Non–Commutative Gauge Theories and the Cosmological Constant

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We discuss the issue of the cosmological constant in non–commutative non-supersymmetric gauge theories. In particular, in orbifold field theories non–commutativity acts as a UV cut-off. We suggest that in these theories quantum corrections give rise to a vacuum energy $\rho$, that is controlled by the non–commutativity parameter $\theta$, $\rho \sim \frac{1}{\theta^2}$ (only a soft logarithmic dependence on the Planck scale survives). We demonstrate our claim in a two-loop computation in field theory and by certain higher loop examples. Based on general expressions from string theory, we suggest that the vacuum energy is controlled by non–commutativity to all orders in perturbation theory.

I. INTRODUCTION

Explaining the small observed value of the cosmological constant is one of the crucial problems in contemporary theoretical physics (see [1–3] for reviews). Although classically the value of the cosmological constant can be tuned to an arbitrarily small value, one cannot do so in a quantum theory. In general quantum fluctuations force one to relate the vacuum energy density to the UV cut-off scale in the theory, $\rho \sim \Lambda^4$. From the theoretical point of view, the most natural thing is to identify $\Lambda$ with the Planck scale. However, this yields a cosmological constant which is enormously bigger than the estimated value.

In supersymmetric theories the vacuum energy is exactly zero, due to the cancellation of fermionic and bosonic fluctuations. However, once supersymmetry is broken, “mixed” contributions of the form $\Lambda^2 \sum (m_B^2 - m_F^2)$ arise (where $m_B$ and $m_F$ are bosonic and fermionic masses respectively). Thus, although supersymmetry might reduce the value of the vacuum energy, the problem remains unsolved.

It was suggested that certain non-supersymmetric theories, called “orbifold field theories” [4], might lead to a small value of the vacuum energy. From the field theory point of view the reason

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is the following: the planar graphs of these theories are exactly the same as the planar graphs of
the parent supersymmetric theories [5]. Therefore, at the large $N$ limit, the value of the vacuum
energy is exactly zero. The problem is that there is no control on the non-planar graphs. In
particular, $U(1)$ contributions lead to a large vacuum energy $\rho \sim \frac{1}{N} \Lambda_P^4$.

In this note we combine the above mentioned approach with the suggestion that non–
commutativity might be a useful ingredient in the solution of the cosmological constant problem
[6,3]. In fact, non–commutativity can play the role of a UV cut–off for non-planar graphs [7].
In a generic non–commutative field theory this fact would not improve the situation. The pla-
nar graphs would contribute exactly as the ordinary theory, namely $\rho_{\text{planar}} \sim \Lambda_P^4$.
However, in non–commutative orbifold theory planar graphs are the same as in the parent supersymmetric
theory [5] and thus do not contribute to the vacuum energy. The contributions from non-planar
graphs are expected to be controlled by the non–commutativity parameter. Thus, in this setup,
there is the possibility to disentangle the cosmological constant problem from the value of the UV
cut–off.

As we shall see, the situation is a bit more involved than that. In fact, it is well known that the
presence of time–like non–commutativity implies a breakdown of unitarity [8], making the field
theory not consistent. Thus we are forced to consider four dimensional field theories with only
two non–commuting space directions $([x^1, x^2] = i \theta \neq 0)$ . In this case the presence of $\theta$ does not
completely regulate the loop integrals. In particular, UV divergences can appear in non–planar
diagram when the Moyal phases involve only internal (i. e. integrated) momenta. Of course this
is always the case for the vacuum bubbles, where there are no external legs. For example, the
first contribution to the vacuum energy coming from a non-planar diagram is at two loop level.
This contribution is, indeed, controlled by the non–commutativity scale, but in theories with
only space-space non–commutativity a soft logarithmic dependence on the Planck scale survives.
This behavior is not surprising: a logarithmic dependence on the UV cut-off is typical of two
dimensional theories and, because of the presence of space-like $\theta$, also in our four dimensional
theories UV divergences can come only from the commutative plane.

