We compute the gravity waves induced by anisotropic stresses of stochastic primordial magnetic fields. The nucleosynthesis bound on gravity waves is then used to derive a limit on the magnetic field amplitude as function of the spectral index. The obtained limits are extraordinarily strong: If the primordial magnetic field is produced by a causal process, leading to a spectral index $n \geq 2$ on super horizon scales, galactic magnetic fields produced at the electroweak phase transition or earlier have to be weaker than $B_\lambda \leq 10^{-27}$ Gauss! If they are induced during an inflationary phase (heating temperature $T \sim 10^{39}$ GeV) with a spectral index $n \sim 0$, the magnetic field has to be weaker than $B_\lambda \leq 10^{-30}$ Gauss! Only very red magnetic field spectra, $n \sim -3$ are not strongly constrained. We also find that a considerable amount of the magnetic field energy is converted into gravity waves.

The gravity wave limit derived in this work rules out most of the proposed processes for primordial seeds for the large scale coherent magnetic fields observed in galaxies and clusters.

**I. INTRODUCTION**

Our galaxy, like most other spiral galaxies is permeated by a magnetic field of the order of $B \sim 10^{-6}$ Gauss with a coherence length of about $\lambda \sim 10$ kpc. Recently, similar magnetic fields have also been observed in clusters of galaxies with coherence scales of up to $\lambda \sim 0.1$ Mpc [1,2]. There is an ongoing debate whether such fields can have been produced by charge separation processes during galaxy and cluster formation [3] or whether primordial seed fields are needed which have then been amplified later by simple adiabatic contraction or by a dynamo mechanism. In the first case, seed fields of $B \sim 10^{-8}$ Gauss are needed while in the second case $B \sim 10^{-20}$ [3] or even $10^{-30}$ Gauss in a universe with low mass density [4] suffice. Several mechanisms have been proposed for the origin of such seed fields, ranging from inflationary production of magnetic fields [5,6] to cosmological phase transitions [7].

Primordial magnetic fields have been constrained in the past in various ways mainly by using their effect on anisotropies in the cosmic microwave background [8–13]. In these works constant magnetic fields and stochastic fields with red spectra $n \sim -3$ [13] have been considered and the limits obtained where of the order of a few $\times 10^{-9}$ Gauss. A simple order of magnitude estimate shows that from the CMB alone one cannot expect to constrain magnetic fields much further: The energy density in a magnetic field is

$$\Omega_B = \frac{B^2}{8\pi\rho_c} \sim 10^{-5}\Omega_\gamma(B/10^{-8} \text{ Gauss})^2,$$

where $\Omega_\gamma$ is the density parameter in photons. We naively expect a magnetic field of $10^{-8}$ Gauss to induce perturbations in the CMB on the order of $10^{-5}$, which are just on the level of the observed CMB anisotropies. It is thus expected that CMB anisotropies cannot constrain primordial magnetic fields to better than, say a tenth of this amplitude.

In this work we show that the gravity waves induced by primordial magnetic fields lead to much stronger constraints, especially for spectral indices $n > -3$. This is due to the fact that the gravity wave spectrum induced by stochastic magnetic fields is always blue (except for $n = -3$ where it is scale invariant) and thus leads to stronger constraints on small scales than on the large scales probed by CMB anisotropies.

The effects of a constant magnetic field on gravity wave evolution and production have been studied in [14]. Here we concentrate on the production of gravity waves, but consider a stochastic magnetic field.

The remainder of this paper is organized as follows: In the next section we define the initial magnetic field spectrum and its evolution in time and we determine the magnetic stress tensor which sources gravity waves. In section 3 we calculate the induced gravity wave spectrum and estimate the effect of back-reaction. In section 4 we derive limits on the primordial magnetic field using the nucleosynthesis limit on gravity waves and discuss our conclusions.

We use conformal time which we denote by $\eta$; the scale factor is $a(\eta)$. Derivatives w.r.t conformal time are denoted by an over-dot, $\ddot{a} = \frac{\dot{a}}{a}$. Greek indices run from 0 to 3, Latin ones from 1 to 3. Spatial (3d) vectors are denoted in bold. The value of the scale factor today is normalized to $a(\eta_0) = 1$. We assume a spatially flat universe throughout.

**II. PRIMORDIAL STOCHASTIC MAGNETIC FIELDS**

In this section we closely follow Ref. [13]. During the evolution of the universe, the conductivity of the inter...
cosity, with cutoff function at later times modes are mainly damped by neutrinos via two effects: on intermediate scales, the field oscillates like \( \cos(\nu_A\eta) \), where \( \nu_A = B^2/(4\pi(p + p)^{3/2}) \) is the Alfvén velocity, and on small scales, the field is exponentially damped due to shear viscosity [16,17]. We will take into account the time dependent damping scale as a time dependent cutoff \( k_d(\eta) \) in the spectrum of \( B_0 \). As we shall see, our constraints come from very small scales where the spectrum is exponentially damped and oscillations can be ignored. We therefore disregard them in what follows.

