Scalar Symmetries of the Hubbard Models with Variable Range Hopping

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Abstract

Examples of scalar conserved currents are presented for trigonometric, hyperbolic and elliptic versions of the Hubbard model with non-nearest neighbour variable range hopping. They support for the first time the hypothesis about the integrability of the elliptic version. The two-electron wave functions are constructed in an explicit form.
The Hubbard models with non-nearest-neighbour hopping first proposed by Gebhard and Ruckenstein [1-4] have received considerable attention since this family of models itself contains the usual Hubbard model on an infinite lattice as well as the Haldane-Shastry [5,6] spin chain as certain limiting cases, and so do the generators of its Yangian symmetry [7]. Here we would like to present some conserved currents belonging to the scalar part of the algebra which provide an evidence for the integrability of more general models with hopping given by elliptic functions. In paper [7] it has been shown that trigonometric and hyperbolic models with hopping matrices

\[ t_{jk} = t(j-k) = \sin^{-1} \frac{\pi}{N}(j-k) \]

and

\[ t_{jk} = t(j-k) = \sinh^{-1} \kappa(j-k) \]

do not imply integrability. One needs also the scalar part of the whole symmetry algebra which has been found in [8] for the Haldane-Shastry spin chain and in [9] for the conventional Hubbard model.

The basic model describes electrons of spin \( \sigma = \pm 1/2 \) created by operators \( c^+_{j\sigma} \) at site \( j \) of a one dimensional lattice. The probability amplitude for hopping between sites \( j \) and \( k \) will be denoted by \( t_{jk} \). The strength of the repulsive interaction of two electrons of different spin on the same lattice site is \( U > 0 \). In these notations, the Hamiltonian reads

\[ H = \sum_{j,k} t_{jk} c^+_{j\sigma} c_{k\sigma} + 2U \sum_j (c^+_{j\uparrow} c_{j\uparrow} - \frac{1}{2})(c^+_{j\downarrow} c_{j\downarrow} - \frac{1}{2}). \]  

We shall consider translational invariant hopping amplitudes, \( t_{jk} = t_{j-k} \).

There is a canonical transformation, namely

\[ c_{j\uparrow} \rightarrow c_{j\uparrow}, \quad c_{j\downarrow} \rightarrow c^+_{j\downarrow}, \quad U \rightarrow -U. \]  

This transformation leaves every Hamiltonian of the form (1) with antisymmetric hopping matrix invariant, but the global spin operators and the Yangian generators are not invariant under (2).

To write down the \( su_2 \) generators of the rotational symmetry of the Hamiltonian (1) and generators of other symmetries, it is convenient to use spin operators formed as linear combinations of products of one creation and one annihilation operator at one site. It is also useful to introduce spin-like operators with indices corresponding to two different sites so as one arranges the pair of operators \( c^+_{j\sigma} c_{k\tau} \) in a \( 2 \times 2 \)-matrix labeled by spin indices \( \sigma \) and \( \tau \), \((S_{jk})^\sigma_\tau = c^+_{j\sigma} c_{k\tau},\)

\[ S^\alpha_{jk} = \text{tr}(\sigma^{*\alpha} S_{jk}), \quad S^0_{jk} = \text{tr}(S_{jk}), \quad S^\alpha_j := S^\alpha_{jj}, \quad S^0_j = S^0_{jj}, \]  

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where the $\sigma^\alpha$ are the Pauli matrices. Note that $(S^\alpha_{jk})^+ = S^\alpha_{kj}$, $(S^0_{jk})^+ = S^0_{kj}$. The spin density and electron density operators are denoted by $\frac{1}{2}S^\alpha_{j}$ and $S^0_{j}$, respectively. The commutators of these operators can be written explicitly,

$$
[S^0_{jk}, S^0_{lm}] = \delta_{kl} S^0_{jm} - \delta_{mj} S^0_{lk},
$$

(4)

$$
[S^0_{jk}, S^0_{lm}] = \delta_{kl} S^0_{jm} - \delta_{mj} S^0_{lk},
$$

(5)

$$
[S^\alpha_{jk}, S^\beta_{lm}] = \delta^{\alpha\beta} (\delta_{kl} S^0_{jm} - \delta_{mj} S^0_{lk}) + i \varepsilon^{\alpha\beta\gamma} (\delta_{kl} S^\gamma_{jm} + \delta_{mj} S^\gamma_{lk}).
$$