We suggest that this situation persists to all orders and that the leading value of the vacuum
energy in non-supersymmetric orbifold field theories is

$$\rho \sim \frac{1}{\theta^2} P(\lambda, \log \sqrt[4]{\theta} \Lambda_P) ,$$

(1.1)

where $P$ is a polynomial of the ’t Hooft coupling $\lambda = g^2 N$ and the log. Note that the leading term
of expression (1.1) is also suppressed by $\frac{1}{N^2}$ with respect to the usual result, since the vacuum
energy generically is $\sim N^2$.

Some remarks are necessary in order to make this statement more precise. First, eq. (1.1)
strongly depends on the UV/IR mixing and on the possibility of re-summing classes of non-planar
diagrams, as suggested in [6]. However, it turns out that this re-summation can not be performed
when the number of bosonic degrees of freedom in the adjoint representation exceeds that of
the adjoint fermions. This pathology was to be expected, since for these non–commutative field
theories the 1-loop dispersion relation signals that the trivial vacuum is quantum–mechanically
unstable [9]. In general, the nature of the vacuum state of the non–commutative field theories
has not been fully understood yet. However, in the case where there are more fermionic fields in
the adjoint representation than bosonic ones, the re-summation proposed by [6] can be performed
without creating any evident instability. Thus, for these theories, we will take a pragmatic ap-
approach: we will consider the expansion around the usual vacuum and analyze the generic degree
of divergence of the vacuum bubbles.

In addition, in the usual commutative case vacuum bubbles are immediately related to the cos-
mological constant because of Lorentz invariance. On the contrary, in non–commutative theories,
this invariance is explicitly broken and the v.e.v. of the matter energy–momentum tensor can take a more general form $\langle T_{\mu\nu} \rangle = -\rho g_{\mu\nu} + \sigma (\theta^{-2})_{\mu\nu}$. In this setup vacuum diagrams correspond
to the combination of the above two terms. We will not separate the contribution related to $\rho$,
because we are interested in the dependence on $\Lambda_P$, that is of the form of eq. (1.1) for both $\rho$ and
$\sigma$. 

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Moreover, there are “phenomenological” obstacles that generally make the relation between non–commutative theories and the real world difficult. A realization of a consistent quantum NC $SU(N)$ model is not at hand at the moment (see however [10] for a recent attempt in this direction). In particular, the baryon number is always gauged in non–commutative theories. In addition, due to UV/IR mixing it is not clear whether a non-supersymmetric non–commutative theory would flow in the IR to its commutative counterpart. A recent analysis of the non-commutative $\mathcal{N} = 2$ model suggests that this indeed might be the case [11,12]. The running of the effective coupling in non-commutative $\mathcal{N} = 1$ SQCD also supports such a scenario [13].

Finally, the cosmological constant receives contributions also from the gravity sector (closed strings) beyond those coming from the gauge theory sector (open string). The non–commutativity we consider does not affect gravitational interactions (closed string amplitudes) and thus $\theta$ cannot act as a regulator for those contributions.

The organization of this manuscript is as follows. In Section 2 we briefly recall the main features of the orbifold field theories derived from the supersymmetric gauge theories. In Section 3 we focus on vacuum diagrams and argue that, in these theories, the non–commutative parameter plays a crucial role in regulating the UV divergences. In Section 4 we use the D-brane picture of the orbifold field theories to discuss the general structure of the vacuum diagrams.

II. ORBITFOLD FIELD THEORIES

Orbifold field theories are obtained by a certain truncation of a supersymmetric gauge theory. For instance, let us consider the special case of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The truncation procedure is as follows: consider a discrete subgroup $\Gamma$ of the $\mathcal{N} = 4$ R-symmetry group $SU(4)$. For each element of the orbifold group, the regular representation $\gamma$ inside $U(|\Gamma|) \otimes SU(N)$ should be specified. Each field $\Phi$ transform as $\Phi \rightarrow r\gamma^\dagger \Phi \gamma$, where $r$ is a representation matrix inside the R-symmetry group. The truncation is achieved by keeping invariant fields. The resulting theory has a reduced amount of supersymmetry, or no supersymmetry at all. It was suggested [4], based on the AdS/CFT conjecture, that the truncated large $N$ theories are finite as the parent $\mathcal{N} = 4$ theory. Later it was proved [5] that the planar diagrams of the truncated theory and parent theories are identical.