To determine the cutoff function \( k_d(\eta) \), we use the results found in [17]. We split the magnetic field into a high frequency and a low frequency component, separated by the Alfvén scale, \( \lambda_A = v_A\eta \), where the Alfvén velocity depends on the low frequency component: \( (B_A^2) = \langle B_0(x)B_0^*(x)\rangle_{\lambda_A} \) [13]. The amplitude of the high frequency component then obeys a damped harmonic oscillator equation, with damping coefficient, \( D(\eta) \), depending on time and on the mean free path of the diffusing particles giving rise to viscosity. In the oscillatory regime, we define the damping scale at each time \( \eta \) to be the scale at which one \( e \)-fold of damping has occurred: \( \int_0^{\eta_d} \frac{d\eta}{D(\eta)} = 1 \). Damping by photon viscosity acts until \( \eta \approx 10^5 \sec \), leading to \( k_d(\eta) = 1.75 \times 10^{10} \eta^{-3/2} \sec^{1/2} \); at later times modes are mainly damped by neutrinos viscosity, with cutoff function \( k_d(\eta) = 4 \times 10^{15} \eta^{-5/2} \sec^{3/2} \). For more details see [18].

We model \( B_0(x) \) as a statistically homogeneous and isotropic random field. The transversal nature of \( B \) then leads to

\[
\langle B_i(k)B_{*j}(q) \rangle = \delta^3(k - q)(\delta^{ij} - \hat{k}^i\hat{k}^j)B^2(k) \quad . (\text{2})
\]

We use the Fourier transform conventions

\[
B_i^j(k) = \int d^3x exp(i\cdot k)B_i^j(x) \quad ,
\]

\[
B_0^j(x) = \frac{1}{(2\pi)^3} \int d^3k \exp(-i\cdot k)B^j_i(k) \quad ,
\]

and \( \hat{k} = k/k, k = \sqrt{\sum_i(k_i^2)} \). If \( B \) is generated by some causal mechanism, it is uncorrelated on super horizon scales,

\[
\langle B_i(x, \eta)B_j(x', \eta) \rangle = 0 \quad \text{for} \quad |x - x'| > 2\eta \quad . (\text{3})
\]

Here it is important that the universe is in a stage of standard Friedmann expansion, so that the comoving causal horizon size is about \( \eta \). During an inflationary phase the causal horizon diverges and our subsequent argument does not apply. In this somewhat misleading sense, one calls inflationary perturbations ‘causal’.

According to Eq. (3), \( (B_i(x, \eta)B_j(x', \eta)) \) is a function with compact support and hence its Fourier transform is analytic. The function

\[
\langle B_i^j(k)B_{*n}(q) \rangle \equiv (\delta^{ij} - \hat{k}^i\hat{k}^j)B^2(k) \quad . (\text{4})
\]

is analytic in \( k \). If we assume also that \( B^2(k) \) can be approximated by a simple power law, we must conclude that \( B^2(k) \propto k^n \), where \( n \geq 2 \) is a even integer. A white noise spectrum, \( n = 0 \) does not work because of the transversality condition which has led to the non-analytic pre-factor \( \delta^{ij} - \hat{k}^i\hat{k}^j \). By causality, there can be no deviations of this law on scales larger than the horizon size at formation, \( \eta_n \).

We assume that the probability distribution function of \( B_0 \) is Gaussian; although this is not the most general random field, it greatly simplifies calculations and gives us a good idea of what to expect in a more general case.

The anisotropic stresses induced are given by the convolution of the magnetic field,

\[
\tau^{ij}(k, \eta) = \frac{1}{4\pi(2\pi)^3a^6} \int d^3q \left[ B_i^j(q)B^j_i(k - q) - \frac{1}{3}B^j_i(q)B^k_i(k - q)\delta^{ij} \right] \quad , (\text{5})
\]

where we have re-introduced the factor \( 1/a^6 \) to transform the present field \( B^i(k) = B^i(k, \eta) \) back to the physical field \( B^i(k, \eta) = B^i(k)/a^3 \). With the use of the projection operator, \( P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j \), we can extract the tensor component of Eq. (5),

\[
\Pi_{ij} = (P^a_aP^b_b - (1/2)P^{ij}P_{ab})\tau^{ab} \quad , (\text{6})
\]