(6)

However, there are other bilinear relations due to the composite nature of these operators. The Hamiltonian (1) now takes the following form

$$
H = \sum_{j,k} t_{jk} S^0_{jk} + U \sum_j \left( (S^0_j - 1)^2 - \frac{1}{2} \right).
$$

(7)

Since the particle number $I^0 = \sum_j S^0_j$ is conserved, only the term $(S^0_j)^2$ is relevant in the interaction part of the Hamiltonian while the other terms commute with $H$ and can be removed by a shift of the chemical potential.

To provide examples of the conserved currents which might exist for some choice of the hopping matrix, consider the ansatz

$$
J = N \sum_{j \neq k} \left[ A_{jk} S^0_{jk} + B_{jk} (S^0_{j} S^0_{k} - \vec{S}_j \vec{S}_k) + D_{jk} (S^0_j + S^0_k) S^0_{jk} + E_{jk} (S^0_j)^2 \right],
$$

(8)

in which $N$ is the number of sites. The condition $[H, J] = 0$ with the use of (4-6) can be cast into the form of two functional equations

$$
4t_{jk} (B_{lk} - B_{jl}) + (t_{jl} D_{lk} - D_{jl} t_{lk}) = 0,
$$

(9)

$$
2 (t_{jk} E_{kl} + t_{kj} E_{jl}) + (t_{jl} D_{kl} + t_{kl} D_{jl}) = 0.
$$

(10)

The coefficients $A$ can be expressed in terms of $B$, $D$ and $E$ as

$$
A_{jk} = -2D_{jk} + (2U)^{-1} [-8t_{jk} B_{jk} + 2t_{kj} E_{jk} - a_{jk}],
$$

(11)

where

$$
a_{jk} = \sum_{l \neq j,k} t_{jl} D_{lk}.
$$
Several boundary equations for $t$, $B$ and $D$ must be satisfied, too:

\[
\sum_{l \neq j,k} (t_{jl}A_{lk} - A_{jl}t_{lk}) = 0, \\
\sum_{k \neq j} (t_{jk}D_{kj} - D_{jk}t_{kj}) = 0, \\
\sum_{k \neq j} (t_{jk}A_{kj} - t_{kj}A_{jk}) = 0.
\]

The first functional equation is just the Calogero-Moser one with known general analytic solution $\psi$:

\[\psi(x) = \frac{\sigma(x + \lambda)}{\sigma(x)\sigma(\lambda)} e^{\nu x}.\]

By using it, one can express all the unknown structures in (9-11) (recall that $t_{jk} = t(j-k)$ etc.) as follows,

\[t(x) = t_0 \psi(x), \quad B(x) = -\frac{d}{4} \psi(x)\psi(-x),\]

\[D(x) = d \left[ \psi'(x) - \frac{h\psi'(\lambda)}{2} + \zeta(\lambda + \nu)\psi(x) \right], \quad E(x) = \frac{d\psi^2(x)}{2} \left[ 1 - h\psi(x+\lambda)\psi(-x-\lambda) \right],\]

\[a(x) = t_0 d\psi(x) \left[ -(N-3)\psi(x) + h_1(N-2)\zeta(x) + (\xi(x) - h_1)(2x\zeta(N/2) - N\zeta(x)) + s \right],\]

\[\xi(x) = \zeta(x + \lambda) - \zeta(x) - \zeta(\lambda), \quad h_1 = h\psi'(\lambda)/2, \quad s = -(N-2)\psi(\lambda) - \sum_{l=1}^{N-1} \psi(l),\]

where $\sigma, \zeta$ and $\varphi$ are the Weierstrass elliptic functions determined by the full periods $\omega_1 = N, \omega_2 = i\kappa$. The other parameters are given by $\lambda = i\alpha$ or $i\alpha + N/2$, $\nu = -2\zeta(N/2)\lambda N^{-1}$, in which $\kappa, d, h$ and $\alpha$ are arbitrary real numbers. Besides this general solution, there are of course the degenerate rational, trigonometric and hyperbolic forms which correspond to one or two infinite periods of the Weierstrass function. The vanishing of the boundary terms can be verified rather easily for the degenerate cases; in general elliptic case one needs to perform long but straightforward calculations which show that the boundary terms also disappear at the above choices. The key formula in these calculations is the following identity

\[\left[ \varphi(y + \lambda) - \varphi(\lambda) \right] \left[ \zeta(x - y) - \zeta(x + \lambda) + \zeta(y) + \zeta(\lambda) \right] + \]

\[\left[ \varphi(x + \lambda) - \varphi(\lambda) \right] \left[ \zeta(y - x) - \zeta(y + \lambda) + \zeta(x) + \zeta(\lambda) \right] = \varphi'(\lambda).\]

