String Theory on AdS Orbifolds

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Abstract

We consider worldsheet string theory on $\mathbb{Z}_N$ orbifolds of $AdS_3$ associated with conical singularities. If the orbifold action includes a similar twist of $S^3$, supersymmetry is preserved, and there is a moduli space of vacua arising from blowup modes of the orbifold singularity. We exhibit the spectrum, including the properties of twisted sectors and states obtained by fractional spectral flow. A subalgebra of the spacetime superconformal symmetry remains intact after the $\mathbb{Z}_N$ quotient, and serves as the spacetime symmetry algebra of the orbifold. One intriguing aspect of the spacetime conformal field theory of the orbifold is that the effective central charge governing its asymptotic density of states differs from the Virasoro central charge.

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1 Introduction

The development of the AdS/CFT correspondence (see [1] for a review, and further references) has given us a wealth of examples of dual realizations of gravity in asymptotically anti-de Sitter space via low energy conformal field theories. A key property exhibited by all of these constructions is holography: the number of degrees of freedom describing physics in a region of space is bounded by the area of that region in Planck units. Unfortunately, the dual description repackages those degrees of freedom in such a way that it is difficult to give a concrete quasilocal description of physics in bulk spacetime, although a number of qualitative checks may be performed, see for example [2, 3, 4, 5, 6, 7]. In particular, one would like to obtain control of a description of black hole formation and evaporation where one sees the horizon, can describe the experience of the infalling observer, etc, in order to finally resolve the puzzles of black hole quantum mechanics.

The present investigation began as an attempt to formulate a situation where one could approach the formation of a black hole state beginning from a description that was string theoretic, local in spacetime, and (as much as possible) perturbative. Our starting point is the description of Giveon, Kutasov and Seiberg [8] of perturbative string theory in $AdS_3 \times S^3 \times T^4$, which is the near horizon geometry of the bound state of $p$ fundamental strings and $k$ NS fivebranes. The string frame metric on the globally $AdS$ spacetime is

$$ds^2 = k \left( -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \right)$$  \hspace{1cm} (1)

A massive object in this spacetime creates a conical deficit angle in the $AdS$ geometry, so that the geometry is asymptotically locally $AdS_3$, but with the angular direction identified under $\phi \sim \phi + 2\pi \gamma$. The relation between the deficit angle and the mass in units of the $AdS$ scale $\ell$ is

$$\ell M = -\frac{1}{2} pk \gamma^2 ,$$  \hspace{1cm} (2)

with $\gamma = 1$ corresponding to globally $AdS$ spacetime, and the limit $\gamma \to 0$ describing the extremal BTZ black hole (with $M = 0$ in our conventions) [9, 10].

In worldsheet string theory, we know how to make a conical singularity in the target space geometry – take the orbifold quotient! After laying out in section 2 our conventions for the target space sigma models on $AdS_3 \equiv SL(2, \mathbb{R})$ and $S^3 \equiv SU(2)$, in sections 3, 4 we describe the spectrum of the $\mathbb{Z}_N$ orbifold of these WZW models.

Supersymmetry requires that one embeds the $\mathbb{Z}_N$ such that it acts simultaneously on $AdS_3$ and $S^3$. The quantization of the $H$ flux through $S^3$ then requires that $N$ divides the number of fivebranes $k$.\footnote{Similar geometries have been considered in [11, 12, 13]. Our construction differs from theirs in that we identify points in spacetime under the monodromy of the singularity, whereas they do not (see section 7.2 for further discussion). The BPS geometries of these works have no restriction on the deficit angle, but do not correspond to exact string solutions; in particular, there is no treatment of the source of stress-energy at the singularity.} We describe these orbifolds in section 4.
They describe a class of conical deficit spacetimes which one might consider as
supersymmetric point objects of mass

\[
\ell M = -\frac{pk}{2N^2}
\]  

embedded in \(AdS_3\). The \(N - 1\) twisted sectors contribute moduli, the blowup modes
and \(B\)-fluxes of the orbifold singularity, that might be thought of as internal exci-
tations of the object. When \(k\) and \(N\) are large, these internal excitations might be
thought of as precursors of the states of black holes. The properties of the mod-
uli space are briefly discussed in section 6, where we also discuss other worldsheet
CFT’s related to the orbifold.

An interesting question, which we will not resolve, is what the worldsheet orbifold
considered here translates to as an operation on the spacetime CFT, which for
generic moduli is the sigma model on the moduli space of instantons [14]. The
orbifold target space is not invariant under \(SL(2,\mathbb{R})\) isometries, as it would be if it
were a CFT vacuum. Rather it resembles more a nontrivial state in a CFT; however,
because of the additional structure of the twisted sectors, it is not any conventional
state in the CFT we started with. Indeed we will see that the BPS spectrum is
different, and in addition there are new moduli beyond the moduli space of string
theory on \(AdS_3 \times S^3 \times T^4\).

Furthermore, the orbifold quotient acts on the \(\mathcal{N} = (4,4)\) superVirasoro algebra
of the spacetime CFT. Section 7 examines the effect of the orbifold projection on
the spacetime superVirasoro symmetries, and the related black hole entropy. The
supersymmetry algebra of the unorbifolded theory consists of a special set of analytic
diffeomorphisms of the asymptotic geometry [15, 16], whose Virasoro central charge
is \(c = 6pk\); the orbifold projection selects a certain \(\mathbb{Z}_N\) invariant subalgebra. The
spacetime orbifold is a representation of this invariant subalgebra rather than the
full Virasoro algebra. It turns out that this invariant subalgebra is itself an \(\mathcal{N} =
(4,4)\) superVirasoro algebra, having central charge \(\tilde{c} = Nc\). The twisted sector
spectrum should consist of representations of the subalgebra which do not lift to
the full superVirasoro algebra. While the symmetry algebra of the orbifold is a
superVirasoro algebra of central charge \(Nc\), the orbifold identification reduces the
horizon area of black holes by a factor of \(N\); we will find that the effective central
charge controlling the black hole entropy is \(c_{\text{eff}} = c/N\). The orbifold appears to fall
into the class of conformal field theories that, like Liouville theory, have a density of
states controlled by an effective central charge that differs from the Virasoro central
charge [17], because the lowest energy state is not the \(SL(2)\) invariant vacuum. Thus
the orbifold provides an important caveat to the relation between Virasoro central
charge and black hole entropy in the \(AdS_3/CFT_2\) correspondence.

We also consider briefly a second class of orbifolds which arises if we relax the
requirement of supersymmetry; then one can allow the orbifold to act purely on
\(AdS_3\), and there is now no restriction on the allowed values of \(N\). Because \(N\) can
be arbitrarily large, one can come arbitrarily close to the BTZ threshold \(M = 0\).
Nevertheless, the spacetime theory has tachyons in twisted sectors (of string scale mass), indicating that the ‘object’ is unstable. We propose this background as a useful arena in which to explore the dynamics of tachyon condensation in the closed string sector; the tachyon instability is confined to the region of the orbifold singularity rather than being spread over all of spacetime, hence one might imagine the orbifold theory describes an unstable point in configuration space that then decays into a final state of strings having the same ADM energy. This situation is thus much like the decay of unstable D-branes [18], a topic of some recent interest. A possible obstruction to this interpretation is the nontrivial K-theory charge carried by fractional D-branes sitting at the orbifold singularity; this charge must disappear (perhaps by a RR Higgs mechanism) after tachyon condensation, or else our proposal for the final state after tachyon condensation is incorrect. The mass of the fractional branes will in general be a function of the tachyon condensate, and could vanish at some point; one could indeed then imagine that the further decay of the system involves condensation of branes that screens away the RR charges. Of course, it is also possible that the system is simply unstable, and decays violently, much as in [19] (see [20] for a recent discussion in the context of AdS spacetimes).

Finally, we note that there may be interesting nonperturbative generalizations of our construction. In the moduli space of the bound state of \( p \) onebranes and \( k \) fivebranes are other perturbative limits [21, 22] for any other charges \( p' \) and \( k' \) such that \( p'k' = pk \). Each of these perturbative limits has a GKS-type description (except \( p \) or \( k \) equal to one), and thus admits an orbifold by \( \mathbb{Z}_N \) where \( N \) divides \( k' \). It would thus appear that one can orbifold by the cyclic group whose order is any proper divisor of \( pk \). In other perturbative limits than the ones where this is naturally defined, the orbifold will appear as an operation that divides by a group larger than would naively be allowed by perturbative string theory, and yet the resulting background must still be consistent. If so, then the set of allowed orbifold constructions in string theory would appear to be larger than heretofore known.

### 2 The Bosonic WZW model for \( SL(2, \mathbb{R}) \)

String theory on \( AdS_3 \) may be formulated in terms of the WZW model [23] on the universal cover of the \( SL(2, \mathbb{R}) \) group manifold. The bosonic WZW action is given by

\[
S[g] = \frac{k}{8\pi} \int_{\Sigma^2} d^2\sigma \, \delta^{\alpha\beta} \text{tr} \left( g^{-1} \partial_\alpha g \, g^{-1} \partial_\beta g \right) + \frac{ik}{12\pi} \int_{M^3} \text{tr}(\omega \wedge \omega \wedge \omega) .
\]

Here \( M^3 \) is a three manifold which has the Euclidean worldsheet \( \Sigma^2 \) as its boundary; \( \omega = g^{-1} dg \) is the Maurer-Cartan form on \( SL(2, \mathbb{R}) \). This action can be expressed in the conventions of [24] as the nonlinear sigma model

\[
S[X] = \frac{k}{2\pi} \int_{\Sigma^2} d^2z \left( G_{\mu\nu} + B_{\mu\nu} \right) \partial X^\mu \partial X^\nu ;
\]
where the length scale $\ell$ of $AdS_3$ that would appear in $G$ and $B$ has been absorbed into $k$ so that $k = \ell^2/\ell_s^2$. We parameterize $AdS_3$ in global ‘cylindrical’ coordinates $X^\mu = (t, \rho, \phi)$. The generators of $SL(2, \mathbb{R})$

$$\tau^1 = \frac{i}{2} \sigma^3 \quad \tau^2 = \frac{i}{2} \sigma^1 \quad \tau^3 = \frac{1}{2} \sigma^2$$

satisfy the algebra

$$[\tau^a, \tau^b] = i \epsilon^{ab}_c \tau^c,$$  \hspace{1cm} (7)

where the metric is $\eta^{ab} = \text{diag}(+1, +1, -1) = -2 \text{tr}(\tau^a \tau^b)$ and $\epsilon^{123} = 1$. In terms of these conventions, an element of $SL(2, \mathbb{R})$ is given by

$$g[X] = e^{2i\theta^a \tau^a} e^{-2i\rho \tau^1} e^{2i\theta^a \tau^a},$$

where $\theta^\pm = (t \pm \phi)/2$. This leads to the following forms for $G$ and $B$

$$G = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2$$
$$B = \sinh^2 \rho \, dt \wedge d\phi.$$  \hspace{1cm} (9)

Taking $\varepsilon$ to be the right handed (with respect to $(t, \rho, \phi)$) volume form on $AdS_3$ this is equivalent to

$$H = \ell^2 dB = -\frac{2}{\ell} \varepsilon.$$  \hspace{1cm} (10)

The WZW action is invariant under the transformation

$$g(z, \bar{z}) \rightarrow \Omega(z)g(z, \bar{z})\bar{\Omega}(\bar{z})^{-1},$$

leading to the currents:

$$J^a(z) = \mathcal{J}_a \tau^a = -\frac{k}{2} \partial g \, g^{-1}, \quad \bar{J}^a(\bar{z}) = \bar{\mathcal{J}}_a \bar{\tau}^a = -\frac{k}{2} \bar{g}^{-1} \bar{\partial} \bar{g}.$$  \hspace{1cm} (12)