tracelessness, orthogonality and symmetry force the correlation function to be of the form

\[
\langle \Pi_{ij}(k, \eta)\Pi_{*im}(k', \eta) \rangle = (\Pi(k, \eta))^2/a^{12}\mathcal{M}^{ijlm}\delta(k - k')
\]

\[
\langle \Pi_{ij}(k, \eta)\Pi_{*ij}(k', \eta) \rangle = \frac{4}{a^8}(\Pi(k, \eta))^2\delta(k - k') \quad , (\text{7})
\]

were we make use of the tensor basis, \( \mathcal{M} \): The correlator of an isotropic symmetric tensor component always has the following structure,

\[
\mathcal{M}^{ijlm}(k) = \delta^{il}\delta^{jm} + \delta^{il}\delta^{jm} - \delta^{ij}\delta^{lm} + k^{-2}(\delta^{ij}k^l\k^m + \delta^{lm}k^i\k^j - \delta^{il}k^j\k^m - \delta^{im}k^i\k^j)
\]

\[
-\delta^{jm}k^i\k^j + k^{-4}k^i\k^j\k^k\k^m \quad . (\text{8})
\]

We now determine the function \( |\mathcal{M}(k, \eta)|^2 \) in terms of the magnetic field. Using Wick’s theorem we have

\[
\langle B_i^j(k)B_{*n}(q)B^m(s)B^m(p) \rangle = \\
\langle B_i^j(k)B_{*n}(q)\rangle \langle B^m(s)B^m(p) \rangle + \\
\langle B_i^j(k)B^m(s) \rangle \langle B_{*n}(q)B^m(p) \rangle + \\
\langle B_i^j(k)B^m(p) \rangle \langle B_{*n}(s)B_{*n}(q) \rangle \quad . (\text{9})
\]
The problem reduces itself to calculating self convolutions of the magnetic field. The power spectrum of Eq. (5) is

\[ \langle \tau^{ij}(k, \eta)\tau^{lm}(k', \eta) \rangle = \frac{a^{-12}}{(4\pi)^{7/2}} \int d^3q \int d^3p \]

\[ \left( [B^i(q)B^j(k-q) - (1/3)\delta^{ij}B^m(q)B_m(k-q)] \right. \]

\[ \left. [B^{i*}(p)B^{m*}(k-p)] - (1/3)\delta^{im}B^{*m}(p)B^{*r}(k-p) \right) . \] (10)

Only the product of the first terms contributes:

\[ \frac{a^{-12}}{4(2\pi)^8} \int d^3q d^3p \left[ B^i(q)B^j(k-q)B^{i*}(p)B^{m*}(k-p) \right] = \delta(k-k') \frac{a^{-12}}{4(2\pi)^8} \int d^3q B^2(q)B^2(|k-q|) \times \]

\[ \left[ (\delta^{il} - \bar{q}^l\bar{q}^i)(\delta^{jm} - (\bar{k}-\bar{q})^j(\bar{k}-\bar{q})^m) + \right. \]

\[ \left. (\delta^{im} - \bar{q}^i\bar{q}^m)(\delta^{jl} - (\bar{k}-\bar{q})^l(\bar{k}-\bar{q})^j) \right] . \] (11)

Using Eqs. (6,7) and \( B_\gamma^*(k) = B_j(-k) \), this leads after some algebra to \( [\Pi]^2 = f(k)^2 \), where

\[ f(k)^2 = \frac{1}{16(2\pi)^8} \int d^3q B^2(q)B^2(|k-q|)(1 + 2\gamma^2 + \gamma^2\beta^2) , \] (12)

with \( \gamma = \hat{k} \cdot \bar{q} \) and \( \beta = \hat{k} \cdot \bar{k} - \bar{q} \).

To continue, we have to specify \( B^2(k) \). For simplicity we assume a simple power law with cutoff \( k_c \) which can depend on time. Clearly \( k_c(\eta) \geq k_0(\eta) \). Motivated from the inflationary magnetic field production we choose \( k_c(\eta_{in}) \sim 1/\eta_{in} \), the primordial magnetic field is coherent up to the horizon size at formation. For magnetic fields produced during the electroweak phase transition the ‘coherence scale’ is substantially smaller [19], \( k_c(\eta_{in}) \gg 1/\eta_{in} \) which would strengthen our limit as we shall see. Since it is unphysical to assume \( k_c(\eta_{in}) < 1/\eta_{in} \), our assumption is conservative. Hence we set

\[ k_c(\eta) = \min(1/\eta_{in}, k_0(\eta)) . \]