One thus can see that the formulas for $t$, $B$, $D$, $E$ and $a$ define scalar conserved currents for two three-parametric families of Hubbard models defined by the sets $(\lambda = i\alpha, \kappa, U/t_0)$.
and \((\lambda = N/2 + i\alpha, \kappa, U/t_0)\). At the points \(\lambda = \omega_1/2, \omega_2/2\) or \((\omega_1 + \omega_2)/2\), i.e. at \(\lambda\) being half-periods of the Weierstrass function \(\wp\), the function \(\psi(x)\) becomes odd and another independent current appears under the canonical transformation (2). It would be of interest to verify by direct calculations that both currents commute as it takes place for the two copies of the Yangian operators in the case of trigonometric and hyperbolic hopping [7]. One can thus conclude that these three-parametric families of Hubbard models might be integrable. An important open question is to confirm this conjecture by constructing the complete set of conserved currents commuting with the Hamiltonian. One can also see that the trigonometric and hyperbolic models of Bares, Gebhard and Ruckenstein fall into these families if one of the periods of the Weierstrass function, \(\omega_1\) or \(\omega_2\), tends to \(\infty\) under an appropriate choice of \(\lambda\).

Let us now calculate exact two-electron wave functions for the Hamiltonian (1) with the general elliptic hopping matrix. The problem is nontrivial only for the \(S = 0\) sector of the model. The wave functions can be written as

\[
\Phi = \sum_{j\neq k} \phi_{jk} c_{j1}^+ c_{k1}^+ |0\rangle + \sum_j g_j c_{j1}^+ c_{j1}^+ |0\rangle, \tag{12}
\]

where \(|0\rangle\) is the vacuum state. The eigenequation \(H\Phi = E\Phi\) can be written in the form

\[
\sum_{s\neq j,k} (t_{js}\phi_{sk} + t_{ks}\phi_{js}) + (t_{jk}g_k + t_{kj}g_j) = E\phi_{jk}, \tag{13}
\]

\[
\sum_{k\neq j} t_{jk}(\phi_{jk} + \phi_{kj}) = (E - 2U)g_j. \tag{14}
\]

Note that \(\phi_{jk}\) can always be decomposed into a sum of the symmetric and antisymmetric parts

\[
\phi_{jk} = \frac{1}{2}(\phi_{jk} + \phi_{kj}) + \frac{1}{2}(\phi_{jk} - \phi_{kj}).
\]

For antisymmetric \(\phi_{jk}\), the equation (14) would be satisfied trivially by \(g_j = 0\), and the solution to the equation (13) would be given by antisymmetrized product of one-electron plane waves. Thus from now on we concentrate on the nontrivial symmetric part, or in other words, assume that \(\phi_{jk}\) is symmetric:

\[
\phi_{jk} = \phi_{kj}.
\]

The ansatz for \(\phi_{jk}\) and \(g_j\) reads

\[
\phi_{jk} = e^{i(p^+_{j}j + p^+_{k}k)} \varphi(j - k) + e^{i(p^+_{k}k + p^+_{j}j)} \varphi(k - j), \tag{15}
\]
\[ g_j = g_0 e^{i(p_1 + p_2)j}, \]

where

\[ \varphi(j) = \varphi_0 \frac{\sigma(j + \tau)}{\sigma(j)}. \]  \hspace{1cm} (16)