A modified $(J^a, \bar{J}^a)$ definition of the currents will be used below. Introducing Cartesian coordinates $(x^1, x^2, x^3) = \sqrt{k} (\rho \cos \phi, \rho \sin \phi, t)$, in the (large $k$) flat space limit the currents become

$$\mathcal{J}^a = i \sqrt{k} \partial x^a \quad \bar{\mathcal{J}}^a = i \sqrt{k} \bar{\partial} x^a.$$  \hspace{1cm} (13)

Consider the behavior of a field $A(z, \bar{z})$ under the infinitesimal transformation

$$-i \delta g = \epsilon(z) g - g \bar{\epsilon}(\bar{z});$$  \hspace{1cm} (14)

the Ward identity then takes the form

$$i \delta A(w, \bar{w}) = \oint_w \frac{dz}{2\pi i} \epsilon_a \mathcal{J}^a(z) A(w, \bar{w}) + \oint_{\bar{w}} \frac{dz}{2\pi i} \bar{\epsilon}_a \bar{\mathcal{J}}^a(\bar{z}) A(w, \bar{w})$$  \hspace{1cm} (15)
under an infinitesimal time translation $\epsilon = \delta t \tau^3$ and $\bar{\epsilon} = -\delta t \tau^3$. One finds the energy operator
\[ E = \mathcal{J}_0^3 + \bar{\mathcal{J}}_0^3 = \oint \frac{dz}{2\pi i} J^3 - \oint \frac{d\bar{z}}{2\pi i} \bar{J}^3 ; \tag{16} \]
an infinitesimal rotation $\epsilon = -\delta \phi \tau^3$ and $\bar{\epsilon} = -\delta \phi \tau^3$ yields the angular momentum
\[ L_{\text{AdS}} = \mathcal{J}_0^3 - \bar{\mathcal{J}}_0^3 . \tag{17} \]
It turns out that the Ward identity implies that the currents $(J^a, \bar{J}^a)$ satisfy OPEs with structure constants of opposite sign. This leads to the convention that $\bar{\mathcal{J}}^3$ lowers the $\mathcal{J}^3$ eigenvalue and does not conform to the conventions for monodromies and parafermion representations in the next two sections. This can be remedied while keeping the same form for $E$ and $L$ in terms of the zero modes by defining the currents as follows
\[ J^3 = \mathcal{J}^3 \quad J^\pm = J^1 \pm i J^2 = \mathcal{J}^\pm \]
\[ \bar{J}^3 = \bar{\mathcal{J}}^3 \quad \bar{J}^\pm = \bar{J}^1 \pm i \bar{J}^2 = \bar{\mathcal{J}}^\pm . \tag{18} \]
The Ward identity then implies the OPEs
\[ J^a(z)J^b(w) \sim \frac{(k/2)\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}J^c(w)}{(z-w)} \]
\[ \bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{(k/2)\eta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{i\epsilon^{abc}\bar{J}^c(\bar{w})}{(\bar{z}-\bar{w})} . \tag{19} \]

The unflowed $SL(2, \mathbb{R})$ representations

We briefly review the $SL(2, \mathbb{R})$ representation content as discussed for example in [25]. As discussed in [26], this content does not enumerate all of the primary states of strings on $AdS_3$. The additional representations, which can be described by spectral flow of those discussed in [25], and their relation to the spectrum of the orbifold will be discussed in the next section. The “unflowed” current algebra primaries of the $SL(2, \mathbb{R})$ WZW model are arranged into three sectors:
\[ D_j^+ \times D_j^- \quad D_j^- \times \bar{D}_j^+ \quad C_j^\alpha \times \bar{C}_j^\alpha . \tag{20} \]
Here $D_j^\pm$ are discrete representations of the universal cover of $SL(2, \mathbb{R})$ which are described as follows:
\[ D_j^\pm = \{ |j, m\rangle \mid m = \pm (j + n) \quad (n \geq 0) \in \mathbb{Z} \} \tag{21} \]
The states with $n = 0$ satisfy $J_0^\pm |j, \pm j\rangle = 0$. Unitarity requires $j \in \mathbb{R}$ with $0 \leq j < k/2$. These bounds are further constrained as described in [8, 26] to $1/2 \leq j < (k-1)/2$. The continuous representations $C_j^\alpha$ are described as follows:
\[ C_j^\alpha = \{ |j, m\rangle \mid m = \alpha + n \quad (0 \leq \alpha < 1) \in \mathbb{R} \quad n \in \mathbb{Z} \} . \tag{22} \]
Unitarity requires $j = 1/2 + is$ where $s \in \mathbb{R}$. Note that the eigenvalue of the quadratic Casimir is bounded from above for $D_j^\pm$ by $-j(j-1) \leq 1/4$. This corresponds to the $\mu^2 > -1/4$ Breitenlohner-Freedman bound for the mass $\mu$ appearing in the Klein-Gordon equation on AdS$_3$. Conversely for $C_j^\alpha$ the quadratic Casimir is bounded from below by $-j(j-1) \geq 1/4$; which implies that the continuous representations describe tachyonic excitations. This will not be true of the spectral flow of these representations to be described below.

3 The Rotational Orbifold

3.1 Twist Ground States

We now consider the orbifold $AdS_3/\mathbb{Z}_N$ formed by a $2\pi/N$ $(N \in \mathbb{Z})$ rotation. As can be seen from the parameterization of the group elements given above, a shift $\phi \rightarrow \phi + \alpha$ may be induced as follows:

$$g(\phi + \alpha) = e^{i\alpha \tau^3} g(\phi) e^{-i\alpha \tau^3}.$$  \hspace{1cm} (23)

Thus the group field $g(z, \bar{z})$ of the WZW model has the monodromy

$$g(e^{2\pi i z}, e^{-2\pi i \bar{z}}) = e^{2\pi i q/N \tau^3} g(z, \bar{z}) e^{-2\pi i q/N \tau^3}$$  \hspace{1cm} (24)

in the presence of the vertex operator $\sigma_q$ which creates the ground state of the $q \in \mathbb{Z}_N$ twisted sector. The holomorphic currents have monodromies

$$J^3(e^{2\pi i z}) = J^3(z)$$

$$J^\pm(e^{2\pi i z}) = e^{\pm 2\pi i q/N} J^\pm(z).$$  \hspace{1cm} (25)

leading to the mode expansions for the $q$th twisted sector

$$J^3(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^3_n$$

$$J^\pm(z) = \sum_{n \in \mathbb{Z}} z^{-n-1 \pm q/N} J^\pm_{n \mp q/N}.$$  \hspace{1cm} (26)

Similarly, the anti-holomorphic currents have the monodromies

$$\bar{J}^3(e^{-2\pi i \bar{z}}) = \bar{J}^3(\bar{z})$$

$$\bar{J}^\pm(e^{-2\pi i \bar{z}}) = e^{\mp 2\pi i q/N} \bar{J}^\pm(\bar{z})$$  \hspace{1cm} (27)

and mode expansions

$$\bar{J}^3(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-1} \bar{J}^3_n$$

$$\bar{J}^\pm(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-1 \pm q/N} \bar{J}^\pm_{n \mp q/N}.$$  \hspace{1cm} (28)
Imposing the condition that $\sigma_q$ is a current algebra primary with $E = L = 0$

\[ J^a_\alpha |\sigma_q\rangle = 0 \quad \forall \alpha \geq 0 \quad \text{and} \quad \bar{J}^a_\beta |\sigma_q\rangle = 0 \quad \forall \beta \geq 0 , \quad (29) \]

the mode algebras implied by the OPEs of the currents may be used to compute the dimension of the twist vertex operators

\[ h(\sigma_q) = \frac{1}{2\pi i} \oint dz \langle \sigma_q | T(z) | \sigma_q \rangle . \quad (30) \]

With $T(z)$ given by the Sugawara construction $T(z) = \frac{1}{k-2} \eta_{ab} J^a J^b(z)$, one finds

\[ h(\sigma_q) = \tilde{h}(\sigma_q) = \frac{k}{k-2} \left( \frac{1}{2} \frac{1}{q/N} \right) (1 - \frac{q}{N}) . \quad (31) \]

Note that in the flat space limit $k \to \infty$, the currents become the translation currents (13) with the $\mathbb{Z}_N$ monodromies (25). The properties of the twist operators agree with those computed in [27].

### 3.2 $SL(2, \mathbb{R})$ Parafermions

A more explicit construction for the twist vertex operators $\sigma_q$ can be given in terms of parafermions. We first introduce a parafermion representation for the currents

\[ J^3 = -\sqrt{k/2} \partial X , \quad J^\pm = \psi^\pm e^{\pm \sqrt{2/k} X} . \quad (32) \]

And similarly for the anti-holomorphic currents. Here $X$ is a holomorphic field with OPE

\[ X(z) X(w) \sim -\ln(z-w) . \quad (33) \]

The parafermions $\psi^\pm$ have OPEs

\[ \psi^+(z) \psi^-(w) \sim k (z-w)^{-2-2/k} , \quad \psi^\pm(z) \psi^\pm(w) \sim 0 , \quad (34) \]

which reflect the OPEs for the $SL(2, \mathbb{R})$ currents (19). The current algebra primaries for the $SL(2, \mathbb{R})$ WZW model are given in terms of parafermions by

\[ \Phi^{SL(2)}_{j m \bar{m}} = \Psi^{SL(2)}_{j m \bar{m}} e^{\sqrt{2/k} (mX + \bar{m}\bar{X})} . \quad (35) \]

Here $(m, \bar{m})$ are the eigenvalues of the zero modes $(J^3_0, \bar{J}^3_0)$ and $j$ is the spin eigenvalue of $SL(2, \mathbb{R})$. Note that, since we are considering the non-compact covering group of $SL(2, \mathbb{R})$, $j$ is not quantized. Also note that, due to modular invariance and the fact that the covering space is simply connected, the holomorphic and anti-holomorphic Casimirs are equal

\[ \eta_{ab} J^a_0 J^b_0 = \eta_{ab} \bar{J}^a_0 \bar{J}^b_0 = -j (j - 1) . \quad (36) \]
This implies, given details of the representation content, that \( m - \bar{m} \in \mathbb{Z} \). Of course this is just a consequence of requiring that the wave function be single valued under a rotation by \( 2\pi \). The dimensions of the primary fields are

\[
h(\Phi_{j\bar{m}}^{SL(2)}) = \bar{h}(\Phi_{j\bar{m}}^{SL(2)}) = \frac{-j(j-1)}{k-2},
\]

(37)

hence the dimensions for the parafermions are

\[
h(\psi_{j\bar{m}}^{SL(2)}) = \frac{-j(j-1)}{k-2} + \frac{m^2}{k},
\]

(38)

and similarly for \( \bar{h} \) with \( m \rightarrow \bar{m} \).

The parafermion twist operators

We now describe the \( \mathbb{Z}_N \) twist operators in terms of the parafermions. It may be verified that, in terms of conformal dimensions and monodromies with the currents, all of the properties of the twist vertex operator are satisfied by

\[
\sigma_q = \psi_{j_q,j_q,j_q}^{SL(2)} \quad \text{where} \quad j_q = \frac{kq}{2N}.
\]

(39)

Thus \( \sigma_q \) is the parafermion for the lowest weight state in \( \mathcal{D}_{j_q}^+ \). In particular using (38), as found in (31) above,

\[
h(\psi_{j_q,j_q,j_q}^{SL(2)}) = \frac{kj_q - 2j_q^2}{k(k-2)} = \frac{k}{k-2} \frac{1}{2} (q/N)(1 - q/N). \]

(40)

The monodromies with respect to the currents may be verified by looking at the OPEs with the primaries

\[
J^\pm(z) \Phi_{j,m,\bar{m}}^{SL(2)}(w) \sim (m \pm j)(z - w)^{-1} \Phi_{j,m \mp 1,\bar{m}}^{SL(2)}(w)
\]

(41)

which implies, through the free OPE (33),

\[
J^+(z) \psi_{j_q,j_q,j_q}^{SL(2)}(w) \sim (z - w)^{-(1-q/N)} \hat{O}^+(w)
\]

\[
J^-(z) \psi_{j_q,j_q,j_q}^{SL(2)}(w) \sim (z - w)^{-q/N} \hat{O}^-(w)
\]

(42)

for some vertex operators \( \hat{O}^\pm \). This may be compared with the result that follows from the mode expansions (26):

\[
J^+(z) |\sigma_q\rangle \sim z^{-(1-q/N)} J^+_q |\sigma_q\rangle
\]

\[
J^-(z) |\sigma_q\rangle \sim z^{-q/N} J^-_{-q/N} |\sigma_q\rangle.
\]

(43)

Finally, the \( J^3 \) current is regular with respect to both \( \sigma_q \) and the parafermions.