The first important fact to keep in mind is that this cutoff scale is always much smaller than the horizon scale. We now parameterize \( B^2(k) \) by

\[ B^2(k) = \begin{cases} \frac{(2\eta)^5}{2} \left( \frac{\lambda^2}{\eta^2} \right)^{n+3} B^2(k) & \text{for } k < k_c \\ 0 & \text{otherwise} \end{cases} \] (13)

The normalization is such that

\[ B_\lambda^2 = \frac{1}{V} \int d^3r (B_0(x)B_0(x+r)) \exp(-r^2/2\lambda^2) , \] (14)

where \( V = \int d^3r \exp(-r^2/2\lambda^2) = \lambda^3(2\pi)^{3/2} \) is the normalization volume. (We have assumed that the cutoff scale is smaller than \( \lambda \).) We will finally fix \( \lambda = 0.1h^{-1}\text{Mpc} \), the largest scale on which coherent magnetic fields have been observed; but the scaling of our results with \( \lambda \) will remain obvious.

The energy density in the magnetic field at some arbitrary scale \( \ell \) is \( \sim B^2 \sim B^2(k)^{3}|k|=1/\ell \propto \ell^{-(n+3)} \). In order not to over-produce long range coherent fields, we must require \( n \geq -3 \). For \( n = -3 \) we obtain a scale invariant magnetic field energy spectrum.

Using Eqs. (13) and (12) we can calculate \( f \). The integral cannot be computed analytically, but the following result is a good approximation for all wave numbers \( k \) [13]

\[ f^2(k, \eta) \approx A \times \begin{cases} k_c(\eta)^{2n+3} & \text{for } n \geq -3/2 \\ k^{2n+3} & \text{for } n \leq -3/2 \end{cases} \] (15)

with

\[ A = \frac{(2\pi)^3}{16} \left( \frac{\lambda/\sqrt{2}}{\gamma^2} \right)^{n+6} B^4_{\lambda} \]

For \( n > -3/2 \), the gravity wave source II is white noise independent of \( n \). Just the amplitude which is proportional to \( (\lambda k_c)^{2n} \) depends on the spectral index. This is due to the fact that the integral (12) is dominated by the contributions from the smallest scales \( k_c^{-1} \). The induced gravity wave spectrum will therefore be a white noise spectrum for all \( n > -3/2 \).

### III. GRAVITY WAVES FROM MAGNETIC FIELDS

We now proceed to calculate the gravity waves induced by the magnetic field stress tensor. The metric element of the perturbed Friedman universe is given by

\[ ds^2 = a^2(\eta)[dh^2 - (\delta_{ij} + 2h_{ij})dx^idx^j] , \]

where \( h_{ij}^0 = 0 \) and \( h_{ij}^0 k^0 = 0 \) for tensor perturbations. The magnetic field sources the evolution of \( h_{ij} \) through

\[ \ddot{h}_{ij} + \frac{a}{a} h_{ij} + k^2 h_{ij} = 8\pi G a^2 \Pi_{ij} . \] (16)

Equation (16) can be solved with the Wronskian method. In terms of the dimensionless variable \( x = k\eta \) the homogeneous solutions are the spherical Bessel functions \( j_0, y_0 \) in the radiation dominated era, and \( j_1/x, y_1/x \) in the matter dominated era respectively. We assume that the magnetic fields were created in the radiation dominated epoch, at redshift \( z_{in} \). Once the gravity wave enters the horizon, \( x > 1 \), additional production becomes negligible and we may match the solution to the homogeneous solutions.

To proceed, we bring Eq. (16) into a somewhat more useful form so that we can work with deterministic functions instead of the random variable \( \Pi_{ij} \). Since the time evolution of the magnetic field spectrum is deterministic, the magnetic field source is perfectly coherent. \( \Pi_{ij}(k, \eta) \)
Inside the horizon, \( \eta < \eta_c \), the expression for the scale factor becomes

\[
\langle \Pi_i^j(k, \eta) \Pi_j^i(k', \eta') \rangle = 4 \delta(k - k') f(k, \eta) f(k', \eta') \left( \frac{a^2(\eta)}{a(\eta)} a(\eta') \right)^4.
\]

The factor of 4 reflects the two helicity modes. From the above it is clear that

\[
\langle \hat{h}^{ij}(k, \eta) \hat{h}^{ij}_*(k', \eta) \rangle = 4 \delta^2(k, \eta) \delta(k - k')
\]

where \( h \) is solution of

\[
\ddot{h} + 2 \frac{\dot{a}}{a} \dot{h} + k^2 h = \frac{8\pi G}{a^2(\eta)} f(k, \eta).
\]

In real space the energy density in gravity waves is given by

\[
\rho_G = \frac{\langle \hat{h}^{ij}_* \hat{h}^{ij} \rangle}{16\pi G a^2}.
\]