Let us consider a specific example [14]. The example is an $U(N) \times U(N)$ gauge theory with 6 scalars in the adjoint of each of the gauge groups and 4 Weyl fermions in the $(N, \bar{N})$ and 4 Weyl fermions in the $(\bar{N}, N)$ bi-fundamental representations. This is the theory that lives on dyonic D3 branes of type 0 string theory and can be also understood, from the field theory point of view, as a $\mathbb{Z}_2$ orbifold projection of $\mathcal{N} = 4$ SYM [15]. Other examples of large $N$ finite non-supersymmetric theories can be found in [16,17].

In all these examples there are more bosonic fields in the adjoint representation than fermionic fields in the adjoint representation. As was already mentioned in the introduction, the non–commutative version of these field theories displays “tachyonic” instabilities [9] and is not useful for our purposes. Thus we wish to give an example of an orbifold field theory with more fermionic fields in the adjoint representation than bosonic ones. We will start from a $U(2N) \mathcal{N} = 1$ SYM with chiral multiplets and no superpotential as a parent theory, and perform a projection similar to the one just described to get a $U(N) \times U(N)$ gauge theory. The content of the “vector multiplet” in the orbifolded theory is a vector in the adjoint of each of the group factor and a fermion in the bi-fundamental representation. In addition we can have $F$ copies of “chiral multiplets” with fermions in the adjoint of each group and scalars in the bi-fundamental. We do not add a superpotential to the action. The planar sector of this theory is in one–to–one correspondence with $\mathcal{N} = 1$ SYM with $F$ chiral multiplets. Note also that when $F > 1$ we have more fermions than bosons in the adjoint representation.
III. FIELD THEORY ANALYSIS

In this section we would like to see how a small vacuum energy can be achieved in a world with large non–commutativity. The trace of the v.e.v. of the energy–momentum tensor is related to the partition function as follows

\[-iV\tau = \log Z,\]  
\[(3.1)\]

where \(V\) is the volume of the system.

For a free field theory, it is possible to perform the Gaussian integration and to calculate \(\tau\). The bosons and fermions contribute with opposite signs. The result is

\[\tau^{(1)} = 2(N_B - N_F)N^2 \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log k^2,\]  
\[(3.2)\]

and can be viewed, diagrammatically, as a sum over configurations of bosonic or fermionic loops. \(N_B\) and \(N_F\) count the bosonic and the fermionic degrees of freedom respectively. In the specific case of the \(U(N) \times U(N)\) theory related to \(N = 4\) one has \(N_B = N_F = 8\) (six scalars plus the two physical degrees of freedom of the gluon for the bosons and two physical d.o.f. for each one of the four Weyl fermions), while for the example related to \(N = 1\) one has \(N_B = N_F = 2 + 2F\). This pattern is general to all orbifold field theories: the number of bosons and fermions is the same and the cosmological constant vanishes at the one loop level.

Let us turn now to the two loop calculation. The various contributions are described in figure (1) below.

![Diagram of two loops contributions to the vacuum energy](image)

FIG. 1. Two loops contributions to the vacuum energy. Solid, wavy, dashed and dotted lines represent spinors, gluons, scalars and ghosts respectively.