8
3.3 Fractional Spectral Flow

The integer spectral flow operator

As described above, for the discrete representations the $SL(2, \mathbb{R})$ spin quantum number $j$ is constrained to the values $1/2 \leq j < (k-1)/2$. This implies, through the mass shell condition, a limit on the dimensions of vertex operators associated with the spaces with which $AdS_3$ may be combined to create a critical string theory. Furthermore there is the expectation that strings of arbitrary level number should be permitted in $AdS_3$. For instance, a classical description of string propagation in $AdS_3$ involves strings that propagate to spatial infinity. These should be associated with a continuous spectrum of physical states. These problems and the problem that the descendants of the above primaries do not lead to a modular invariant partition function were resolved relatively recently in [26], by considering the role of spectral flow in $SL(2, \mathbb{R})$. Spectral flow by $w$ units introduces a shift in the worldsheet quantum numbers

$$L_0 \rightarrow L_0 - J_0^3 w - \frac{k}{4} w^2,$$

$$J_0^3 \rightarrow J_0^3 + \frac{k}{2} w,$$  \hspace{1cm} (44)

and similarly for $\bar{L}_0$, $\bar{J}^3_0$. In the parafermion representation, the operator that implements this spectral flow is given by

$$t_w = e^w \sqrt{\frac{k}{2}} (X + \bar{X})$$  \hspace{1cm} where  \hspace{1cm} $w \in \mathbb{Z}$.  \hspace{1cm} (45)

The result of the introduction of these operators is to extend the primary states to include all vertex operators of the form

$$\Phi_{w,jm\bar{m}}^{SL(2)} = \Psi_{jm\bar{m}}^{SL(2)} e^{\sqrt{2/k}(m+\frac{1}{2}w)X+(\bar{m}+\frac{1}{2}w)\bar{X}}.$$  \hspace{1cm} (46)

This leads to the emergence of discrete states of arbitrary level number as well as continuous states that are not tachyonic.

The fractional spectral flow operator

The spectrum of the orbifold consists of vertex operators which comprise a closed mutually local OPE algebra which includes the twist vertex operators. For the primary states this translates to

$$\exp \left( i \left( \frac{2\pi}{N} \right) L \right) |\Phi_{w,jm\bar{m}}^{SL(2)}\rangle = |\Phi_{w,jm\bar{m}}^{SL(2)}\rangle$$  \hspace{1cm} (47)

\footnote{In [28], the spectral flow operators are referred to as twist operators; we prefer to refer to them as spectral flow operators, reserving the term twist operators to refer to the twisted sector vertex operators of the $\mathbb{Z}_N$ orbifold.}
which implies the condition \( m - \bar{m} \in N\mathbb{Z} \). The twisted sectors consist of states resulting from the OPEs of the surviving untwisted sector states with the twist vertex operators. Consider the most singular term in the OPE of an unflowed lowest weight primary of spin \( j = j_q \) with the twist operator of the \((N - q)\) sector

\[
\Phi_{0j_qj_q}^{SL(2)}(z, \bar{z}) \sigma_{N-q}(w, \bar{w}) = e^{q/N} \sqrt{k/2}(X(z) + \bar{X}(\bar{z})) \sigma_q(z, \bar{z}) \sigma_{N-q}(w, \bar{w})
\]

\[
\sim \frac{C_{q,N-q}}{|z-w|^{4h(\sigma_q)}} \sigma_N(w, \bar{w}) e^{q/N} \sqrt{k/2}(X(w) + \bar{X}(\bar{w})) + \ldots \quad (48)
\]

Note that \( \sigma_N \) has dimension zero

\[
h(\sigma_N) = h(\Psi_{k/2, k/2}^{SL(2)}) = 0 . \quad (49)
\]

It is natural to associate it with the identity operator. We have thus introduced a fractional spectral flow operator into the spectrum

\[
t_{q/N} = e^{q/N} \sqrt{k/2}(X + \bar{X}) \quad \text{where} \quad q \in \mathbb{Z}_N . \quad (50)
\]

Fractional spectral flow generates primary states of the orbifold theory that are of the form

\[
\Phi_{pj_m\bar{m}}^{SL(2)}/\mathbb{Z}_N = \Psi_{j_m\bar{m}}^{SL(2)} e^{\sqrt{2/k}((m+\frac{k}{2N})X + (\bar{m}+\frac{k}{2N})\bar{X})} . \quad (51)
\]

Here \( m - \bar{m} \in N\mathbb{Z} \) and \( p \in \mathbb{Z} \). The twisted sector of the primary is given by \( q = N - (p \mod N) \) as may be verified by the monodromy with the currents.

The unflowed primary states, being representations of the zero mode algebra, correspond in the classical limit to geodesics of the geometry. As explained in [26] the geodesics corresponding to the discrete representations of the WZW model are timelike and those of the continuous representations are spacelike. Spectral flow of a geodesic which passes through the origin by \( w \in \mathbb{Z} \) stretches the geodesic in the timelike direction, allowing spacelike geodesics corresponding to tachyonic primaries to become timelike, and forms a string worldsheet wrapped \( w \) times around the origin as a surface of revolution of the stretched geodesic. The operation of fractional spectral flow introduces strings of this type that wind a fractional number of times, and are closed only by virtue of the orbifold identification. The example of fractional spectral flow on a timelike geodesic for \( N = 6, w = 0 \) and \( q = 2 \) is shown in figure 3.3.

4 The \((AdS_3 \times S^3)/\mathbb{Z}_N\) Orbifold

4.1 Bosonic Structure

To construct a critical bosonic string theory \((c = 26)\) that has a well defined flat space limit, the WZW model describing string theory on \( AdS_3 \) must be combined
with another CFT. Here we will consider some aspects of the bosonic string theory on a $\mathbb{Z}_N$ orbifold of $AdS_3 \times S^3 \times \mathcal{N}$ associated with a combined rotation of $AdS_3$ and $S^3$. Here $\mathcal{N}$ is the target space of a bosonic CFT which has central charge $c_\mathcal{N} = 20$. String theory on $S^3$ may be described in terms of a WZW model on the $SU(2)$ group manifold. The description of this WZW model is very similar to that provided above for $SL(2, \mathbb{R})$. One distinction is in the central charges of the bosonic theories

$$c_{SL(2)} = \frac{3k_{SL(2)}}{k_{SL(2)} - 2}, \quad c_{SU(2)} = \frac{3k_{SU(2)}}{k_{SU(2)} + 2}. \quad (52)$$

This arises from the difference in the value of the quadratic Casimirs of the adjoint representations of the respective groups. Here we will consider only $c_{SL(2)} + c_{SU(2)} = 6$ for the bosonic theory, so that $k_{SL(2)} - 2 = k_{SU(2)} + 2$. The $SU(2)$ group manifold may be parameterized by three Euler angles. One of these angles, which will be denoted $\phi_{SU(2)}$, may be shifted by the transformation

$$g(\phi_{SU(2)} + \alpha) = e^{i\alpha \sigma^3/2} g(\phi_{SU(2)}) e^{-i\alpha \sigma^3/2}. \quad (53)$$

The orbifold we would like to consider identifies $AdS_3 \times S^3$ as follows

$$(\phi_{SL(2)}, \phi_{SU(2)}) \sim (\phi_{SL(2)} + 2\pi/N, \phi_{SU(2)} - 2\pi/N). \quad (54)$$

The twisted sector states on this target space may be described by considering the twisted spectra on the rotational orbifolds of the two group manifolds separately and then pairing the $q$ twisted sector of $SL(2, \mathbb{R})/\mathbb{Z}_N$ with the $N - q$ twisted sector of $SU(2)/\mathbb{Z}_N$. 

Figure 1: Fractional spectral flow of a timelike geodesic producing a string in the $q = 2$ twisted sector for the $N = 6$ orbifold.
The $SU(2)/\mathbb{Z}_N$ CFT

We briefly review the $SU(2)/\mathbb{Z}_N$ theory [29]. An asymmetric version of this orbifold is described in [30]. When the level $k$ and spin $j$ appear in this paragraph they correspond to $k_{SU(2)}$ and $j_{SU(2)}$. The current algebra primaries of the $SU(2)$ WZW model are expressed in terms of parafermions as

$$\Phi^{SU(2)}_{jm\bar{m}} = \Psi^{SU(2)}_{jm\bar{m}} e^{i\sqrt{2/k}(mY + \bar{m}\bar{Y})}.$$  \hspace{1cm} (55)

Here $Y$ is a free boson associated with the holomorphic current $J^3_{SU(2)}$:

$$J^3_{SU(2)} = i\sqrt{k/2}\partial Y.$$  \hspace{1cm} (56)

And similarly for $\bar{Y}$. As for the $SL(2,\mathbb{R})$ case, the holomorphic and antiholomorphic Casimirs are equal. The dimension of the primaries, as computed using the Sugawara stress tensor $T(z) = \frac{1}{k+2}\delta_{ab}J^aJ^b(z)$, is given by $h(\Phi^{SU(2)}_{jm\bar{m}}) = \frac{j(j+1)}{k+2} - \frac{m^2}{k}$.

$$h(\Psi^{SU(2)}_{jm\bar{m}}) = j(j+1) - \frac{m^2}{k}.$$  \hspace{1cm} (57)

And similarly for $\bar{h}$ with $m \rightarrow \bar{m}$. The twist vertex operators of the $SU(2)/\mathbb{Z}_N$ orbifold are given by the lowest weight state in the $D^{SU(2)}_{j_q}$ representation. That is

$$\sigma^{SU(2)}_q = \Psi^{SU(2)}_{j_q, -j_q, -j_q} \quad \text{where} \quad j_q = \frac{kq}{2N}.$$  \hspace{1cm} (58)

Note that the $j = j_q$ representation exists since the level $k$ of the orbifold theory is restricted to $k_{SU(2)} \in N\mathbb{Z}$; $SU(2)/\mathbb{Z}_N$ is a compact manifold with a volume that is $1/N$ of that of $SU(2)$, and the $H$ flux threading it must be an integer. Also note that, as for the $SL(2,\mathbb{R})/\mathbb{Z}_N$ orbifold, $m - \bar{m} \in N\mathbb{Z}$. The dimensions of the twist operators are thus given by

$$h(\sigma^{SU(2)}_q) = kj_q - \frac{2j_q^2}{k(k+2)} = \frac{k}{k+2} \frac{1}{2} \left( q/N \right) \left( 1 - q/N \right)$$  \hspace{1cm} (59)

which, as for $SL(2,\mathbb{R})/\mathbb{Z}_N$ above, agrees with the calculation [27] in the $k \rightarrow \infty$ flat space limit. The twist operators imply the existence of fractional spectral flow operators

$$t^{SU(2)}_{q/N} = e^{iq/N\sqrt{k/2}(Y + \bar{Y})} \quad \text{where} \quad q \in \mathbb{Z}_N.$$  \hspace{1cm} (60)

However there is no independent spectral flow quantum number $w$ characterizing the representations of the (compact) $SU(2)$ WZW model, since integer spectral flow maps the current algebra representations into themselves.
4.2 The Supersymmetric Orbifold