The factor \( 1/a^2 \) comes from the fact that \( \dot{h} \) denotes the derivative w.r.t. conformal time. Fourier transforming this relation we obtain with Eq. (18)

\[
\frac{d\rho_G(k)}{d\log(k)} = \frac{k^3 \dot{h}^2}{a^2(2\pi)^6 G}
\]

such that

\[
\frac{d\Omega_G(k)}{d\log(k)} = \frac{d\rho_G(k)}{\rho_c d\log(k)} = \frac{k^3 \dot{h}^2}{a^2 \rho_c (2\pi)^6 G},
\]

where \( \rho_c = 3H_0^2/(8\pi G) \) denotes the critical density today and \( H_0 = 3.2h_0 \times 10^{-18}\text{sec}^{-1} \) is the Hubble parameter, \( 0.5 < h_0 < 0.8 \).

We first consider the case \( n \leq -3/2 \) for which \( f \) is independent of \( \eta \) (see Eq. (15)). The source term in Eq. (19) is relevant only on super-horizon scales, \( x < 1 \). Additional gravity wave production inside the horizon can be neglected. We only consider wavelengths which enter the horizon during the radiation dominated era, \( 1/k < \eta_c \). Inside the horizon, \( x \geq 1 \), a good approximation to the numerical solution for \( \eta < \eta_c \) is

\[
h(x) = -\frac{8\pi G f(k) \eta_c^2 \sin x}{\Omega_{rad}^2} \frac{\sin x}{x} \log(x_{in})
\]

where we have used the expression for the scale factor

\[
a(\eta) = H_0 \eta \left( \frac{H_0 \eta}{4} + \sqrt{\Omega_{rad}} \right)
\]

\[
a(\eta) \sim \Omega_{rad} \eta/\eta_{eq} \text{ for } \eta < \eta_{eq} = 2(\sqrt{2} - 1)\sqrt{\Omega_{rad}/H_0} \sim 1.7 \times 10^{15}\text{sec}.
\]

Here \( \eta_{eq} \) is conformal time at matter and radiation equality and \( \Omega_{rad} = 4.2h_0^2 \times 10^{-5} \) is the density parameter of radiation (the photons and three types of massless neutrinos). We have set \( \Omega_{rad} = 1 \) and we have neglected a possible cosmological constant which modifies the evolution of the scale factor only at very late times, \( z < 2 \) and therefore is irrelevant for the results presented here.

After horizon crossing, when further production can be neglected, the energy density in gravity waves of fixed comoving scale \( 1/k \) just scales like radiation and the density parameter in gravity waves produced by the magnetic field can be expressed as

\[
\frac{d\Omega_G(k)}{d\log(k)} \approx \frac{k^3 \dot{h}^2(\eta_{eq})}{a^2(\eta_{eq}) \rho_{rad}(\eta_{eq})(2\pi)^6 G_{\Omega} \Omega_{rad}}
\]

\[
\approx \frac{(d\Omega_{B}(k))^2}{\Omega_{rad} \rho_{eq}^2} 24 \log^2(x_{in})
\]

\for \(-3 < n < -3/2\)

\[
\Omega_B(k_c) = \int_0^{k_c} \frac{dk}{k} \frac{d\Omega_B(k)}{d\log(k)}
\]

\[
\approx \frac{\Omega_{B}(1/\eta_{in})^{12}(n + 3)}{\Omega_{rad}^{1/\eta_{in}}} \text{ for } -3 < n < -3/2,
\]

where we have used the expressions

\[
\frac{d\Omega_B(k)}{d\log(k)} = \frac{B^2}{8\pi \rho_c} \frac{(k_c)^{n+3}}{2^{(n+3)/2} \Gamma((n+3)/2)}
\]

\[
\Omega_B(k_c) = \int_0^{k_c} \frac{dk}{k} \frac{d\Omega_B(k)}{d\log(k)}
\]

\[
= \frac{B^2}{8\pi \rho_c} \frac{(k_c)^{n+3}}{2^{(n+5)/2} \Gamma((n+5)/2)}
\]

for the magnetic field density parameter. In the formula for \( \Omega_B \) we have neglected the logarithmic dependence \( \Omega_{rad} \).