The parameters \( p_1 \) and \( p_2 \) are connected with \( \tau \) by the conditions of the periodicity of \( \phi_{jk}, \phi_{j+N,k} = \phi_{j,N+k} = \phi_{jk}, \)

\[ e^{ip_1N+\eta_1\tau} = 1, \quad e^{ip_2N-\eta_1\tau} = 1, \]  \hspace{1cm} (17)

where \( \eta_1 = 2\zeta(N/2) \). The problem is to find all the parameters \( p_1, p_2, g_0 \) and \( \tau \) from the eigenequations (13-14) if the ansatz (15-16) is correct. Recall that the elliptic hopping matrix in general is given by

\[ t(j-k) = t_0 \psi(j-k) = t_0 e^{\nu(j-k)} \frac{\sigma(j-k+\lambda)}{\sigma(j-k)\sigma(\lambda)}, \]  \hspace{1cm} (18)

where the factor \( \nu = -2\zeta(N/2)\lambda N^{-1} \) is chosen so as to satisfy the periodicity condition, \( t(j-k+N) = t(j-k) \). With the use of (16) and (18) the second eigenequation (14) can be cast into the form

\[ (E - 2U)g_0 = 2t_0 \varphi_0 S_1(p_1, p_2, \tau), \]  \hspace{1cm} (19)

where

\[ S_1(p_1, p_2, \tau) = \sum_{s \neq 0} e^{\nu s} \frac{\sigma(s + \lambda)}{\sigma^2(s)\sigma(\lambda)} \left[ e^{-ip_2 s} \sigma(s + \tau) + e^{-ip_1 s} \sigma(s - \tau) \right]. \]

The first eigenequation can be written as

\[ F(\tau, p_1, p_2, j, k) + F(\tau, p_1, p_2, k, j) + F(-\tau, p_2, p_1, j, k) + F(-\tau, p_2, p_1, k, j) \]

\[ = -g_0 \left[ t(j-k) e^{i(p_1+p_2)k} + t(k-j) e^{i(p_1+p_2)j} \right] + E\phi_{jk}, \]

where

\[ F(\tau, p_1, p_2, j, k) = \sum_{s \neq j,k} t(j-s) e^{i(p_1 s + p_2 k)} \varphi(s-k), \]

\[ = e^{i(p_1 j + p_2 k)} \sum_{q \neq 0, k-j} t(-q) e^{ip_1 q} \varphi(q + j - k). \]
Let us now calculate the sum in the last expression with the use of the explicit forms of \(t(j), \varphi(j)\) (16), (18), and introduce the notation

\[
S(l) = \sum_{\lambda \neq 0, l} \frac{\sigma(q - \lambda) \sigma(q - l + \tau)}{\sigma(q) \sigma(\lambda) \sigma(q - l)} e^{(-\nu+i\rho)q},
\]

where \(l = k - j \in \mathbb{Z}\), and the function of a continuous argument \(x\),

\[
G(l, x) = \sum_{q=0}^{N-1} \frac{\sigma(q - \lambda) \sigma(q - l + \tau + x)}{\sigma(q) \sigma(\lambda) \sigma(q - l + x)} e^{(-\nu+i\rho)q}.
\]

The function \(G(l, x)\) is double quasiperiodic,

\[
G(l, x + 1) = e^{\nu - i\rho} G(l, x), \quad G(l, x + \omega_2) = e^{\eta_2(\tau - \lambda)} G(l, x),
\]

where \(\eta_1 = 2\zeta(N/2)\) and \(\eta_2 = 2\zeta(\omega_2/2)\) (recall that \(\omega_2\) is the second period of all the Weierstrass functions here). It has a simple pole at \(x = 0\) with decomposition near it of the form

\[
G(l, x)|_{x \to 0} = -\frac{\sigma(-l + \tau)}{\sigma(-l)} \left( x^{-1} + \zeta(-l + \tau) - \zeta(-l) - \zeta(\lambda) \right)
+ \frac{\sigma(l - \lambda) \sigma(\tau)}{\sigma(l) \sigma(\lambda)} \left( x^{-1} + \zeta(l - \lambda) - \zeta(l) + \zeta(\tau) \right) e^{l(-\nu+i\rho)} + S(l).
\]