The supersymmetric WZW model

We now consider the superstring on the orbifold \((AdS_3 \times S^3)/\mathbb{Z}_N \times N\). To form a critical \(c = 15\) theory, the CFT on \(N\) is required to have \(c_N = 6\). Only the case \(N = T^4\) will be described in detail; the generalization is straightforward. The OPEs and Sugawara stress tensor are described in what follows for both the \(SU(2)\) and \(SL(2, \mathbb{R})\) supersymmetric level \(k\) WZW model [31] with the Killing metric \(g_{ab}\) and quadratic Casimir \(Q\). The WZW supercurrent \(C^a\) is expressed in terms of the total current \(J^a\) and the fermions \(\psi^a\) as

\[
C^a = \psi^a + \theta J^a .
\] (61)

Here \(\theta\) is the holomorphic worldsheet Grassmann coordinate. \(J^a\) and \(\psi^a\) satisfy the OPEs:

\[
J^a(z) J^b(w) \sim \frac{(k/2) g^{ab}}{(z-w)^2} + \frac{i \epsilon_{abc} J^c(w)}{(z-w)}
\]

\[
J^a(z) \psi^b(w) \sim \frac{i \epsilon^{abc} \psi^c(w)}{(z-w)}
\]

\[
\psi^a(z) \psi^b(w) \sim \frac{(k/2) g^{ab}}{(z-w)}
\]

Subtracting the contribution to \(J^a\) which comes from the fermionic piece of the SUSY WZW action produces the bosonic current

\[
j^a = J^a + \frac{i}{k} \epsilon^{abc} \psi^b \psi^c ,
\] (63)

which leads to the OPEs

\[
j^a(z) j^b(w) \sim \frac{(\tilde{k}/2) g^{ab}}{(z-w)^2} + \frac{i \epsilon^{abc} j^c(w)}{(z-w)}
\]

\[
j^a(z) \psi^b(w) \sim 0 .
\] (64)

Where \(\tilde{k} = k - Q\). The stress tensor may be obtained as usual:

\[
T = \frac{1}{k + Q} j^a j^b g_{ab} - \frac{1}{k} \psi^a \partial \psi^b g_{ab} ,
\] (65)

and the Virasoro central charge is

\[
c = \frac{3 \tilde{k}}{k + Q} + \frac{3}{2} .
\] (66)

The Virasoro supercurrent is given by

\[
G = \frac{2}{k} \left( g_{ab} \psi^a j^b - \frac{i}{3k} \epsilon_{abc} \psi^a \psi^b \psi^c \right) .
\] (67)
The superstring on $\text{AdS}_3 \times S^3 \times T^4$

We briefly review here the description of the superstring on $\text{AdS}_3 \times S^3 \times T^4$ as it appears in [8, 30]. The fermions and the total and bosonic currents associated with the $\hat{SL}(2, \mathbb{R})$ current algebra will be denoted by $(\psi^A, J^A, j^A)$ respectively. For $\hat{SU}(2)$ they will be denoted by $(\chi^a, K^a, k^a)$. The (canonically normalized) $\hat{U}(1)^4$ fermions and current will be denoted $\lambda^j$ and $i\partial F^j$. Note that the levels of the associated bosonic WZW current algebras are shifted as described above. That is, for the superstring

$$c_{SL(2)} = \frac{3(k_{SL(2)} + 2)}{k_{SL(2)}} + \frac{3}{2}, \quad c_{SU(2)} = \frac{3(k_{SU(2)} - 2)}{k_{SU(2)}} + \frac{3}{2}. \quad (68)$$

The condition $c = 15$ then leads to $k = k_{SL(2)} = k_{SU(2)}$. The ten fermions may be written in terms of five canonically normalized free bosons $H_I$ where $I \in (1, \ldots, 5)$ as follows

$$
\begin{align*}
  i\partial H_1 &= -i\frac{2}{\pi} \psi^1 \psi^2 = J^3 - j^3 \\
  i\partial H_2 &= -i\frac{2}{\pi} \chi^1 \chi^2 = K^3 - k^3 \\
  i\partial H_3 &= -2 \frac{2}{\pi} \psi^3 \chi^3 \\
  i\partial H_4 &= -i \lambda^1 \lambda^2 \\
  i\partial H_5 &= -i \lambda^3 \lambda^4 .
\end{align*}
$$

(69)

The spacetime supercharges are constructed as in [32]:

$$Q_\alpha = \oint \frac{dz}{2\pi i} e^{-\varphi/2} S_\alpha(z) , \quad (70)$$

where $\varphi$ is a boson of the $\beta, \gamma$ superghost system and

$$S_\alpha = \exp \left( \frac{i}{2} \epsilon_I H_I \right) \quad (71)$$

is one of the 32 spin fields possible before the imposition of the GSO projection and the condition of BRST invariance. Here $\alpha = (\epsilon_1 \ldots \epsilon_5)$ where $\epsilon_I = \pm 1$. The GSO projection amounts to imposing

$$\prod_{I=1}^{5} \epsilon_I = 1 . \quad (72)$$

BRST invariance is not guaranteed due to the presence of the term cubic in the fermions in $G(z)$ (67). The cancellation of the term of order $z^{-3/2}$ in the $G(z)S_\alpha(0)$ OPE leads to the condition

$$\prod_{I=1}^{3} \epsilon_I = 1 . \quad (73)$$

Thus there are 8 ‘left-moving’ supercharges on $\text{AdS}_3 \times S^3 \times T^4$ as compared to 16 in flat space (and 8 more from the right-movers).
Massless and spacetime chiral states

As shown explicitly in [8], the worldsheet $\hat{SL}(2, \mathbb{R}) \times \hat{SU}(2) \times \hat{U}(1)^4$ current algebra is associated with an $N = 4$ superconformal algebra in spacetime. The global modes of the spacetime Virasoro and current algebras are just the charges of the associated total worldsheet currents. Thus for $\hat{SL}(2, \mathbb{R})$ and $\hat{SU}(2)$:

$$ L_0 = \oint \frac{dz}{2\pi i} J^3(z) \quad \quad L_\pm = \oint \frac{dz}{2\pi i} J^\pm(z) $$

(74)

and

$$ T_0^a = \oint \frac{dz}{2\pi i} K^a(z) \quad . $$

(75)

Chiral primary states of this spacetime algebra were described in [8] [30] in terms of the unflowed worldsheet current algebra primaries as follows. The worldsheet primaries of the WZW supercurrents are given by the primaries of the bosonic ( level $(k + 2)$ for $SL(2, \mathbb{R})$ and $(k - 2)$ for $SU(2)$ ) WZW model

$$ \Phi_{j,m}^{SL(2)} \Phi_{j',m'}^{SU(2)} e^{i p \cdot F + i \bar{p} \cdot \bar{F}} \quad . $$

(76)

Here $(p, \bar{p})$ is a vector in an even, self-dual Narain lattice $\Gamma^{4,4}$. Physical states consist of bosonic current algebra descendants and fermionic excitations of these primaries which are in the BRST cohomology. Suppressing the anti-holomorphic quantum numbers, the $\hat{SL}(2, \mathbb{R})$ primary states satisfy [8]

$$ [L_n, \Phi_{j,m}^{SL(2)}] = (n(j - 1) - m) \Phi_{j,m+n}^{SL(2)} $$

(77)

where $L_n$ are generators of the spacetime Virasoro algebra. Thus these primaries can be seen to be modes of an operator with spacetime scaling dimension $h = j$. Unitarity of the spacetime superconformal algebra requires that $h \geq j_{SU(2)}$, where $j_{SU(2)}$ is the spin associated with the spacetime $\hat{SU}(2)$ current algebra $T_n^a$. This is, of course, also the spin of the total worldsheet $\hat{SU}(2)$ current $K^a$. The chiral primary operators in the NS sector saturate this bound and satisfy $p = 0$ and $N = 1/2$, where $N$ is the total level of the oscillator excitations. Physical states which are spacetime chiral primaries correspond to massless excitations as can be seen from the NS sector mass-shell condition

$$ \frac{-j(j - 1)}{k} + \frac{j'(j' + 1)}{k} + \frac{p \cdot p}{2} + N = \frac{1}{2} \quad . $$

(78)

This implies $j' = j - 1$.\(^5\) There are eight massless physical NS states for a given $j$ as well as $m, m'$ (which are suppressed)

$$ \mathcal{V}_j^m = e^{-\varphi} \lambda^i \Phi_{j,m}^{SL(2)} \Phi_{j'}^{SU(2)} $$

\(^5\)Note that a different convention is used here for the $SL(2, \mathbb{R}) j$ quantum number than is used in [8, 30].
\[ W_j^\pm = e^{-\varphi} \left[ \psi \Phi_j^{SL(2)} \right]_{j \pm 1} \Phi_j^{SU(2)} \]
\[ \chi_j^\pm = e^{-\varphi} \Phi_j^{SL(2)} \left[ \chi \Phi_j^{SU(2)} \right]_{j \pm 1} \]

Here the quantum numbers outside the brackets refer to \( j_{SL(2)} \) and \( j_{SU(2)} \), the spins associated with the quadratic Casimirs of the total currents \( J^a \) and \( K^a \) respectively. Details of the construction of these states may be found in [30]. The additional two states

\[ W_0^j = e^{-\varphi} \left[ \psi \Phi_j^{SL(2)} \right]_j \Phi_j^{SU(2)} \]
\[ \chi_0^j = e^{-\varphi} \Phi_j^{SL(2)} \left[ \chi \Phi_j^{SU(2)} \right]_j \]  

are not in the BRST cohomology. Of these massless states only \( W^-_j \) and \( \chi^+_j \) are spacetime chiral primaries, that is are modes of spacetime operators which satisfy \( h = j_{SL(2)} = j_{SU(2)} \). Ramond sector states are found by applying the spacetime supercharges to these NS states [32].

In all of these vertex operators, the current algebra primaries \( \Phi_{j'm'\bar{m}'}^{SU(2)} \) and \( \Phi_{jm'\bar{m}}^{SL(2)} \) span the space of wavefunctions on \( AdS_3 \times S^3 \) for spins less than \( k/2 \), while the fermions \( \psi, \chi, \lambda \) carry the polarizations of the various supergravity modes.

**Superparafermions**

Due to the contribution of the fermions to the total currents \( J^a \) and \( K^a \) of \( SL(2, \mathbb{R}) \) and \( SU(2) \), respectively, it is convenient to write the current algebra primaries in terms of superparafermions times exponentials of the bosons related to the total currents \( J^3 \) and \( K^3 \). For example for the \( \hat{SU}(2) \) superparafermions

\[ \Phi_{jm'\bar{m}}^{SU(2)} = \hat{\Psi}_{jm'\bar{m}}^{SU(2)} \exp\left[ i\sqrt{2\kappa}\left( m\mathcal{Y} + \bar{m}\mathcal{\bar{Y}} \right) \right]. \]

Where we have bosonized the total current as \( K^3 = i\sqrt{k/2} \partial\mathcal{Y} \). The bosonic current \( k^3 = i\sqrt{k/2} \partial\mathcal{Y} \) of the parafermion construction of (56) (referred to as \( J_{SU(2)}^3 \) there) and the boson \( H_2 \) of (69) are then rewritten in terms of \( \mathcal{Y} \) and another boson \( \mathcal{H}_2 \) (both canonically normalized) via\(^6\)

\[ Y = \sqrt{\frac{k}{k}} \mathcal{Y} - \sqrt{\frac{k}{k}} \mathcal{H}_2 \]
\[ H_2 = \sqrt{\frac{k}{k}} \mathcal{H}_2 + \sqrt{\frac{k}{k}} \mathcal{Y} \]  

One also has the following relation between the level \( k \) superparafermions and the parafermions of the bosonic \( SU(2) \) level \( k - 2 \) WZW model

\[ \hat{\Psi}_{jm'\bar{m}}^{SU(2)} = \Psi_{jm'\bar{m}}^{SU(2)} \exp\left[ -i\frac{4}{kk}\left( m\mathcal{H}_2 + \bar{m}\mathcal{\bar{H}}_2 \right) \right]. \]