If \( n > -3/2 \) the result changes due to the fact that \( f \) now depends on time via \( k_c(\eta) = \min(1/\eta_{in}, k_d(\eta)) \). Clearly, \( k_d(\eta_{in}) > 1/\eta_{in} \) by causality. We define the time \( \eta_{visc} \) to be the moment when the damping scale becomes smaller than \( \eta_{in} \), \( k_d(\eta_{visc}) = 1/\eta_{in} \). From that time on, the function \( f \) decays like a power law,

\[
f^2(k, \eta) \propto k_c^{n+3} \propto f^2(k, \eta_{visc}) \eta^{(2n+3)}
\]

where \( \alpha \) is a positive power describing the growth of the viscosity scale. Hence the source term starts decaying faster than \( 1/a^2 \), and the additional gravity wave production after \( \eta_{visc} \) is sub-dominant. We neglect it in our attempt to derive an upper limit for primordial magnetic fields. For \( n > -3/2 \), Eq. (22) is then simply modified by \( -\log(x_{in}) \rightarrow \log(x_{visc}/x_{in}) \). Taking also into account
that up to $\eta_{\text{visc}}$ the cutoff scale is $k_c(\eta) = 1/\eta_{\text{in}}$, hence $f^2(k, \eta) \propto k^2 e^{-3} = 1/\eta_{\text{in}}^3$, we obtain
\[
\frac{d\Omega_G(k)}{d\log(k)} \simeq \left(\frac{d\Omega(k)}{d\log(k)}\right)^2 (k\eta_{\text{in}})^{-3-2n} 24 \log^2(x_{\text{visc}}/x_{\text{in}}),
\]
for $n > -3/2$ \quad (28)
\[
\Omega_G \simeq \frac{\Omega_B^2(1/\eta_{\text{in}})}{\Omega_{\text{rad}}} 8(n+3)^2, \quad (29)
\]
for $n > -3/2$.

Since we have neglected the logarithms, the final formula for gravity wave production is nearly the same for all values of the spectral index (cf. Eqs. (29) and (25)).

In these formulas back-reaction, namely the decrease of magnetic field energy due to the emission of gravity waves is not included. Therefore Eqs. (24,25) and (28,29) are reasonable approximations only if $\Omega_G \lesssim \Omega_B$. In the opposite case, which is realized whenever
\[
\Omega_B(1/\eta_{\text{in}}) \gtrsim \frac{\Omega_{\text{rad}}}{\max(2n+6,3)4(n+3)} = \Omega_B(n) \equiv \Omega_B(n)
\]
the magnetic field energy is fully converted into gravity waves. In Fig. 1 the values $\Omega_G$ and $\Omega_B(1/\eta_{\text{in}})$ as functions of the spectral index are shown for two different choices of the creation time for the primordial magnetic field: the electroweak transition, $\eta_{\text{in}} \sim 4 \times 10^6$ sec and inflation $\eta_{\text{in}} \sim 8 \times 10^{-3}$ sec, for a magnetic field amplitude $B_\lambda = 10^{-9}$ Gauss.

From Fig. 1 we see that $\Omega_G$ as calculated above dominates over $\Omega_B(1/\eta_{\text{in}})$ for all spectral indices larger than $-2.9$ in the inflationary case and $-2.5$ for electroweak magnetic field production, for an amplitude of $B_\lambda = 10^{-9}$ Gauss. This is due to the fact that we have neglected back-reaction which leads to a loss of magnetic field energy. Clearly, the magnetic field cannot convert more than all its energy into gravity waves. However, in the situation shown in Fig. 1, it does actually convert most of its energy into gravity waves before it is dissipated by plasma viscosity, since gravity wave production happens before and at horizon crossing while viscosity damping is active only on scales which are well inside the horizon. When our calculation gives $\Omega_G > \Omega_B(k_c = 1/\eta_{\text{in}})$ we can take into account back-reaction by simply setting $\Omega_G \sim \Omega_B(k_c = 1/\eta_{\text{in}})$. We shall use this approximation for $\Omega_G$ in what follows.

Fig. 1 also shows that, since the value of the magnetic field density parameter at which conversion into gravity waves is quasi complete is so close to the nucleosynthesis limit, $\Omega_B(1/\eta_{\text{in}})h_0^2 \sim 1.12 \times 10^{-6} \equiv \Omega_{\text{lim}}h_0^2$, the two curves $\Omega_Bh_0^2$ and $\Omega_B(1/\eta_{\text{in}})h_0^2$ cross close to $\Omega_{\text{lim}}h_0^2$. This means that the gravity wave limit for magnetic fields is very close to the limit obtained by setting $\Omega_G = \Omega_B(1/\eta_{\text{in}})$.

\[
B^2(q)B^2(|k - q|) = 0 \quad \text{for all} \quad 0 \leq q \leq k_c.
\]