On the other hand, this function can be written in the form

\[
G(l, x) = G_0(l) e^{rx} \frac{\sigma_1(x + \mu)}{\sigma_1(x)},
\]

where \(\sigma_1\) is the Weierstrass sigma function with quasiperiods \((1, \omega_2)\) and \(G_0(l)\) is a constant factor. The parameters \(r\) and \(\mu\) can be found from the quasiperiodicity conditions,

\[
\begin{align*}
\tau(p_1) &= (2\pi i)^{-1} \left( \eta_{12} \nu - i\rho_1 - (\tau - \lambda) \eta_{11} \right), \\
\mu(p_1) &= (2\pi i)^{-1} \left( -\omega_2 \nu + i\rho_1 + (\tau - \lambda) \eta_{2} \right),
\end{align*}
\]

where \(\eta_{11} = 2\zeta_1(1/2)\) and \(\eta_{12} = 2\zeta_1(\omega_2/2)\). Comparing the decompositions of both forms near \(x = 0\), one finds the sum \(S(l)\) explicitly,

\[
S(l) = -\frac{\sigma(-l + \tau)}{\sigma(-l)} \left( \zeta_1(\mu) + r - \zeta(-l + \tau) + \zeta(-l) + \zeta(\lambda) \right)
+ \frac{\sigma(l - \lambda) \sigma(\tau)}{\sigma(l) \sigma(\lambda)} \left( \zeta_1(\mu) + r - \zeta(l - \lambda) + \zeta(l) - \zeta(\tau) \right) e^{l(-\nu+i\rho)}.
\]
The explicit form of $F(\tau, p_1, p_2, j, k)$ now reads with the use of equations (22), (23) as follows,

$$
F(\tau, p_1, p_2, j, k)
= -\frac{\sigma(j - k + \tau)}{\sigma(j - k)} \varphi_0 t_0 e^{i(p_1 + p_2)k} \left( \zeta_1(\mu) + r - \zeta(j - k) + \zeta(j - k) + \zeta(\lambda) \right)
+ \varphi_0 \sigma(\tau) t(j - k) e^{i(p_1 + p_2)k} \left( \zeta_1(\mu) + r + \zeta(j - k + \lambda) - \zeta(j - k) - \zeta(\tau) \right).
$$

Now, taking explicit summation of all four $F$’s with different arguments in equation (20), one finds its compact form

$$
-\phi_{jk} t_0 \left[ \zeta_1(\mu) + \zeta_1(\tilde{\mu}) + r + \tilde{r} + 2\zeta(\lambda) \right]
+ \varphi_0 \sigma(\tau) \left[ t(j - k) e^{i(p_1 + p_2)k} + t(k - j) e^{i(p_1 + p_2)j} \right] \left[ \zeta_1(\mu) - \zeta_1(\tilde{\mu}) + r - \tilde{r} - 2\zeta(\tau) \right]
= E\phi_{jk} - g_0 \left[ t(j - k) e^{i(p_1 + p_2)k} + t(k - j) e^{i(p_1 + p_2)j} \right],
$$

where $\tilde{r} = r(-\tau, p_2), \tilde{\mu} = \mu(-\tau, p_2)$. Comparing the coefficients in both sides, one obtains two equations

$$
E = -t_0 \left[ \zeta_1(\mu) + \zeta_1(\tilde{\mu}) + r + \tilde{r} + 2\zeta(\lambda) \right], \quad (24)
$$
$$
g_0 = -\varphi_0 \sigma(\tau) \left[ \zeta_1(\mu) - \zeta_1(\tilde{\mu}) + r - \tilde{r} - 2\zeta(\tau) \right]. \quad (25)
$$

These equations define $E$ and $g_0/\varphi_0$ in terms of $p_1, p_2$ and $\tau$. Plugging them into (19) results in the equation for determining the phase shift $\tau$. Together with the relations between $\tau, p_1$ and $p_2$ (17) this equation allows one to determine all the parameters of the ansatz (14) for two-electron wave functions.

To summarize, we have found new scalar conserved currents for the Hubbard model with an elliptic hopping matrix and its trigonometric and hyperbolic degenerations. We proposed the ansatz for the two-electron $S = 0$ wave function and proved that it allows one to determine momenta and phase shift of two-electron states. It would be of interest to investigate the question of the completeness of our solution. In the case of the Heisenberg chain, it is known that the analogous ansatz gives complete description of all two-magnon states. At this time, it is not clear how to generalize this ansatz for the case of three or more electrons, and to find higher conserved currents, but our results give clear indications for the integrability of elliptic families of the 1D Hubbard models.
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References