\(^6\)Recall our notation \( \hat{k} = k - Q \). Thus, for \( SU(2) \), \( Q = 2 \) and \( \hat{k} = k - 2 \); while for \( SL(2, \mathbb{R}) \), \( Q = -2 \) and \( \hat{k} = k + 2 \).
The bosonic currents and associated fermions are given by

\[ k^1 \pm ik^2 = \psi^\pm \exp \left[ \pm i \sqrt{\frac{2}{k}} \left( Y - \frac{1}{\sqrt{2}} H_2 \right) \right] \]
\[ \chi^1 \pm i\chi^2 = \sqrt{k} \exp \left[ \pm i \left( \frac{1}{\sqrt{2}} Y + \sqrt{\frac{k}{2}} H_2 \right) \right] , \tag{84} \]

and the supercurrent is written as

\[ \sqrt{k} G = \psi^+ e^{-i\sqrt{k/2} H_2} + \psi^- e^{+i\sqrt{k/2} H_2} + \sqrt{2} \chi^3 \partial Y . \tag{85} \]

Similarly one may write \( X \) and \( H_1 \) of the \( SL(2, \mathbb{R}) \) supersymmetric WZW model in terms of bosons \( X, H_1 \) of the analogous superparafermionic construction for \( SL(2, \mathbb{R}) \). The advantage of the superparafermion description is that it makes manifest the \( \mathbb{Z}_k \) symmetry of the supersymmetric \( SU(2) \) WZW model, which acts only on the boson \( Y \) which bosonizes the (total) current \( J^3_{SU(2)} \) (whereas the bosonic parafermions have a \( \mathbb{Z}_{\tilde{k}} \) symmetry acting on \( Y \)). Furthermore the superparafermion operators are by construction primary fields of the superVirasoro algebra. Thus the twist operators for the superstring on the orbifold \((AdS_3 \times S^3)/\mathbb{Z}_N \times N\) (where \( N \) divides \( k \)) will be written in terms of the superparafermions \( \hat{\Psi}^{SL(2)}_{jmn} \) and \( \hat{\Psi}^{SU(2)}_{jmn} \).

**The \( (AdS_3 \times S^3)/\mathbb{Z}_N \times T^4 \) orbifold**

We now consider the superstring on the orbifold \((AdS_3 \times S^3)/\mathbb{Z}_N \times T^4\). The twist vertex operators for the superstring which have the proper monodromy with respect to the currents and have a single valued OPE with the worldsheet supercurrent \( G(z) \) are constructed as follows. The twist operators on \( SL(2, \mathbb{R})/\mathbb{Z}_N \) are given by the superparafermions

\[ \Omega^q_{SL(2)} = \hat{\Psi}^{SL(2)}_{j_q j_q j_q} = \Psi^{SL(2)}_{j_q j_q j_q} \exp \left[ i \sqrt{\frac{k}{k}} \frac{q}{N} \left( H_1 + \bar{H}_1 \right) \right] ; \tag{86} \]

here again \( j_q = kq/2N \), and \( \tilde{k} = k - Q \).\(^7\) Similarly the twist operators on \( SU(2)/\mathbb{Z}_N \) are given by

\[ \Omega^q_{SU(2)} = \hat{\Psi}^{SU(2)}_{j_q j_q j_q} = \Psi^{SU(2)}_{j_q j_q j_q} \exp \left[ i \sqrt{\frac{k}{k}} \frac{q}{N} \left( H_2 + \bar{H}_2 \right) \right] . \tag{87} \]

The dimension of both the superparafermions is \( h(\Omega_q) = q/2N \); thus the twist operators on the \((AdS_3 \times S^3)/\mathbb{Z}_N \times T^4 \) orbifold are

\[ \Omega^{\text{orb}}_q = \Omega^q_{SL(2)} \Omega^{SU(2)}_{N-q} , \quad q = 1, \ldots, N - 1 ; \tag{88} \]

\(^7\)Note however the shift in notation – in the supersymmetric case the order of the cyclic orbifold group \( N \) divides the level \( k \) of the total current, rather than the bosonic level \( k \). Hence one must suitably modify the formulae (40), (59) for the parafermion dimensions.
each of these states has vanishing spacetime energy $\mathcal{L}_0$ and $S^3$ angular momentum $T_0^3$. The condition for primary fields to be invariant under the action of the orbifold identification

$$\exp \left[ i (2\pi/N) (L_{SL(2)} - L_{SU(2)}) \right] \Phi_{jm\bar{m}}^{SL(2)} \Phi_{j'm'\bar{m}'}^{SU(2)} = \Phi_{jm\bar{m}}^{SL(2)} \Phi_{j'm'\bar{m}'}^{SU(2)}$$  \hspace{1cm} (89)

implies the condition $(m - \bar{m}) - (m' - \bar{m}') \in N\mathbb{Z}$. Here the generators of rotation are given by

$$L_{SL(2)} = J_3^0 - \bar{J}_3^0 \hspace{1cm} L_{SU(2)} = K_3^0 - \bar{K}_3^0 .$$  \hspace{1cm} (90)

Consider the OPE of the twist vertex operators with the spin fields used to construct the spacetime supercharges

$$\Omega_q^{ORB}(z) S_\alpha(0) \sim z^{\pm 1/2} z^{q/N(\epsilon_1 - \epsilon_2)/2} : \Omega_q^{ORB}(z) S_\alpha(0) : ;$$  \hspace{1cm} (91)

this implies $\epsilon_1 = \epsilon_2$ and thus, from (73) above, $\epsilon_3 = 1$. Thus only 4 ‘left-moving’ supercharges survive the orbifold projection out of the original 8 on $AdS_3 \times S^3 \times T^4$ (and 4 more from the right-movers). The associated spin fields can be indexed by the $i \partial H_1$ and $i \partial H_4$ charges

$$S_{\epsilon_1 \epsilon_4} = e^{\frac{i}{2}(\epsilon_1 (H_1 + H_2) + H_3 + \epsilon_4 (H_4 + H_5))} .$$  \hspace{1cm} (92)

Note that the twist operators (88) commute with half of the supersymmetry generators, thus the corresponding states (and their fractional spectral flows) are BPS.

**The supersymmetric (fractional) spectral flow operator**

The integer spectral flow operator introduced in [26] was extended to the superstring in [28]. Express the total currents $J^3$ and $K^3$ in terms of bosons $X$ and $Y$

$$J^3 = -\sqrt{k/2} \partial X \hspace{1cm} K^3 = i \sqrt{k/2} \partial Y$$  \hspace{1cm} (93)

The holomorphic part of the integer spectral flow operator for the superstring is

$$t_w = e^{w \sqrt{k/2}(X + iY)}$$  \hspace{1cm} (94)

Note that $h(t_w) = 0$ and that $t_w$ is mutually local with respect to the spacetime supercharges. Integer spectral flow $\mathcal{O}_{BPS} \rightarrow t_w \mathcal{O}_{BPS}$, $w \in \mathbb{Z}$, extends the range of $SL(2)$ and $SU(2)$ spins $j$ of BPS operators (79) beyond the window $\frac{1}{2} \leq j < \frac{k-1}{2}$. The orbifold again admits fractional spectral flow under (94) with $w = p/N$, $p \in \mathbb{Z}$, describing oscillating strings of the sort depicted in figure 3.3. The BPS single particle spectrum of the orbifold thus consists of the surviving untwisted sector supergravity states (79) (i.e. those with $(m - \bar{m}) - (m' - \bar{m}') \in N\mathbb{Z}$), together with their fractional spectral flows by (94) (with $w = p/N$); furthermore there are the twist ground states (88) and their fractional spectral flows.
The continuous representations $C_j^\alpha$ of $SL(2, \mathbb{R})$ spin $j = \frac{1}{2} + is$ appear in the spectrum of the $AdS_3 \times S^3 \times T^4$ theory in nonzero spectral flow sectors, where they describe a continuum of long strings moving in or out from the boundary of $AdS_3$ with radial momentum $s$. One might think that the $\mathbb{Z}_N$ orbifold identification lowers the threshold of the continuum of long string states by a factor of order $N$, since a long string need only wind part of the way around the $AdS_3$ angular direction; however, the effect is less dramatic. From the analysis of [26, 28], a continuous representation of spin $j = \frac{1}{2} + is$ can be spectrally flowed $w$ units and then used to dress a physical vertex operator $e^{-\phi/2} \hat{\Psi}_{j q j q j q}^{SL(2)} \mathcal{O}_{\text{int}}$ (where $\mathcal{O}_{\text{int}}$ is a vertex operator in the remaining $S^3 \times T^4$ sigma model), for which the mass shell condition is

$$-\frac{j(j-1)}{k} - w m - \frac{k}{4} w^2 + h_{\text{int}} - \frac{1}{2} = 0; \quad (95)$$

the spacetime energy of this state is then

$$\mathcal{L}_0 = J_0^3 = \frac{k}{2} w + m = \frac{k w}{4} + \frac{1}{w} \left( \frac{1 + 4 s^2}{4 k} + h_{\text{int}} - \frac{1}{2} \right). \quad (96)$$

Before the orbifold, only $w \in \mathbb{Z}$ is allowed, and the threshold of the continuum occurs for $w = 1$, $s = h_{\text{int}} = 0$, at $\mathcal{L}_0 = \frac{(k-1)^2}{4k}$. The orbifold allows fractional spectral flow of this $w = 1$ state to a fractional winding $w = 1 - \frac{p}{N}$, provided that one spectrally flows an amount $-p/N$ in $SU(2)$. This flows $h_{\text{int}} = 0$ to $h_{\text{int}} = \frac{k}{4}(1 - w)^2$ according to the $SU(2)$ version of (44); the spacetime energy of the threshold state becomes

$$\mathcal{L}_0 = \frac{1}{w} \left[ \frac{(k-1)^2}{4k} - \frac{k}{2} w(1 - w) \right]. \quad (97)$$

The second term reduces the energy cost somewhat, but the threshold is still at energies of order $k$, not $k/N$, because the lowered energy cost in $AdS_3$ winding is compensated by the cost of $S^3$ momentum.

### 4.3 The $AdS_3/\mathbb{Z}_N$ Orbifold of the Superstring

The orbifold of $AdS_3$ alone breaks supersymmetry in the superstring. The twist operators are the superparafermions $\hat{\Psi}_{j q j q j q}^{SL(2)}$ of $SL(2, \mathbb{R})$. Their dimension is $kq/2N < 1/2$ and thus the NS sector twist operators describe tachyonic excitations of the orbifold point; the ground state is unstable. The twisted sector RR ground state operators take the form

$$e^{-\phi/2} \hat{\Psi}_{j q j q j q}^{SL(2)} \exp \left[ \epsilon_1 \left( \frac{1}{2} \sqrt{\frac{2}{k}} \mathcal{X}^1 + \frac{i}{2} \sqrt{\frac{\tilde{k}}{k}} \left( \frac{k}{k} - \frac{2q}{N} \right) \mathcal{H}_1 + \frac{i}{2} \mathcal{H}_2 \right) + \frac{i}{2} \mathcal{H}_3 + \epsilon_4 \frac{i}{2} (\mathcal{H}_4 + \mathcal{H}_5) \right]$$

(98)

where we have suppressed the right-moving free bosons. These are massless gauge fields coupling to the conserved RR charge of fractional D-branes (see section (5)). At the orbifold point, the vacuum is unstable, but the fractional D-branes couple to K-theoretic topological charges; the D-branes themselves are stable. As mentioned in the introduction, if the configuration is to decay to some superposition of states in the standard $AdS_3 \times S^3$ string theory, the fate of these charges must be understood.
Another class of excitations of the conical defect are fractional D-branes. D-branes at an orbifold singularity $\mathbb{R}^n/\Gamma$ are classified by the representations of $\Gamma$, which characterize the orbifold action on Chan-Paton structure [33, 34, 35, 36]. Branes in a given irreducible representation are pinned to the orbifold point; however, assembling irreps into the regular representation, a moduli space develops that allows them to be moved off the orbifold point – in the case of $\mathbb{Z}_k$, one needs $k$ fractional D-branes to put one brane on each leaf of the covering space away from the origin.

