term retained in the usual one-$\Delta m^2$ approximation, (2.13) may not be larger than the term that would describe this transition if $\theta_{13} = 0$, namely

$$P(\nu_e \to \nu_\tau) = \sin^2(2\theta_{12}) \sin^2 \theta_{23} \sin^2 \phi_{21} \quad \text{for} \quad \theta_{13} = 0$$

(6.1)

This is the same as the expression for $P(\nu_e \to \nu_\mu) = P(\nu_\mu \to \nu_e)$, eq. (4.4) under the same assumption, $\theta_{13} = 0$ with the interchange of $\cos^2 \theta_{23}$ and $\sin^2 \theta_{23}$.
VII. MATTER EFFECTS FOR NEUTRINO OSCILLATIONS WITH TWO RELEVANT $\Delta m^2$ SCALES

In many experiments matter effects can be relevant. This is the case with solar neutrinos, atmospheric neutrinos, and future possibilities for $O(10^3)$ km baseline neutrino oscillation experiments using neutrino factories [40]- [49]. In these cases, oscillation probabilities are modified by the interaction of the neutrinos in the matter: $\nu_\mu$ and $\nu_\tau$ have the same forward scattering amplitude, via $Z$ exchange, while $\nu_e$ has a different forward scattering amplitude off of electrons, involving both $Z$ and $W$ exchange. This leads to a matter-induced oscillation effect when electron neutrinos are involved in the oscillations.

In this case one needs to solve the evolution equation which includes the effects of the interactions with matter, which reads (for a generic two-generation case)

$$i \frac{d}{dx} \begin{pmatrix} \nu_e \\ \nu_\alpha \end{pmatrix} = \left( \frac{1}{2E} UM^2 U^\dagger + V \right) \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

(7.1)

where

$$M^2 = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \quad V = \begin{pmatrix} V_e & 0 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

(7.2)

Here $V = V_e = \sqrt{2} G_F N_e$ where $N_e$ is the electron number density and we have $\sqrt{2} G_F N_e [\text{eV}] = 7.6 \times 10^{-14} Y_e \rho [\text{g/cm}^3]$, where $\rho$ is the mass density and $Y_e$ is the average electron fraction of the matter.

Since only relative phases are important for oscillations, we can subtract $(1/4E)(m_1^2 + m_2^2) + (1/\sqrt{2}) G_F N_e$ from the diagonal, and the evolution equation becomes:

$$i \frac{d}{dx} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} -A(x) & B \\ B & A(x) \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix}$$

(7.3)

with

$$A(x) = \frac{\Delta m^2}{4E} \cos(2\theta) - \frac{G_F}{\sqrt{2}} N_e(x)$$

(7.4)

$$B = \frac{\Delta m^2}{4E} \sin(2\theta)$$

(7.5)

For the case of constant density this leads to an oscillation probability

$$P(\nu_a \rightarrow \nu_b) = \sin^2(2\theta_m) \sin^2(\omega L)$$

(7.6)

where

$$\omega = \sqrt{A^2 + B^2} = \frac{\Delta m^2}{4E} \left[ \sin^2(2\theta) + \left( \cos(2\theta) - \frac{2\sqrt{2} G_F N_e E}{\Delta m^2} \right)^2 \right]^{1/2}$$

(7.7)
gives the effective squared mass difference, divided by 4\(E\), in matter, and \(\theta_m\) is the relevant effective mixing angle in matter, specified by

\[
\sin^2(2\theta_m) = \frac{\sin^2(2\theta)}{\sin^2(2\theta) + \left(\cos(2\theta) - \frac{2\sqrt{2}G_F N_e E}{\Delta m^2}\right)^2} \tag{7.8}
\]

Thus, the resonance condition is

\[
E = 13 \text{ GeV} \left(\frac{\Delta m^2}{3 \times 10^{-3} \text{ eV}^2}\right) \left(\frac{3 \text{ g/cm}^2}{\rho}\right) \left(\frac{1/2}{Y_e}\right) \cos 2\theta \tag{7.9}
\]

where we have introduced scaling factors normalized by typical values of the density and the fraction \(Y_e = Z/A\) in the upper mantle.

Letting the vacuum oscillation length \(L_{\text{vac}}\) be defined as \(L_{\text{vac}} = 4\pi E/|\Delta m^2|\), the effective oscillation length \(L_m\), in matter, defined by \(\omega L_m = \pi\), is

\[
L_m = L_{\text{vac}} \left[\sin^2(2\theta) + \left(\cos(2\theta) - \frac{2\sqrt{2}G_F N_e E}{\Delta m^2}\right)^2\right]^{-1/2} \tag{7.10}
\]

We recall that, as is evident from these formulas, the oscillation probability in matter depends on the sign of \(\cos 2\theta\), i.e., whether \(\theta\) is in the first or second octant, given that one takes \(\Delta m^2 > 0\).

