Noncommutative Quantum Mechanics and rotating frames

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Abstract

We study the effect of noncommutativity of space on the physics of a quantum interferometer located in a rotating disk in a gauge field background. To this end, we develop a path-integral approach which allows defining an effective action from which relevant physical quantities can be computed as in the usual commutative case. For the specific case of a constant magnetic field, we are able to compute exactly the noncommutative Lagrangian and the associated shift on the interference pattern for any value of \( \theta \).

Introduction

The interest in noncommutative space, recently aroused in connection with developments in string theory [1]-[3], rapidly spread on other domains going from Quantum field theories and Quantum mechanics to Condensed matter physics [4]-[20] (See [21] for a complete list of references). Concerning quantum mechanical problems, since noncommutative physics can be connected with the dynamics of charged particles in a magnetic field (the Landau problem), many interesting results have been presented, going from the Aharonov-Bohm effect to the Quantum Hall effect [7]-[20].

The purpose of the present work is two fold. On the one hand, we want to discuss the specific quantum mechanical problem of a charged particle in a rotating disk, in the

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presence of an electromagnetic field, when space is the anticommutative plane. This is an interesting problem related to the Aharonov-Bohm effect, relevant to the physics of superconducting interferometers.

On the other hand, we want to develop a simple procedure to handle, within the path-integral approach, noncommutative quantum mechanical problems. The idea is to provide the Feynman path-integral alternative to the wave equation approach developed in [7]-[9]. This last approach is based in taking into account noncommutativity of the base space by using the so called * product when the potential in the Hamiltonian acts on the wave function. Now, at the Hamiltonian level, this amounts to an appropriate shift in the coordinate dependence of the potential (and no change in momenta) so that, finally, noncommutativity is encoded in the shifted potential through the noncommutative parameter $\theta_{ij}$. Our approach starts precisely at this point and makes use of the Feynman recipe for constructing, in phase space, the transition amplitude $Z$ for a quantum system in noncommutative space as an integral over trajectories. Now, for simple (quadratic both in $p$ and $x$) potentials, one can integrate over momenta ending with $Z$ written as a path-integral over $x$, with an effective action where noncommutativity manifest just through the parameter $\theta_{ij}$ appearing in the effective action.

Let us start by defining the Moyal $*$-product of functions on the noncommutative plane,

$$ (f * g)(x) = \exp \left( \frac{i}{2} \theta_{ij} \partial_x \partial_y \right) f(x)g(y) \bigg|_{y=x} $$

Here $\theta_{ij} = \theta \varepsilon_{ij}$ ($i, j = 1, 2$), with $\theta$ a real parameter with dimensions of $(length)^2$. The Moyal bracket is then defined as

$$ \{f(x), g(x)\} = (f * g)(x) - (g * f)(x) $$

Now, for $f = x^1$ and $g = x^2$, eq.(2) takes the form

$$ \{x^1, x^2\} = i\theta $$

which can be connected, using the Moyal-Weyl correspondence, with the operator algebra approach to noncommutative quantum mechanics where one starts from the commutation relation

$$ [\hat{x}^1, \hat{x}^2] = i\theta $$

The Heisenberg algebra is completed with the commutation relations (we put $\hbar = 1$)

$$ [\hat{p}^1, \hat{p}^2] = 0 $$

$$ [\hat{x}^i, \hat{p}^j] = \delta^{ij} $$

Note that eq.(4) coincides with the canonical commutation relation for a massless particle of unit charge when a constant magnetic field $B = 1/\theta$ is present.
The classical system

Let us consider a particle with mass $m$ and charge $q$ located in a disc rotating with constant angular velocity $\omega$, in the presence of a gauge field background $A_i$. It is an interesting system since, as we shall see, rotational effects are connected to magnetic ones, and the rotating disk introduces topological features equivalent to that resulting from a confined magnetic flux. In a region of a rotating frame that is not simply connected, the inertial forces can be cancelled without completely cancelling the inertial vector potential, and its presence can be detected in a quantum interference experiment as with the Aharonov-Bohm effect [22]. We shall construct here the Hamiltonian of such a system and then analyse the associated quantum problem in noncommutative space. Dynamics of such a classical system is governed by the Lagrangian

$$L = -m\sqrt{g_{ij}\dot{x}^i\dot{x}^j} - qA_i\dot{x}^i$$

(7)