In flat spacetime, the tension of a $D_p$-brane is $\mu_p = (g_s \ell_s^{p+1})^{-1}$; however, in the presence of the nontrivial background (9), the energetics is modified due to binding energy with the background fundamental strings and NS fivebranes [22]. For example, in the IIB theory, the left-moving energies of D-branes wrapping the $T_4$ are

$$L_0 = \frac{1}{4k} \sum_{i=1}^{4} \left( \frac{w^D_3}{r_i} \right)^2. \quad (99)$$

Here $w^D_3 = \frac{1}{6} \epsilon_{i j k l} w_{i j k l}^{D_3}$ is the winding charge of D3-branes, and $w^D_1$ is the wrapping charge of D1-branes; $r_i$ are the radii of a rectangular $T_4$, and $v_4 = r_1 r_2 r_3 r_4$. We can then orbifold $AdS_3 \times S^3$ by $\mathbb{Z}_N$, and passing to the IIA theory to turn odd branes into even ones, place fractional branes along the orbifold singularity, which is extended along one direction of the $S^3$; similar D-branes stretched across a great circle of $S^3$ have been considered recently in [37]. The size of the $S^3$ is $\sqrt{k}$ in string units, so (99) should be multiplied by $\sqrt{k}$. Similarly, in the IIB theory one has D-branes that lie along the one dimension of the orbifold singularity in $S^3$ and also wrap zero, two, or all four directions of $T_4$

$$L_0 = \frac{1}{4\sqrt{k}} \left( \frac{w^D_1}{\sqrt{v_4}} + w^D_3 \sqrt{v_4} \right)^2 + \frac{1}{4\sqrt{k}} \sum_{i<j} \left( * w_{ij}^{D_3} \frac{\sqrt{v_4}}{r_i r_j} + w_{ij}^{D_3} \frac{r_i r_j}{\sqrt{v_4}} \right)^2. \quad (100)$$

Assembling fractional branes into regular representations does not result in a true moduli space for the brane in $(AdS_3 \times S^3)/\mathbb{Z}_N$; due to the gravitational redshift of anti-de Sitter space, it costs energy to move an object away from the origin. However, the associated energy cost decreases from string scale for fractional branes to the scale set by the $AdS$ radius of curvature.

### 6 Moduli Space

In appropriately scaled variables, in the limit $k \to \infty$ the sigma model tends to flat spacetime. Consequently, the supersymmetric orbifold of the previous section degenerates to $\mathbb{R}^{1,1} \times (\mathbb{R}^4/\mathbb{Z}_N)$. String theory on the orbifold $\mathbb{R}^4/\mathbb{Z}_N$ has $4(N-1)$ moduli corresponding to the resolution of the orbifold singularity (c.f. [38, 39]). The resolution by blowing up inserts $N-1$ two-spheres. The moduli parametrize
the hyperKähler blowup modes together with the B-flux through the two-spheres; in particular, the orbifold point corresponds to a set of collapsed two-spheres each threaded by a half unit of B-flux [38].

From the construction of the twisted sector spectrum, we see that the AdS$_3 \times S^3/\mathbb{Z}_N$ orbifold also has a set of $4(N-1)$ massless modes corresponding to the operators (88). The orbifold effectively acts only on the coset theory $SL(2,\mathbb{R})/U(1) \times SU(2)\times U(1)$, which is an $\mathcal{N} = (4,4)$ worldsheet superconformal field theory. The $\mathcal{N} = 4$ algebra [40] is generated by the stress tensor $T(z)$, four supercurrents $G^{\alpha\bar{\alpha}}(z)$, and the $SU(2)$ R-symmetry current $J^{\alpha\bar{\beta}}(z)$ (and corresponding antiholomorphic currents); the supercurrents additionally transform as a doublet under a global $SU(2)_l \times SU(2)_r$ symmetry of the small $\mathcal{N} = (4,4)$ superconformal algebra, and are R-symmetry singlets. Since the $N-1$ quartets of massless fields all preserve $\mathcal{N} = 4$ worldsheet supersymmetry, all these deformations are exactly marginal (the trace of the stress tensor lies in the same supermultiplet as the anomaly in the $SU(2)$ R-symmetry, which is not renormalized). Four additional massless multiplets arise from the untwisted sector and are universal, being obtained by worldsheet spectral flow [41] from the identity sector (see [42] for a recent review). These are the spin one-half representations of the $SU(2)$ R-symmetry current algebra, associated in spacetime to the supergravity multiplet. However, this last multiplet is not part of the moduli space of the orbifold – the expectation value of the zero momentum supergravity modes are not moduli on a noncompact space. All told, the $4(N-1)$ massless fields parametrize the moduli space $O(4,N-1)/O(4) \times O(N-1)$. It would be interesting to know whether there are any global identifications (other than the integer periodicity in the B flux).

For $k \gg N \gg 1$, the orbifold looks locally very much like $\mathbb{R}^{1,1} \times (\mathbb{R}^4/\mathbb{Z}_N) \times T^4$, and an approximate description of the resolved orbifold singularity as an ALE space is appropriate (up to distances of order $\sqrt{k}\ell_s$ from the orbifold point). The metric on the ALE space is

$$ds^2 = V^{-1}(\vec{x})(d\tau + \vec{\omega} \cdot d\vec{x})^2 + V(\vec{x})d\vec{x} \cdot d\vec{x}$$

$$V(\vec{x}) = \sum_{i=1}^N |\vec{x} - \vec{x}_i|^{-1},$$

(101)

where $\vec{\nabla} V = \vec{\nabla} \times \vec{\omega}$. This should accurately describe the vicinity of the blowup when the deformation is not too far from the orbifold point in moduli space (which is $\vec{x}_i = 0$ for all $i$). The $\vec{x}_i$ parametrize the three metric deformations of the collapsed spheres, and can roughly be thought of as the locations of the poles of the homology two-spheres of the resolved manifold (the fourth modulus being the B-flux through these homology two-spheres). These deformations might be thought of as certain kinds of breathing modes of the object in AdS$_3$. It is interesting that the object can have a large number of internal excitations, related to its mass.
An aside on related models

The orbifold action by the maximal discrete symmetry $Z_k$ on the $(SL(2)/U(1)) \times (SU(2)/U(1))$ sigma model was considered in a related context in [43]. The first factor $(SL(2)/U(1))$ is the Euclidean 2d black hole or ‘cigar’ sigma model of [44, 45], and also the theory of $SL(2, \mathbb{R})$ superparafermions; the second factor $(SU(2)/U(1))$ is the the $X^k$ Landau-Ginsburg theory [46, 47], or equivalently the $SU(2)$ superparafermion theory. The orbifold $[(SL(2)/U(1)) \times (SU(2)/U(1))]/Z_k$ was shown in [43, 48, 49] to give a CFT with a target space which is asymptotically $S^3 \times \mathbb{R}$, corresponding to the throat geometry of $k$ fivebranes. It was argued in these works that the target spacetime is algebraically described by the deformed $Z_k$ singularity

$$X^k + Y^2 + Z^2 = \mu. \tag{102}$$

What is the relation among these various theories? The operators/states of the $SU(2)$ WZW model can be decomposed into a parafermion part and a free field part arising from the bosonization of $J^3$:

$$\Phi_{j,m}^{SU(2)} = \hat{\Psi}_{j,m}^{SU(2)} \exp \left[ i \sqrt{\frac{2}{k}} (nY + \bar{n}\bar{Y}) \right] \tag{103}$$

with $m = n$, $\bar{m} = \bar{n}$, and $m, \bar{m} = -j, ..., j$. On the other hand, the tensor product theory $(SU(2)/U(1))_{pf} \times U(1)_{circ}$ consists of the operators on the RHS of (103), with $m, \bar{m}$ independent of $n, \bar{n}$, and $(m - \bar{m}) \in k\mathbb{Z}$ (and similarly for $n - \bar{n}$). The parafermion operators respect a $Z_k^{pf} \times \tilde{Z}_k^{pf}$ symmetry, under which the operators carry the quantum numbers $(l, l') = (m + \bar{m}, m - \bar{m}) \mod k$ [50]. Similarly considering a $Z_k^{circ} \times \tilde{Z}_k^{circ}$ subgroup of the $U(1) \times U(1)$ symmetry of the free boson, the $SU(2)$ WZW theory is the orbifold of the parafermion times $U(1)$ theory by the diagonal vectorlike $Z'_k = (Z_k^{pf} \times Z_k^{circ})_{diag}$

$$SU(2) = \left[ \left( \frac{SU(2)}{U(1)} \right)_{pf} \times U(1)_{circ} \right]/Z'_k \tag{104}$$

under which the states (103) carry $Z'_k$ charge $(m + \bar{m} - n - \bar{n})$. The orbifold by $Z'_k$ sets $m = n$, $\bar{m} = \bar{n}$; the twisted sectors relax the condition $m - \bar{m} \in k\mathbb{Z}$ to $m - \bar{m} \in \mathbb{Z}$. The relation between the $SL(2, \mathbb{R})$ WZW model and the $\frac{SL(2, \mathbb{R})}{U(1)} \times \mathbb{R}$ cigar plus time background is not as straightforward; the parafermion states have spin $j$ taking continuous values $0 \leq j < \frac{k-1}{2}$. Nevertheless, the $Z_k$ symmetry of (54) (for $k = N$) acts on the embedded parafermion model in precisely the same way as in [43, 48, 49].

There are thus two ways of getting at the orbifold theory we are discussing in this paper. Starting with the target

$$(SL(2))_{WZW} \times \left( \frac{SU(2)}{U(1)} \right)_{pf} \times U(1)_{circ} \times T^4, \tag{105}$$

$8$The monomial $X^q$ of the Landau-Ginsburg field is the superparafermion $\hat{\Psi}_{j_0,j_0,j_0,j_0}^{SU(2)}$. 22
one may construct $AdS_3 \times S^3$ by orbifolding by $\mathbb{Z}_k'$ and then performing the orbifold by the $\mathbb{Z}_k$ of (54), which acts on the embedded $SU(2)$ and $SL(2)$ parafermions of the respective WZW models. Alternatively, one can first orbifold by this latter $\mathbb{Z}_k'$, to make a variant of the CHS model as in [43, 48, 49], and then quotient by the first symmetry $\mathbb{Z}_k$.

There are correspondingly a couple of ways to view the deformation along the moduli space of the orbifold singularity. One is (101), which gives an approximate description of the geometry near the orbifold singularity for not too large a blowup. On the other hand, for $k = N \gg 1$, the description of [43, 48, 49] in terms of Landau-Ginsburg models is perhaps more appropriate, and the geometry near the object is stringy.

The division of the target space into superparafermion and $U(1)$, as in (105), is the starting point for a more general orbifold using the construction of [28]. There, spacetime CFT vacua preserving at least eight supersymmetries are built using $c = 15$ worldsheet superconformal field theories of the form

$$AdS_3 \times U(1)_k \times \left( \frac{\mathcal{M}}{U(1)} \right),$$

where $\mathcal{M}/U(1)$ is $\mathcal{N} = 2$ supersymmetric on the worldsheet, and the subscript $k$ is the ‘level’ of the $U(1)$ supercurrent algebra (the radius squared of the target space circle in units of the self-dual radius). In this paper we have concentrated on perhaps the simplest example $\mathcal{M}/U(1) = SU(2)/U(1) \times T^4$, but clearly one may generalize. A $\mathbb{Z}_N$ orbifold preserving half the supersymmetry may be constructed along the lines of sections 3 and 4, provided $N$ divides $k$.

7 The Spacetime CFT

There is an intimate relation between the asymptotic $AdS$ geometry of the string background and the conformal invariance of its dual CFT description [15, 51, 16, 8]. The orbifold provides some new twists on the standard story. We begin with an analysis of the symmetry algebra of the spacetime CFT, then proceed to show how various length scales in the spacetime geometry and the counting of black hole states are affected by the orbifold. It will turn out that the orbifold quotient affects the effective central charge in the asymptotic density of states and in the symmetry algebra in opposite ways – the former is decreased by a factor of $N$, while the latter is increased by a factor of $N$. 