We next recall the formulas for matter effects on oscillation probabilities in the three-flavor case with the one-\(\Delta m^2\) dominance approximation. Here, the evolution of the weak eigenstates is given by

\[
i\frac{d}{dx} \nu = \left(\frac{1}{2E} U M^2 U^\dagger + V\right) \nu \tag{7.11}
\]

where

\[
\nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{7.12}
\]

\[
M^2 = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \tag{7.13}
\]

\[
V = \begin{pmatrix} \sqrt{2}G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.14}
\]

Subtracting \(m_1^2\) from the diagonal, \(M^2\) becomes
\[ M^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{sol} & 0 \\ 0 & 0 & \Delta m^2_{atm} + \Delta m^2_{sol} \end{pmatrix} \] (7.15)

In order to calculate the oscillation probabilities for long-baseline terrestrial neutrino oscillation experiments and for analysis of atmospheric neutrino data, it is convenient to transform to a new basis defined by (e.g. [44])

\[ \nu = R_{23} \tilde{\nu} \] (7.16)

The evolution of \( \tilde{\nu} \) is given by

\[ \tilde{\mathcal{H}} = \frac{1}{2E} K R_{13} K^* R_{12}^* M^2 R_{12}^i K^i + V \] (7.17)

In the one-\( \Delta m^2 \) approximation, this can be reduced to

\[ \tilde{\mathcal{H}} \simeq \begin{pmatrix} \frac{1}{2E} s_{13}^2 \Delta m^2_{32} + \sqrt{2} G_F N_e & 0 & \frac{1}{2E} s_{13} c_{13} \Delta m^2_{32} e^{-i\delta} \\ 0 & 0 & 0 \\ \frac{1}{2E} s_{13} c_{13} \Delta m^2_{32} e^{i\delta} & 0 & \frac{1}{2E} c_{13}^2 \Delta m^2_{32} \end{pmatrix} \] (7.18)

It can be seen now that in the basis \(( \nu_e, \tilde{\nu}_\mu, \tilde{\nu}_\tau )\) the three-flavor evolution equation decouples, and it is enough to treat the two-flavor case. We define \( S \) and \( P \) by

\[ \begin{pmatrix} \nu_e \\ \nu_\mu \\ \tilde{\nu}_\tau \end{pmatrix} (x) = S \begin{pmatrix} \nu_e \\ \nu_\mu \\ \tilde{\nu}_\tau \end{pmatrix} (0) \] (7.19)

and

\[ P \equiv |S_{13}|^2 = 1 - |S_{33}|^2 \] (7.20)

Transforming back to the flavor basis \(( \nu_e, \nu_\mu, \nu_\tau )\), the probabilities of oscillation become

\[ P(\nu_e \rightarrow \nu_\mu) = P(\nu_\mu \rightarrow \nu_e) = s_{23}^2 P \] (7.21)
\[ P(\nu_e \rightarrow \nu_\tau) = c_{23}^2 P \] (7.22)
\[ P(\nu_\mu \rightarrow \nu_\tau) = s_{23}^2 c_{23}^2 [2 - P - 2Re(S_{22} S_{33})] \] (7.23)

If in (7.18) we subtract from the diagonal the quantity \((1/4E) \Delta m^2_{32} + (1/\sqrt{2}) G_F N_e\), we see that it is then necessary to solve the evolution equation for a two-flavor neutrino system as in equation (2.10), where in \( A \) and \( B \), \( \Delta m^2 = \Delta m^2_{32} \) and \( \theta = \theta_{13} \). For the case of constant density, \( S = e^{-i\tilde{H}L} \), so that \( S_{33} = e^{-i\delta L}(\cos \omega L - i(A/\omega) \sin \omega L) \) and \( P \) is given by equations (7.6)-(7.8). Explicitly for \( \nu_\mu \rightarrow \nu_e \),

\[ P(\nu_\mu \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_\mu) = \sin^2(2\theta_{13,m}) \sin^2(\theta_{23}) \sin^2(\omega_{32,m} L) \] (7.24)

In this case, as in the two-flavor analysis, the interaction with matter makes the oscillations sensitive to the sign of \( \Delta m^2_{atm} \). For antineutrinos, the matter potential has the same