For nonrelativistic velocities in an inertial frame, we can write (we take for the moment $c = 1$)

$$L = \frac{1}{2}mv^2 + q\vec{v} \cdot \vec{A}$$

(8)

We want to discuss the case of a constant magnetic field. In the ordinary (commutative) case, this can be very simply achieved by considering a gauge field of the form

$$A_i = \varepsilon_{ijk}B_jx_k$$

(9)

and then identifying $\vec{B}$ with a constant magnetic field (say in the $z$ direction) computed from $F_{12}$ (which coincides with the curl of $\vec{A}$). Now, in noncommutative space, the appropriate field strength, which in fact changes covariantly under noncommutative $U(1)$ gauge transformations, should be defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iq\{A_\mu, A_\nu\}$$

(10)

so that the gauge potential (9) yields a field strength with the only non-vanishing component in the form

$$F_{12} = 2B (1 + q\theta B/2)$$

(11)

As expected, $F_{12}$ coincides, to zeroth order in $\theta$, with $2B$. The term linear in $\theta$ modifies the commutative result giving, however, a field strength that is still constant and, a fortiori, gauge invariant.

Inserting (9) in Lagrangian (8) one has

$$L = \frac{1}{2}mv^2 + q\vec{v} \cdot \vec{B} \times \vec{r}$$

(12)

In order to write the Lagrangian in the rotating frame (with constant angular velocity $\omega$ which we take parallel to $\vec{B}$) one has just to change $\vec{v} \rightarrow \vec{v} + \vec{\omega} \times \vec{r}$ getting,

$$L = \frac{1}{2}mv^2 + m\vec{v} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2}m(\vec{\omega} \times \vec{r})^2 + q\vec{r} \cdot (\vec{v} \times \vec{B}) + q\vec{B} \cdot (\vec{r} \times (\vec{\omega} \times \vec{r}))$$

(13)
Now, if we define a vector field $\vec{V}$ such that

$$\vec{V} = \vec{\omega} \times \vec{r}$$  \hspace{1cm} (14)

so that the canonical momentum reads

$$\vec{P} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m\vec{V} + q\vec{A}$$  \hspace{1cm} (15)

With this, the Hamiltonian in the rotating frame takes the form

$$H = \frac{1}{2m}(\vec{P} - m\vec{V} - q\vec{B} \times \vec{r})^2 - \frac{m}{2}\vec{V}^2 - q\vec{V} \cdot (\vec{B} \times \vec{r})$$  \hspace{1cm} (16)

**The quantum system in noncommutative space: the path-integral approach**

We are now ready to investigate the path-integral approach to the quantum problem described by Hamiltonian (16), in noncommutative space. To this end, we shall proceed to the construction of the quantum transition amplitude using the Feynman integral over trajectories. As it is well known, this approach replaces the analysis of the wave equation for a system with quantum Hamiltonian $\hat{H}$ (we put $\hbar = 1$)

$$i\frac{\partial \psi(\vec{x}, t)}{\partial t} = \hat{H}\psi(\vec{x}, t)$$  \hspace{1cm} (17)

by the phase space path-integral $Z$ giving the transition amplitude between some given initial and final states

$$Z = \int D\vec{p}D\vec{x} \exp \left( i(\vec{p}\dot{\vec{x}} - H(\vec{p}, \vec{x})) \right)$$  \hspace{1cm} (18)

where $H(\vec{p}, \vec{x}) = \langle \vec{p} | \hat{H} | \vec{x} \rangle$.

In order to adapt the path-integral approach to the case of noncommutative space, let us consider, for definiteness, the simple case in which the system corresponds to a particle of mass $m$, in a potential $V(\vec{x})$, so that the quantum Hamiltonian is $\hat{H} = \vec{p}^2 / 2m + V(\vec{x})$. Within the Moyal product approach to noncommutative space theories, Schrödinger equation for such a system should be written as

$$i\frac{\partial \psi(\vec{x}, t)}{\partial t} = \frac{1}{2m}\vec{p}^2 \psi(\vec{x}, t) + V(\vec{x}) \ast \psi(\vec{x}, t)$$  \hspace{1cm} (19)