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7.1 The Spacetime Superconformal Algebra

The spacetime conformal field theory of the onebrane-fivebrane system is a representation of the \( N = 4 \) superVirasoro algebra \[40\]^9

\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n^3 \delta_{n,-m}
\]

\[
[G_{a\alpha}^r, G_{b\beta}^s] = \delta^{ab}\left(2\delta^{\alpha\beta}L_{r+s} - 2(r-s)\sigma_{i}^{\alpha\beta}T_{r+s}^i + \frac{c}{12} 4r^2 \delta_{r,-s}\delta^{\alpha\beta}\right)
\]

\[
[T^i_m, T^j_n] = i \epsilon^{ij}_{k} T^k_{m+n} + \frac{c}{12} m \delta_{m,-n}\delta^{ij}
\]

(107)

\[
[T^i_m, G_{a\alpha}^r] = -\frac{i}{2} e^{ab} \sigma_{a\beta} G_{m+r}^{b\beta}
\]

\[
[L_m, G_{a\alpha}^r] = \left(\frac{1}{2} m - r\right) G_{m+r}^{a\alpha}
\]

\[
[L_m, T^i_n] = -n T^i_{m+n}.
\]

The central terms on the RHS of the first two lines differ slightly from standard CFT conventions due to the \( AdS_3 \) convention that the \( SL(2, \mathbb{R}) \) invariant vacuum state has energy \( L_0 = -\frac{c}{24} \) rather than zero. Expressions for the generators of this algebra in terms of the fields of the sigma model on (Euclidean) \( AdS_3 \times S^3 \) have been given in \[8, 52, 53\]. The rule of thumb is that any (anti)holomorphic algebra on the worldsheet is related to a corresponding algebra in spacetime; roughly, long strings near the boundary of \( AdS_3 \) have the spacetime algebra pulled back onto the worldsheet.

The \( \mathbb{Z}_N \) orbifold of spacetime projects the generators (107) onto the subalgebra which commute with the \( \mathbb{Z}_N \) action generated by

\[
\exp\left[\frac{2\pi i}{N}(L_0 - \tilde{L}_0 - T^3_0 + \tilde{T}^3_0)\right].
\]

(108)

For example, of the Virasoro and R-current generators \( L_n \) and \( T^3_n \), one keeps only those with \( n \in N\mathbb{Z} \); similarly \( G_{r}^{a\pm} \) survive for \( r \mp \frac{1}{2} \in N\mathbb{Z} \) (so that \( G_{r}^{a\pm} \) are the surviving supersymmetries), and we also keep \( T^i_n \) for \( n \mp 1 \in N\mathbb{Z} \). Note that the effect of the orbifold on the spacetime CFT is not simply to project onto the states that are invariant under (108); this would allow multiparticle states invariant under (108) built out of particles that were not individually invariant under (108) (for instance, operators in the Virasoro enveloping algebra built out of products of Virasoro raising operators whose total level is a multiple of \( N \) but whose component raising operators have levels that are not multiples of \( N \)). Such states are not present in the orbifold.

In the twisted sector of the orbifold, no new (anti)holomorphic currents appear on the worldsheet, and therefore one expects that no new symmetries of spacetime will appear. The \( \mathbb{Z}_N \) projected superVirasoro algebra is then the full set of symmetries

^9At the particular point in the moduli space realized by the perturbative worldsheet description, the spacetime CFT is singular; the representation theoretic aspects of this situation are not well understood. We thank D. Kutasov for emphasizing this point.
of the spacetime. The surviving generators satisfy the commutation relations

\[
\left[ \frac{1}{N} \mathcal{L}_{mN}, \frac{1}{N} \mathcal{L}_{nN} \right] = (m - n) \frac{1}{N} \mathcal{L}_{(m+n)N} + \frac{N_c}{12} m^3 \delta_{m,-n} \\
\left[ \frac{1}{\sqrt{N}} \mathcal{G}_{mN+\frac{1}{2}}, \frac{1}{\sqrt{N}} \mathcal{G}_{nN-\frac{1}{2}} \right] = \delta^{ab} \left( 2 \frac{1}{N} \mathcal{L}_{(m+n)N} - 2(m-n+\frac{1}{N})T^3_{(m+n)N} + \frac{N_c}{12} (m + \frac{1}{N})^2 \delta_{m,-n} \right) \\
\left[ \frac{1}{\sqrt{N}} \mathcal{G}_{mN+\frac{1}{2}}, \frac{1}{\sqrt{N}} \mathcal{G}_{nN+\frac{1}{2}} \right] = \delta^{ab} \left( -2(m-n)T^+_{(m+n)N+1} \right) \\
\left[ T^3_{mN}, T^3_{nN} \right] = \frac{N_c}{12} m \delta_{m,-n} \delta^{ij} \\
\left[ T^+_{mN+1}, T^-_{nN-1} \right] = iT^3_{(m+n)N} + \frac{N_c}{12} (m + \frac{1}{N}) \delta_{m,-n} \delta^{ij} \\
\left[ T^3_{mN}, T^+_{nN+1} \right] = iT^+_{(m+n)N+1} \\
\left[ T^3_{mN}, \frac{1}{\sqrt{N}} \mathcal{G}^{a\pm}_{nN+\frac{1}{2}} \right] = \pm \frac{i}{2} \sqrt{N} \mathcal{G}^{b\pm}_{(m+n)N+\frac{1}{2}} \\
\left[ T^\pm_{mN+1}, \frac{1}{\sqrt{N}} \mathcal{G}^{a\mp}_{nN+\frac{1}{2}} \right] = \frac{i}{2} \sqrt{N} \mathcal{G}^{b\pm}_{(m+n)N+\frac{1}{2}} \\
\left[ \frac{1}{N} \mathcal{L}_{mN}, \frac{1}{N} \mathcal{L}_{nN} \right] = -nT^i_{(m+n)N} \\
\left[ \frac{1}{N} \mathcal{L}_{mN}, \frac{1}{N} \mathcal{L}^i_{nN} \right] = 4nT^i_{(m+n)N}.
\]

Thus we can define generators

\[
\tilde{\mathcal{L}}_n = \frac{1}{N} \mathcal{L}_{nN} \\
\tilde{\mathcal{G}}^{a\pm}_{nN+\frac{1}{2}} = \frac{1}{\sqrt{N}} \mathcal{G}^{a\pm}_{nN+\frac{1}{2}} \\
\tilde{T}^3_{nN} = T^3_{nN} \\
\tilde{T}^\pm_{nN+\frac{1}{2}} = T^\pm_{nN+\frac{1}{2}}
\]

which satisfy a form of the $\mathcal{N} = 4$ superVirasoro algebra for $\tilde{c} = Nc$, spectrally flowed [41] by $1/N$ units from Ramond boundary conditions. Note that the ground state of the orbifold is not invariant under the $SL(2,\mathbb{R})$ generated by $\tilde{L}_0, \tilde{L}_{\pm}$.

### 7.2 Energy and Entropy

The relations between the string scale $\ell_s$, three-dimensional Planck scale $\ell_p^{(3)}$, and $AdS$ curvature scale $\ell$ of the geometry are as follows:

\[
\frac{\ell_s}{\ell_p^{(3)}} = 4p\sqrt{k} \quad , \quad \frac{\ell}{\ell_p^{(3)}} = 4pk \quad , \quad \frac{\ell}{\ell_s} = \sqrt{k}.
\]

In addition, the six-dimensional string coupling is $g_6^2 = k/p$.\textsuperscript{10}

\textsuperscript{10}The six-dimensional string coupling is $g_6^2 = g_0^2 \ell_s^4/V_4$, where $V_4$ is the volume of $T^4$ or K3; there is a duality $g_6 \rightarrow 1/g_6$, and a condition $V_4/\ell_s < p/k$ for the description in terms of fundamental strings to be valid [8, 54].
The target space metric (suppressing the $T^4$ factor) is locally $AdS_3 \times S^3$

$$ds^2 = \left( -(r^2/\ell^2 + 1) \, dt^2 + dr^2 + r^2 \, d\phi^2 \right) + \ell^2 \left( \cos^2 \chi \, d\theta^2 + d\chi^2 + \sin^2 \chi \, d\psi^2 \right) \quad (112)$$

with the radial coordinate $r$ related to $\rho$ of the metric (9) by $r = \ell \sinh \rho$. The orbifold identification $(\phi, \psi) \sim (\phi + \frac{2\pi}{N}, \psi - \frac{2\pi}{N})$ results in a spacetime with a conical defect. Beginning with the metric (112), and making the orbifold quotient, one should rewrite the metric in terms of angular coordinates with period $2\pi$; this is accomplished by defining new coordinates

$$\tilde{\phi} = N\phi \; , \; \tilde{t} = Nt \; , \; \tilde{r} = r/N \; , \; \tilde{\psi} = (\psi + \phi) \; , \quad (113)$$

so that the metric takes the form

$$ds^2 = \left( -(\tilde{r}^2/\ell^2 + N^{-2}) \, d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 \, d\tilde{\phi}^2 \right)$$

$$+ \ell^2 \left( \cos^2 \chi \, d\theta^2 + d\chi^2 + \sin^2 \chi \, (d\tilde{\psi} - \frac{1}{N}d\tilde{\phi})^2 \right) \quad (114)$$

The geometry has a conical defect at the origin of the $AdS$ spatial directions. In terms of these new coordinates, there are off-diagonal $d\tilde{\phi}d\tilde{\psi}$ terms in the metric; the spacetime is not really a direct product of $AdS_3$ and $S^3$. This metric structure is somewhat different from the near-horizon geometry of rotating black holes and supersymmetric conical defects [55, 11, 12, 13], which also have the local $AdS_3 \times S^3$ form (112); for example, the BPS conical defect geometry is written in terms of globally well defined (i.e. periodic with period $2\pi$) coordinates on the three-sphere $\tilde{\theta}, \tilde{\psi}$ related to the coordinates in (112) by

$$\tilde{t} = t/\gamma \; , \; \tilde{\phi} = \phi/\gamma \; , \; \tilde{r} = \gamma r \; , \; \tilde{\theta} = \theta + t/\ell \; , \; \tilde{\psi} = \psi + \phi \; ; \quad (115)$$

in other words, in addition to $d\tilde{\phi}d\tilde{\psi}$ cross terms in the metric, there are angular velocity terms $d\tilde{\theta}d\tilde{\psi}$. The $AdS_3$ cone angle and $S^3$ angular velocity are both proportional to $\gamma$.

The general BTZ metric may be written

$$ds_3^2 = -N_t^2 \, dt_{BTZ}^2 + N_\phi^2 \, d\phi_{BTZ}^2 + r_{BTZ}^2 (d\phi_{BTZ} - N_\phi \, dt_{BTZ})^2$$

$$N_t^2 = \frac{r_{BTZ}^2}{\ell^2} - 8\ell_p^{(3)} M + \frac{16\ell_p^{(3)} M_{AdS}}{r_{BTZ}^2} \quad (116)$$

$$N_\phi = \frac{4\ell_p^{(3)} L_{AdS}}{r_{BTZ}^2} \; .$$

Geometries with $M > 0$ are black holes, while those with $M < 0$ are conical defects. The deficit angle in the $AdS_3$ geometry of the orbifold is $2\pi (1 - 1/N)$; comparing (114) with (116), we see that this corresponds to a mass

$$\ell M = -\frac{k p}{2N^2} \quad (117)$$
below the BTZ black hole threshold $M = 0$. We get closest to the BTZ threshold by
taking $N = k$, in which case $\ell M = -p/2k = -\frac{1}{2}\ell \theta^2$. One is thus still a macroscopic
distance in configuration space from the BTZ black hole threshold. Note that the
effect of the orbifold is to rescale the mass of the ‘vacuum’ by $1/N^2$. Another way
to see this is to use the residual superVirasoro symmetry (109). The ‘vacuum’ is
the $1/N$ spectral flow from the extremal Ramond sector ground state with $\tilde{L}_0 = 0$,
$\tilde{T}_0^3 = \frac{\tilde{c}}{12} = \frac{\tilde{c}}{12N}$. Spectral flow induces the shifts

$$
\begin{align*}
\tilde{L}_0' &= \tilde{L}_0 + \frac{\tilde{c}}{24} \eta^2 + \tilde{T}_0^3 \eta \\
\tilde{T}_0^3' &= \tilde{T}_0^3 + \frac{\tilde{c}}{12} \eta,
\end{align*}
$$

so that the orbifold ‘vacuum’ with $\eta = -1/N$ has charge $\tilde{T}_0^3 = 0$ and energy $\tilde{L}_0 =
-\frac{\tilde{c}}{12N^2}$. Note that this energy can also be written as $L_0 = -\frac{\tilde{c}}{12}$, the energy of the
$SL(2)$ invariant state before the orbifold.