The quadratic nature of the coupling of $B$ to gravity waves actually damps the magnetic field energy at least on all wave numbers $q > k_{\text{lim}}/2$, where $k_{\text{lim}}$ is the smallest wave number for which Eq. (12) is still satisfied,
\[
k_{\text{lim}} \lambda (\log(k_{\text{lim}}/\eta_{\text{in}}))^{\frac{1}{n+3}} \simeq \left(\frac{\rho_{\text{rad}}}{24\rho(B_\lambda)}\right)^{\frac{1}{n+3}} \sqrt{2},
\]
for $n < -3/2$
\[
\left[0.7 \times 10^{-16}(10^{-20}\text{Gauss}/B_\lambda)^2\right]^{\frac{1}{n+3}} \sqrt{2},
\]
for $n > -3/2$
\[
k_{\text{lim}} \lambda \simeq \left[\frac{24\rho(B_\lambda)}{\log^2(\eta_{\text{visc}}/\eta_{\text{in}})}\right]^{\frac{1}{n+3}} \sqrt{2},
\]
for $n < -3/2$
\[
\left[0.7 \times 10^{-16}(10^{-20}\text{Gauss}/B_\lambda)^2\right]^{\frac{1}{n+3}} \sqrt{2},
\]
for $n > -3/2$. 

\[\text{FIG. 1. We show } \Omega_G h_0^2 \text{ and } \Omega_B(1/\eta_{\text{in}})h_0^2 \text{ as functions of the spectral index } n \text{ for two different times of primordial field creation: the electroweak transition ( } \Omega_Bh_0^2 \text{ dashed, blue and } \Omega_B(1/\eta_{\text{in}})h_0^2 \text{ short-dashed, red), and inflation ( } \Omega_Gh_0^2 \text{ dotted, blue and } \Omega_B(1/\eta_{\text{in}})h_0^2 \text{ long-dashed, red) for } B_\lambda = 10^{-9} \text{ Gauss. The nucleosynthesis limit, } \Omega_{\text{lim}}h_0^2 < 1.1 \times 10^{-6} \text{ is also indicated [20]}.}\]
During the matter dominated era gravity wave production is somewhat less efficient [13].

**IV. LIMITS AND CONCLUSIONS**

The first and simplest limit for primordial magnetic fields produced before nucleosynthesis comes from the fact that the energy density which they contribute may not change the expansion law during nucleosynthesis. This limit can be cast in [20]

\[ \Omega_B(k_c(\eta_{\text{nuc}})) h_0^2 \leq 1.12 \times 10^{-6} \equiv \Omega_{\text{lim}} h_0^2 . \]

From Eq. (27) we have

\[ \Omega_B(k_c(\eta_{\text{nuc}})) = \frac{B_2^2}{8 \pi c} \left( \frac{k_c(\eta_{\text{nuc}}) \lambda}{\eta_{\text{nuc}}} \right)^{n+3} \frac{\eta_{\text{nuc}}}{\rho_c} \frac{\rho_{\text{rad}}}{\rho_c} \frac{H_0^2}{\eta_{\text{nuc}}^2} \]

\[ \approx \frac{4.5 h_0^2}{2} \times 10^3 (5.9 \times 10^6)^n \left( \frac{B_2}{10^{-9} \text{Gauss}} \right)^2 \left( \frac{\lambda}{10^{13} \text{sec}} \right)^{n+3} \]

where we have inserted \( k_d(\eta_{\text{nuc}}) \approx \sqrt{30 \pi T_{\text{rad}} \rho_c/\eta_{\text{nuc}} m_p \rho_{\text{rad}} H_0^2} \approx 10^5/\eta_{\text{nuc}} \approx 6 \times 10^{-7} \text{sec}^{-1} \) (for details see Refs. [17,13,18]).

Together with the above constraint this gives already an interesting limit on primordial magnetic fields with spectral indices \( n > -2 \), as shown in Fig. 2 (solid line). For causal mechanisms of seed field production, \( n \geq 2 \), it even implies \( B_2 < 10^{-22} \text{Gauss} \).

![FIG. 2. We show the nucleosynthesis limit on \( B_2 \) (black, solid) as a function of the spectral index, \( n \) together with the limit from gravity waves if the primordial field is produced at the electroweak transition (red, short-dashed) or during inflation (blue, long-dashed) for \( \lambda = 0.1h^{-1} \text{Mpc} \times 10^{13} \text{sec} \).

Nevertheless, the limit implied from the production of gravity waves is more stringent, since the gravity waves have been produced at very early times, when the magnetic field damping scale was much smaller than \( \lambda_d(\eta_{\text{nuc}}) \sim 1.7 \times 10^{9} \text{sec} \).