In the parent non–commutative theory, the sum of all contributions vanishes as required by supersymmetry. In the case of the orbifold theory, the situation is different. Some fields are in the bi-fundamental representation whereas others are in the adjoint and the Feynman rules for the two representations are clearly different. For example: the vertex of a gluon coupled to fundamental matter is

\[V_{\text{fund}} = g \times T^{a}_{ij} \times \exp \left( \frac{i}{2} p \theta q \right),\]  
\[(3.3)\]

while for a gluon coupled to an adjoint matter the vertex is

\[V_{\text{adj}} = \frac{1}{2} g \times \left( \text{tr} \left[ T^{a}, T^{b} \right] T^{c} \times \cos \left( \frac{i}{2} p \theta q \right) + i \text{ tr} \left\{ T^{a}, T^{b} \right\} T^{c} \times \sin \left( \frac{i}{2} p \theta q \right) \right).\]  
\[(3.4)\]

For a full list of Feynman rules, in the case of adjoint matter see [18,19]. Let us start by focusing on the planar graphs. The sum of these contributions takes the following form
\[ \tau_{\text{planar}}^{(2)} = \frac{1}{2} \, N_B \left( N_B - N_F \right) \, N^2 (i \lambda) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \, \frac{-i}{p^2 \, q^2} , \]  

(3.5)

The vacuum energy vanishes in the planar limit, as required by the properties inherited from the original supersymmetric theory. The contribution related to the non–planar diagrams breaks this cancellation. In 't Hooft’s two index notation, the reason for this non-vanishing contribution is the presence of non-planar diagrams which exist only for the case of fields in the adjoint representation. On the contrary, fields in the bi-fundamental representation cannot give rise to any non-planar vertex. Thus, the only 2-loop non-planar diagrams come from the adjoint sector of the theory and the sum of these contributions is

\[ \tau_{\text{non-planar}}^{(2)} = -\frac{1}{2} \lambda \, N_A^A \left( N_A^A - N_F^A \right) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{1}{p^2 \, q^2} \, e^{ip \theta q} , \]  

(3.6)

where \( N_A^A \) is the number of bosons in the adjoint representation and \( N_F^A \) is the number of fermions in adjoint representation. In the specific example of the \( \mathcal{N} = 4 \) daughter theory, \( N_A^B = 8 \) and \( N_F^A = 0 \) and hence \( N_A^A - N_F^A > 0 \), whereas in the example of the \( \mathcal{N} = 1 \) daughter theory \( N_A^B = 2, N_F^A = 2F \) and hence \( N_A^A - N_F^A < 0 \) for \( F > 1 \). As we shall see, this difference in sign will be important later on.

It is convenient (and useful for comparison with string theory amplitudes) to use the Schwinger parameterization in order to calculate (3.6)

\[ \tau_{\text{non-planar}}^{(2)} = -\frac{1}{2} \frac{\lambda}{(4\pi)^2} \, N_A^A \left( N_A^A - N_F^A \right) \int_0^\infty dt_1 dt_2 \frac{1}{t_1 t_2 (t_1 t_2 + \frac{\pi^2}{t})} \]  

(3.7)

The integral (3.7) is both UV divergent (small \( t \)) and IR divergent. The IR divergences are not manifest in (3.7), but exist due to massless degrees of freedom (see (3.6)). In order to regulate (3.7), both UV and IR cut-offs are needed. In principle a regulator which preserves gauge invariance should be used. However, since we know that in the supersymmetric case the sum of all the contributions should vanish, all the results reported in this work are determined unambiguously by using a naive cut-off. Once the integral is regulated, the vacuum energy is suppressed by \( \frac{1}{\theta^2} \).

Note also that the UV divergences are only logarithmic due to the presence of \( \theta \). Let us ignore for a moment the logarithmic IR divergences. The result is

\[ \tau^{(2)} \sim -\lambda \frac{N_A^A (N_A^A - N_F^A)}{\theta^2} \left( \log \sqrt{\theta} \Lambda_F \right)^2 \]  

(3.8)

We would like to give another example. The example is a 4-loops non-planar diagram which involves quartic interactions. Although we do not consider all other 4-loop contributions (and even all 3-loop contributions), we compute this diagram since it captures the nature of diagrams with maximal non-planarity. Maximal non-planarity means that the determinant of the intersection matrix, describing which propagators are crossing, is non-vanishing. In the next section we will give a string picture of the field theory perturbation expansion in the non–commutative case and it will be more clear why these diagrams have a special status. The non-planar 2-loop diagrams and the 4-loop example given in figure (2) below are of this type.