BTZ black holes may be constructed by dumping energy into the system. In the
unorbifolded theory, these have an entropy

$$
S = \frac{A}{4\ell_p^{(3)}} = \pi \left[ \frac{\ell(\ell M + L_{AdS})}{2\ell_p^{(3)}} \right]^{\frac{1}{2}} + \pi \left[ \frac{\ell(\ell M - L_{AdS})}{2\ell_p^{(3)}} \right]^{\frac{1}{2}} = 2\pi \left( \sqrt{pk\tilde{L}_0} + \sqrt{pk\bar{\tilde{L}}_0} \right).
$$

(119)

The orbifold identification reduces the horizon area by a factor of $N$, and rescales
the mass $M$ and $AdS_3$ angular momentum $L_{AdS}$ by a factor of $N^{-2}$ (which is easily
seen from the coordinate rescalings (113) of the BTZ metric (116)); the Virasoro
generators are also rescaled by a factor of $N$, see equation (109). Thus after the
orbifold quotient we expect the entropy to be

$$
S \sim 2\pi \left( \sqrt{pk\tilde{L}_0/N} + \sqrt{pk\bar{\tilde{L}}_0/N} \right).
$$

(120)

In an effective string picture [56], one might think of the reduction in entropy as
being due to the fact that the energy gap in left and right moving excitations of the
long string is increased by a factor of $N$. Note that, for extremal black holes (e.g.
$\tilde{L}_0 = 0$), the quantity under the square root is a product of integers, since $N$ divides
$k$ and $\tilde{L}_0$ takes integer values due to the quantization of $L_{AdS}$.

Quantum gravity effects in $AdS_3$ are thus suppressed after orbifolding effectively
by $\ell/\ell_p^{(3)} = 4pk/N$. If we were starting to see the black hole phase in the perturba-
tive orbifold, we would have to have $4pk/N$ massless 2d scalar fields and their
superpartners, whereas we only get $4(N-1)$ from the orbifold. The best we can
do is when $k = N$, where the orbifold has $4(k-1)$ massless superfields confined to
the orbifold singularity, while black holes have the entropy of $4p$ scalars, and $p \gg k$
for weak string coupling. There thus appears to be a kind of correspondence princi-
ple [57] operating, where we approach the black hole density of states precisely when
we pass to strong coupling.\(^{11}\) Of course, the nonsupersymmetric orbifold (where we
quotient only \(AdS_3\)) gets arbitrarily close to the BTZ threshold, since \(N\) need not divide \(k\), but the number of tachyonic multiplets is \(N - 1\), so the system is highly
unstable. In this case it is not clear how to disentangle the large number of light
modes near the orbifold singularity, which one might want to associate with black
hole degrees of freedom, from the instability of the background spacetime there.

Even though we cannot approach the density of states of the BTZ black hole in
a regime where low-energy supergravity is reliable, the \(N = k\) orbifold does have a
large number of degrees of freedom concentrated in the vicinity of the orbifold
singularity, and it will be interesting to see to what extent the properties of the
orbifold parallel those of the black hole that arises at higher mass.

Brown and Henneaux [15] have shown that any three dimensional geometry that
is asymptotically locally \(AdS_3\) has a set of asymptotic diffeomorphisms that obey the
Virasoro algebra with central charge \(c_{BH} = 3\ell/2L_p^3 = 6pk\). Strominger [16] argued
that the entropy of \(AdS_3\) black holes could be written in terms of that central charge
and the conformal weights as

\[
S = 2\pi \left( \sqrt{c_{\text{eff}} L_0 / 6} + \sqrt{\tilde{c}_{\text{eff}} \tilde{L}_0 / 6} \right)
\]

with \(c_{\text{eff}} = c_{BH}\). The orbifold considered here is somewhat peculiar; the spacetime
CFT is a representation of the truncated algebra (109) of central charge \(\tilde{c} = NC_{BH}\),
while the black hole entropy (120) has a density of states controlled by the effective
central charge \(c_{\text{eff}} = c_{BH}/N\). Thus neither the central charge of the spacetime
Virasoro algebra of the orbifold theory, nor the effective central charge in the entropy
formula, is equal to the Brown-Henneaux value.

The arguments of [16] are rather general, claimed to apply to any spacetime which
has an asymptotic \(AdS_3\) structure. Assumptions underlying that analysis include the
existence of a conventional unitary CFT, with \(SL(2)\) invariant ground state, which
serves as the definition of quantum gravity for a particular superselection sector (i.e.
asymptotic geometry of spacetime). The relation between central charge and the
asymptotic density of states \(\rho(M)\) then follows from the application of a modular
transformation to the finite temperature (Euclidean torus) partition function of
the spacetime CFT, followed by extraction of the leading contribution, which is
dominated by the lowest energy state of mass \(M_{\text{min}}\) in the transformed sum over
states (c.f. [17, 58]):\(^{12}\)

\[
Z_{\text{torus}}(\tau, \bar{\tau}) = \int dM dL_{\text{AdS}} \rho(M, L_{\text{AdS}}) \exp \left[ i\pi \tau (\ell M + L_{\text{AdS}}) + i\pi \bar{\tau} (\ell M - L_{\text{AdS}}) \right] \\
\sim \left| \exp \left[ -i\pi M_{\text{min}} / \tau \right] \right|^2 .
\]  

\(^{11}\)There is an interesting classical limit \(N \to \infty, \rho \gg k \gg N\), where the geometry (114) naively
becomes the extremal BTZ black hole. However, this is achieved by sending the Planck scale to
zero, so the limit is rather degenerate. In units of the 10d Planck scale, the black hole states are
really infinitely far away, as the above analysis shows.

\(^{12}\)In the conventions of this paper, where extremal black holes have zero mass.
Then performing an integral transform to isolate the density of states, a saddle point approximation to the integral yields the result \( \rho(M, L_{AdS}) = \exp[S] \), where \( S \) is given by (121) with \( c_{\text{eff}} = -12M_{\text{min}} \). In a unitary CFT, the lowest energy state is typically the \( SL(2) \) invariant vacuum, for which \( M_{\text{min}} = -c/12 \) with \( c \) equal to the Virasoro central charge; then the entropy is (121) with \( c_{\text{eff}} = c \).

It is not clear how much of this structure is present for the orbifold. The orbifold acts on the \( AdS_3 \) directions, and this is not an operation that is simply stated in a candidate dual CFT. The GKS formalism allows one to see the structure of perturbative excitations around a particular state in the spacetime CFT – the conical singularity. Usually the background is the \( SL(2) \) invariant vacuum of the CFT; here it is an excited state which is not \( SL(2) \) invariant. Much of the structure of the spacetime CFT is inaccessible to the GKS construction, because most of its states are macroscopically different from the one described by the worldsheet CFT. In the case of the orbifold, both a description of the states of the BTZ black hole regime and of an \( SL(2) \) invariant vacuum are lacking. It appears in the GKS formalism as though the effect of the orbifold is to allow only a subalgebra of Brown-Henneaux diffeomorphisms, which has the effect of increasing the effective central charge of the remaining subalgebra; on the other hand, the restrictions imposed by the orbifold identification would appear to reduce the number of microstates that could contribute to the entropy.

So we seem to have reached a conundrum – either there is some extension of the conformal algebra entirely invisible to the perturbative GKS construction, that restores the relation between Virasoro central charge and entropy; or the density of states is not that of a unitary conformal field theory. However, there is another possibility, which we believe is the resolution of the apparent discrepancy. In unitary conformal field theories where the lowest energy state is \( not \) the \( SL(2) \) invariant vacuum, but rather has energy \( M_{\text{min}} > -c/12 \), the asymptotic density of states is controlled by the effective central charge \( c_{\text{eff}} = -12M_{\text{min}} < c \) [17]. A canonical example is Liouville theory, for which \( c_{\text{eff}} = 1 \) regardless of the value of the central charge \( c \) in the Virasoro algebra. The orbifold background is \( not \) \( SL(2) \) invariant, and has energy \( M = -\tilde{c}/12 \cdot \frac{1}{N^2} \) with respect to the surviving Virasoro algebra generated by the \( \tilde{L}_n \). Assuming that this is the lowest energy state of the conformal field theory, one predicts an asymptotic density of states (121) with \( c_{\text{eff}} = \tilde{c}/N^2 = 6pk/N \), consistent with equation (120).

Following this line of reasoning, we predict that the orbifold represents a novel class of unitary spacetime conformal field theories, whose ground state is \( not \) invariant under the global \( SL(2) \) subalgebra of the conformal algebra, and whose entropy is less than that given by a naive application of the ‘Cardy formula’ (121) in which the effective central charge \( c_{\text{eff}} \) is taken to be the Virasoro central charge of the spacetime conformal field theory. Rather, \( c_{\text{eff}} \) is \( N^2 \) times smaller than the central charge \( \tilde{c} \) of the spacetime conformal algebra. Moreover, \( \tilde{c} \) is itself \( N \) times larger than the Brown-Henneaux value \( 3\ell/2\ell_p^{(3)} = 6pk \), because the orbifold projection removes all but every \( N^{th} \) generator of the Brown-Henneaux algebra from the spec-
trum. Related to all this is an interpretation of the unexcited orbifold background as the lowest energy state of the spacetime CFT, with energy $-\frac{c}{12} \frac{1}{N^2}$ relative to the extremal BTZ geometry.

8 Summary

We have shown that the $SL(2)$ and $SU(2)$ (super)parafermion theories [25, 50, 29] provide a natural formulation of the $\mathbb{Z}_N$ orbifold of string theory on $AdS_3 \times S^3$ (times $T^4$ or $K3$). The spacetime so constructed is a kind of embedding of an ALE singularity into $AdS_3 \times S^3$. Twisted sector vertex operators are built out of the parafermion highest weight states, and the spectral flow operation used [26, 28] to fill out the single-string spectrum extends to a fractional spectral flow generating radially oscillating strings that close only up to the orbifold identification. The $N-1$ twisted sectors provide $4(N - 1)$ new moduli of the spacetime CFT, analogous to the hyperKähler blowup modes of an ALE singularity.

We also explored the symmetry algebra of the spacetime CFT and the associated black hole entropy formula. It would appear that the orbifold spacetime is a representation of a fractionally moded $\mathcal{N} = (4, 4)$ superconformal algebra; the fractional moding can be obtained by $1/N$ spectral flow from Ramond boundary conditions. The central charge $\tilde{c}$ of this algebra is larger than the effective central charge $c_{\text{eff}}$ that appears in the entropy formula, as happens in Liouville theory [17]. The fact that the orbifold ‘vacuum’ is not invariant under the $SL(2)$ subalgebra of the spacetime conformal algebra, as in Liouville theory, is consistent with this interpretation, and yields the correct entropy if we assume that this state is the lowest energy state of the spacetime CFT.

The twisted sector states provide a set of excitations confined to the orbifold singularity. There is a macroscopic number of such excitations when $N$ is large, and it will be interesting to see to what extent these excitations mimic the behavior of the black hole states that appear at higher energy. Work in this direction is in progress.

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