Setting \( \Omega_{G} = \Omega_B(1/\eta_{in}) \) whenever the result of Eqs. (25,29) is larger than this limit which is the simplest way to take into account back-reaction, \( \Omega_{G} h_0^2 < 1.12 \times 10^{-6} \) yields the constraint for primordial magnetic fields created at \( \eta_{in} \). The result for \( \Omega_{G} \) becomes larger than \( \Omega_B(1/\eta_{in}) \) at the limiting value \( \Omega_{\text{lim}} \) imposed from nucleosynthesis for all spectral indices

\[ n > -3 + \sqrt{\frac{\Omega_{\text{rad}}}{8 \Omega_{\text{lim}}}} \sim -1 . \]

Then the magnetic field damping due to gravity wave productions is very important on sufficiently small scales. But also for smaller values of the spectral index, \( n > -3 \), we have \( \Omega_{G} \approx \Omega_B(1/\eta_{in}) \) for \( \Omega_{G} \approx \Omega_{\text{lim}} \) and there is still a considerable amount of magnetic field damping due to gravity wave production.

The results for primordial magnetic fields produced at inflation and at the electroweak scale are shown in Fig. 2 (dashed lines). Primordial magnetic fields produced before nucleosynthesis are very strongly constrained. The obtained limit can be approximated by

\[ B_2/10^{-9}\text{Gauss} < 700h_0 \times (\eta_{in}/\lambda)^{(n+3)/2} N(n) \quad (32) \]

where \( N(n) \equiv \sqrt{2 \Gamma \left( \frac{n+5}{2} \right)} \sim 1 . \)

The nucleosynthesis bound becomes stronger for smaller cutoff scales, larger \( k_c \), according to Eq. (32) it scales like \( (k_c \lambda)^{-2(n+3)/2} \).

If the seed field is produced during an inflationary phase at GUT scale temperatures, where conformal invariance can be broken e.g. by the presence of a dilaton, the induced fields must be smaller than \( B_2 \sim 10^{-20} \text{Gauss} \) for \( n > -2 \). If seed fields are produced after inflation, their spectrum is constrained by causality. Deviation from a power law with \( n \geq 2 \) can only be produced on sub-horizon scales, \( k > 1/\eta_{in} \). Therefore our limit obtained by setting \( B(k) = 0 \) on sub-horizon scales, \( \eta_{in} > 1 \), is the most conservative choice consistent with causality.

Mechanisms which still can produce significant seed fields are either ‘ordinary’ inflation, if the spectral index \( n \lesssim -2 \) or a late inflationary phase at the electroweak scale (or even later) where a seed field with \( n \lesssim 0 \) can have amplitudes of \( B_2 \sim 10^{-20} \text{Gauss} \).

It is also interesting to note that magnetic fields which contribute an energy density close to the nucleosynthesis bound, loose a considerable amount (if not all) of their energy into gravity waves, which might be detectable. In fact the space born interferometer approved by the European Space Agency and NASA, called LISA which has his most sensitive regime where it can detect \( \Omega_{\text{rad}} h_0^2 \sim 10^{-11} \) around \( 10^{5} \text{Hz} \sim 1/\eta_{\text{weak}} \) [20] will either detect or
rule out all magnetic seed field with spectral index $n \gtrsim -0.5$ produced around or before the electroweak phase transition. In fact, if LISA does not detect a gravity wave background, the constraint analogous to Eq. (32) for $\eta_h \lesssim 4 \times 10^8$ sec yields

$$B_X \lesssim 10^{-20} \text{Gauss} \quad \text{for all indices } n > -0.5$$

for all mechanisms producing seed fields before or at the electroweak phase transition.

The process of back-reaction is non-linear and from the calculations presented here it is not clear in which way the longer wavelengths relevant for the magnetic seed fields but which do not contribute substantially to the magnetic field energy are affected by it. It may well be that they are not significantly damped.

We conclude that magnetic seed fields have to be produced relatively late, or after nucleosynthesis to evade the discussed bounds. Our gravity wave bound is not relevant for magnetic fields which are produced on sub-horizon scales. But for a coherence length $\lambda \sim 0.1$ Mpc to enter the horizon, this requires a temperature of creation $T \lesssim 100$ eV. The only mechanism found so far which could lead to seed fields is recombination, where large scale coherent fields of the order of $B \sim 10^{-20}$ Gauss can be induced by magneto-hydrodynamic effects, and the difference in the viscosity of electrons and ions [21], a charge separation mechanism. Our work strongly constrains processes of quantum particle production (during e.g. an inflationary phase) as origin for the observed magnetic fields and favors more conventional processes like charge separation in the late universe.

**Acknowledgment:** We thank Pedro Ferreira, Michele Maggiore and Roy Maartens for helpful discussions. This work is supported by the Swiss NSF.

---