FIG. 2. A four loop example which involves three quartic vertices.
Upon integration over $q$ of the form
\[\int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \left( \frac{1}{q^2} \right)^2 \frac{1}{q^2 e^{ip\theta q}} \]
which yields a contribution that is suppressed by higher powers of $N$ and $\theta$ with respect to the two loop one (figure (1)). In addition, due to UV/IR mixing effects, there is a contribution to (3.9) of the form
\[
\Lambda_p^2 \left( \log(1 + \frac{1}{\theta^2 m^2 \Lambda_p^2}) \right)^2,
\]
which we can neglect. This is justified if we assume that $\theta m \Lambda_p \gg 1$ (where $m$ is the IR cut-off) and, in this limit, only negative power of $\Lambda_p$ are generated.

The general rule is that higher genus maximal non-planar diagram are suppressed by both powers of $N^2$ and $\theta$.

So far we ignored the infra-red divergences appearing in the integrals over Schwinger parameters. The reason is that these IR divergences might disappear after re-summation of higher order contributions, as suggested by [6]. Consider the non-planar diagram in figure (3) below.

![Higher loop diagram](image)

FIG. 3. Higher loop contribution to the vacuum energy. The dashed lines represents scalars. The filled bubbles represent a loop with either bosons or fermions.

This is a typical higher loop diagram. Denoting the internal momenta in the dashed loops by $q_i$ and the momentum in the “overall” loop by $p$, the potentially divergent contribution takes the form
\[
((N_B^A - N_F^A)\lambda)^n \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \cdots \frac{d^4 q_n}{(2\pi)^4} \left( \frac{1}{p^2} \right)^n \frac{1}{q_1^2} \frac{1}{q_2^2} \cdots \frac{1}{q_n^2} e^{ip\theta \sum_i q_i},
\]
(3.11)

Upon integration over $q_i$, (3.11) takes the following form
\[
((N_B^A - N_F^A)\lambda)^n \int \frac{d^4 p}{(2\pi)^4} \left( \frac{1}{p^2 e^{ip\theta}} \right)^n \sim
\]
\[
((N_B^A - N_F^A)\lambda)^n \int dt_1 dt_2 \left( \frac{t_1}{t_2} \right)^{n-1} \frac{1}{t_1 t_2 (t_1 t_2 + \frac{q^2}{4})}. \]
(3.12)

At any order $n$ the contribution (3.12) seems to be severely IR and UV divergent divergent. However, before integrating over $t_1, t_2$ let us re-sum all orders. This is possible only when $N_F^A >
\(N_B^A\), since only in this case does the series have alternating signs, allowing us to arrive to a meaningful finite answer. Otherwise we will obtain a divergent answer. The result is

\[
\sum_{n=1}^{\infty} ((N_B^A - N_B^B) / \lambda)^n \int dt_1 dt_2 \left( \frac{t_1}{t_2} \right)^{n-1} \frac{1}{t_1 t_2 (t_1 t_2 + \frac{\theta^2}{4})} \sim -\lambda \frac{1}{\theta^2} \log(\sqrt{\theta} \Lambda) \tag{3.13}
\]

The integral is (3.13) logarithmically divergent. It suggests that upon summation all orders the vacuum energy is indeed (1.1).

Following [6], we suggest that the scenario that was mentioned above is general: in each case where non-planarity introduces IR divergences, these divergences would disappear upon re-summation (or would result in at most as logarithmic IR divergences). The algorithm is the following: each non-planar graph can be “opened” by cutting a propagator, then it can be glued to itself \(n - 1\) times and finally it can be “closed” by gluing the ends of the propagator and integrating over the momentum flow. A re-summation of all orders graphs should result with at most logarithmic divergences. This procedure makes sense only in theories with \(N_B^B > N_B^A\).

Another important issue is the existence of mixed planar and non-planar graphs. There is a danger that in such a case there will be quadratic UV divergences. However, since the parent theory is supersymmetric, this problem is avoided. The reason is that a generic mixed diagram can be divided to its planar and non-planar parts connected by propagators. Let us assume, for simplicity, that the propagators are bosonic. Each propagator carries a factor of \(1/p^2\). The planar part must be \(\sim p^2 \log p / \Lambda\), since this sector is supersymmetric and therefore diverges, at most, logarithmically. It means that the two propagators and the planar piece together contributes \(\sim 1/p^2 \log p / \Lambda\). Therefore they can be substituted by a single propagator (for the sake of finding the \(\theta\) dependence of the whole graph). It means that a non-planar graph connected to a planar piece behaves as a fully non-planar graph. This argument holds also for planar and non-planar graphs which are connected by fermionic propagators or by ghosts. In the case of fermionic propagators, the planar part of the graph diverges logarithmically due to supersymmetry. In the case of ghosts, the reasoning is different: the planar graph must diverge at most logarithmically in order to preserve gauge invariance.

**IV. STRING THEORY CALCULATIONS**

It is well known that non–commutative field theories can be nicely embedded in string theory [20]. This means that, also in the non–commutative case, it is possible to perform a particular decoupling limit on various string quantities and recover the corresponding field theoretic results. Indeed the string setup seems to be the most natural one for studying non–commutativity. In fact, at a field theory level the presence of non–commutativity requires to modify the form of the interactions terms in the microscopic Lagrangian. In a string setup, on the contrary, the building blocks, like the 3-string vertex operator, are unmodified [24], and, in order to recover non–commutativity, it is sufficient to expand the usual theory around a slightly different vacuum containing a constant \(B\)-field. In the decoupling limit \((\alpha' \rightarrow 0)\), a particular combination of \(B\) and \(\alpha'\) is kept fixed and gives rise to the non–commutative parameter \(\theta_{\mu \nu}\).

From the world-sheet point of view the presence of \(B\) just affects the commutations relations among the string modes [21] and the nontrivial modification is concentrated in the zero-modes part. Because of this, it was possible to derive [24], at least in the bosonic case, a master formula which contains any loop string amplitude in presence of a background \(B\)-field. The same decoupling limit used at tree level can be applied on this formula and many explicit checks [22,24] showed that in this way one can recover exactly the Feynman diagrams of various field theories. Thus, the embedding of non–commutative field theory in a string setup is not only useful at a conceptual level, but also represents a simplifying technique for the computation of perturbative amplitudes. This is the same pattern that emerged after the first string revolution in the study of the usual gauge and gravity interactions [25].
In this section we will use the general result derived at the string level to study the main features of the higher loops vacuum bubbles. Even if the string formula we will use takes into account only bosonic degrees of freedom, it is able to capture the relevant properties of the non-planar diagrams. At each order in the string perturbative expansion, all the vacuum graphs are re-summed in few Riemann surfaces. Remembering that we need to consider only oriented open strings, at the first order we encounter just the annulus which, in the \( \alpha' \to 0 \) limit, degenerates into a planar diagram. At the next order we have two possibilities: a planar disk with two holes and a non-planar surface where two propagators have to cross, see Figure (4).

![Figure 4](image)

**FIG. 4.** The oriented two–loops surfaces. The one at the left has 3 borders, while the one at the right has only one border.

In general the string result depends on two topological properties of the surfaces: the number of loops \( h \) (or the number of independent momenta flowing in the propagators) and the number of borders (or equivalently the intersection matrix \( J_{IJ} \) among the internal momenta). In a compact form the contribution to the vacuum bubbles can be written as

\[
V^{\theta}_{0;h} = \left[ \sqrt{\det(M)} \right]^{1-h} \int \left[ \det \left( -\frac{A}{2} \right) \right]^{-1/2} [dm]_h ,
\]

where the integration is taken over the moduli of the Riemann surface and \( M \) is the open string metric \( M_{\mu\nu} = g_{\mu\nu} - (Bg^{-1}B)_{\mu\nu} \). A contains the period matrix \( \tau_{IJ} \), the non–commutativity parameter \( \Theta^{\mu\nu} = 2\pi\alpha' \left( \frac{1}{g+1}B \frac{1}{g-1} \right)^{\mu\nu} \) and the intersection matrix \( J_{IJ} \)

\[
A_{IJ}^{\mu\nu} = 2\alpha' (2\pi i\tau_{IJ}) (M^{-1})^{\mu\nu} - i\theta^{\mu\nu} J_{IJ} .
\]

In order to make the comparison with the field theory result simpler, one can rescale the string coordinates and simply use \( \eta^{\mu\nu} \) as open string metric and \( \theta^{\mu\nu} = 2\pi\alpha' B^{\mu\nu} \) as noncommutative parameter. It is important to stress that, in (4.1), the determinant is taken over the space of both Lorentz (\( \mu \)) and loop indices (\( I \)). Just to give a flavor of how this expression can be made explicit we report the expression for the measure in the case of bosonic string written in the Schottky parameterization of the world–sheet surface and refer to [26] for the derivation and the explicit form of the other quantities

\[
[dm]_h = \frac{1}{dV_{abc}} \prod_{I=1}^{h} \left[ \frac{dk_I d\xi_I d\eta_I}{k_I^2 (\xi_I - \eta_I)^2 (1 - k_I)^2} \right] \prod_{\alpha} \left[ \frac{\prod_{n=1}^{\infty} (1 - k^{\alpha}_n)^{-d+2}}{(1 - k^\alpha)^2} \right] .
\]

For each hole one needs to introduce 3 real parameters \( (\eta_I, \xi_I, k_I) \) that specify its position and its width. The \( k^\alpha_\mu \)'s are the multipliers of the other elements of the Schottky group and can be written as function of the elementary parameters \( (\eta_I, \xi_I, k_I) \), related to the generators of the group. Finally \( dV_{abc} \) remembers that, because of the projective invariance, one has to fix the values of three of these variables. Thus at \( h \) loops, one has \( 3h - 3 \) integrations which is exactly the number of propagator one has in a \( \phi^3 \)–like vacuum bubble. In fact for each propagator one can introduce a Schwinger parameter and the moduli of the Riemann surface are generically connected to these parameters by relations like \( f(\eta_I, \xi_I, k_I) \sim \exp (-t/\alpha') \). The exact form of such relations depends on the specific corner of the moduli space one is looking at. The contribution of each corner resum the results of the Feynman diagrams with the same topology. This one-to-one mapping between string and field theoretic result is pictorially intuitive. For instance, it is possible to recover also
diagrams with quartic interactions, both in the usual [27] and in the non–commutative case [23], when some of the string moduli are related to Schwinger parameters of finite length in unit of $\alpha'$, which means they are vanishing in the field theory limit. Actually, the case of diagrams with only four–point vertices is the easiest, because all the factors of $\alpha'$ in the string formula cancel after the change of variable from the string moduli to the Schwinger parameter is performed [28]. Thus it is sufficient to consider only the leading order in $\alpha'$ in all the string expressions appearing in (4.1).

For instance, let see how the diagrams in Figure (4), reduces to the field theory diagrams of Figure (1). In this case $A$ is a $2 \times 2$ matrix with respect to the loop indices. In the $SL(2, R)$ gauge $\xi_1 = \infty$, $\eta_1 = 0$ and $\xi_2 = 1$, the leading of the matrix $A$ entering in the non–planar diagram is

$$A = 2\alpha' \left( -\eta^{\mu\nu} \log k_1 \eta^{\mu\nu} \log \eta_2 + \frac{\alpha}{2\alpha'} \log k_2 \right) + \ldots$$  \hfill (4.4)$$

while for the planar diagram, one get a similar result, but without $\theta_{\mu\nu}$ terms in the off-diagonal entries since the intersection matrix is zero. In the case of quartic interactions, the propagator in the middle, related to $\log \eta_2$, has vanishing length $\alpha' \log \eta_2 \sim 0$, while $-\alpha' \log k_1 = t_i$. In particular, for the non–planar bubbles that correspond to a surface with $h = 2$, but with only one border ($b = 1$), one gets

$$V_{h=2,b=1} \sim \int dt_1 dt_2 \frac{1}{t_1 t_2 (t_1 t_2 + \frac{\alpha^2}{\theta})}. \hfill (4.5)$$

As we noticed, this integral has at most logarithmic UV divergences. In fact, the UV behavior is encoded in the region of small $t_i$'s. The presence of the space–space non–commutative parameter is sufficient to change the usual dependence on the Planck scale $\Lambda_P^4$ into a milder $1/\theta^2 \log \sqrt{\theta} \Lambda_P$. This pattern generalizes to all the maximal non–planar diagrams, that is those with arbitrary $h$, but always with one border (this request implies that $h$ is even). In fact, for quartic interactions one gets

$$V_{h,b=1} \sim \int \prod_{i=1}^{2h-2} dt_i \bigg/ P_0^{(h)}(t_i) \left( \sum_{j=0}^{h/2} \frac{\alpha^2}{\theta} j P_j^{(h-2j)}(t_i) \right), \hfill (4.6)$$

where $P_j^{(a)}$ are polynomials of order $a$ in the Schwinger parameters and are linear in each $t_i$. Here $P_0^{(0)}(t_i) = 1$. Thus, the presence of a term $\sim \theta^h$ regulates the contribution coming from the Gaussian integration over the plane where the non–commutativity is present. The other two directions give the usual result, but again the UV divergences are of logarithmic type $\sim dt_i/t_i$.

The presence of three–point vertices makes it more difficult to write a general formula for the vacuum bubbles and to read from it the singular behaviour for small $t_i$'s. In fact, in these diagrams there are more propagators in comparison to the case of four–point vertices and, thus, more Schwinger parameters. For instance, at two loops, one needs to associate a Schwinger parameter also to $\eta_2$ ($-\alpha' \log k_i = t_i + t_3$ and $-\alpha' \log \eta_2 = t_3$). This means that the change of variable from the string moduli to the field theory $t_i$'s generates extra factors of $1/\alpha'$ that need to be compensated by considering also the subleading terms in the string expression of $1/\sqrt{\det A}$. This expansion has two effects: it generates higher powers of the polynomial appearing in the denominator of (4.6) and a new polynomial at the numerator, that takes into account the dependence of the three-point vertices on the inflowing momenta. For instance, a generic 4–loop diagram with six three–point vertices will look like

$$V_{4,b=1} \sim \int \prod_{i=1}^{9} dt_i \frac{Q^{(9)}(t_i)}{P_0^{(4)}(t_i)} \left[ \sum_{j=0}^{7} \left( \frac{\alpha^2}{\theta} j P_j^{(4-2j)}(t_i) \right) \right]^x,$$ \hfill (4.7)

where $1 \leq x \leq 4$. It is clear that the most divergent contributions come from the terms with $x = 1$. We explicitly computed some diagrams of this kind and always found that $t$'s present
in the polynomial $Q^{(9)}$ cancel the power-like divergence of the denominator. We think that this pattern is general, because the presence of $\Lambda_P^2$ terms in the non-commutative computation would imply the existence of an unwanted $\Lambda_P^6$ contribution in the commutative case.

What we said up to now is valid for the maximal non-planar diagrams, that have only one border. From the string master formula it is clear that the diagrams with $b > 1$ are more divergent. In this case the determinant of the intersection matrix is vanishing and thus the regulating term in (4.6) with $\theta^b$ and no $t_i's$ is absent. In general, the maximal power of $\theta$ present is equal to the rank of the first minor with nonvanishing determinant. The diagram considered in Figure (3) is of this kind, like all those entering in the resummation procedure outlined in the previous section, after the first one that is maximal non-planar. As we already suggested, after resummation these power-like divergences should disappear. Clearly other diagrams with more than one border are those containing a planar part. In this case it is clear that the bosonic contribution will contain a power-like dependence on $\Lambda_P$; however, as we argued in the previous section, these terms disappear once also the fermion diagrams are taken into account, since the planar part of the whole diagram is supersymmetric.

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