Then, noncommutativity just manifests through the potential term $V \ast \psi$. Now, as proposed in [7]-[9], one can replace this $\ast$ product by an ordinary one, provided the argument in $V$ is shifted according to

$$V(\vec{x}) \ast \psi(\vec{x}) = V(\vec{x} - \vec{\tilde{p}})\psi(\vec{x})$$  \hspace{1cm} (20)

with

$$\vec{\tilde{p}} = \frac{1}{2}\theta\varepsilon_{ij}p_j$$  \hspace{1cm} (21)
so that the equivalent wave equation reads
\[ i \frac{\partial \psi(x,t)}{\partial t} = \frac{1}{2m} \vec{p}^2 \psi(x,t) + V(x - \vec{p}) \psi(x,t) \equiv \hat{H}_{\text{eff}} \psi(x,t). \]

Eq.(22) can be seen as a “normal” (ordinary space) Schrodinger equation for a system with a modified Hamiltonian $\hat{H}_{\text{eff}}$. Then, one can apply the usual Feynman recipe to the system with Hamiltonian $\hat{H}_{\text{eff}}$, and write the transition amplitude in the form
\[ Z = \int D\vec{x} D\vec{p} \exp \left( i (\vec{p} \cdot \vec{x} - \hat{H}_{\text{eff}}(\vec{p},\vec{x})) \right) \]
with $H_{\text{eff}}(\vec{p},\vec{x}) = \langle \vec{p} | \hat{H}_{\text{eff}} | \vec{x} \rangle$. It is just in the shifted potential term in $H_{\text{eff}}$ where noncommutativity manifests. Depending on the form of the potential, which depends now on $\vec{p}$ because of the shift (20), the integral over momenta could be done in close form, leading to a Lagrangian version of $Z$.

In the case of Hamiltonian (16), the shift (20) amounts to
\[ A_i = -\varepsilon_{ij} B x_j \rightarrow -\varepsilon_{ij} B (x_j - \tilde{p}_j) \]
\[ V_i = -\varepsilon_{ij} \omega x_j \rightarrow -\varepsilon_{ij} \omega (x_j - \tilde{p}_j) \]

As a result, Hamiltonian $H_{\text{eff}}$ can be written in the form
\[ H_{\text{eff}} = \frac{1}{2m} \left( 1 + qB\theta/2 \right)^2 \left( \vec{p} - \frac{q\vec{A}}{1 + qB\theta/2} \right)^2 + \frac{1}{2} \omega p^2 \theta - \vec{p} \cdot \vec{\omega} \times \vec{r}. \]

In the present case, this expression can be used to define effective mass and charge resulting from deformation of space at the noncommutative scale [9],
\[ m_{\text{eff}} = \frac{m}{(1 + qB\theta/2)^2} \]
\[ q_{\text{eff}} = \frac{q}{1 + qB\theta/2}. \]

Being $H_{\text{eff}}$ quadratic in $\vec{p}$, one can integrate out the momenta in (23), this yielding to the Lagrangian version of the path-integral $Z$. The answer is
\[ Z = \int D\vec{x} \exp \left( i \int dt L_{\text{eff}} \right) \]
where the effective Lagrangian $L_{\text{eff}}$ is given by
\[ L_{\text{eff}} = \frac{1}{2m} \left( mv_i - \frac{(qB (1 + qB\theta/2) + m\omega)\varepsilon_{ij} x_j)^2}{m\omega \theta + (1 + qB\theta/2)^2} \right) - \frac{1}{2m} (qBx_i)^2. \]

Note that this is the exact expression for the Lagrangian, to all orders in $\theta$. As expected, it reduces to the classical one for $\theta = 0$. It is important to stress that applying the
noncommutative transformation defined in eq.(24) to the classical Lagrangian (13) does not yield the effective Lagrangian eq.(28). This is due to the fact that the former is obtained after path-integrating the momenta (this implying that factors in the numerator of the shifted Hamiltonian appear as denominators in the Lagrangian) and not just by a simple shift in the \( x \) variables.

Up to first order in \( \theta \), the effective Lagrangian can be written, in terms of vector fields, as

\[
L_{\theta}^0 = \frac{1}{2} m (v_i + V_i + 2 q A_i)(v_i + V_i) - \frac{1}{2} q m \theta_{jk} \left( \frac{\partial_j A_i}{m} \right)(v_i + V_i) + \frac{q}{m} A_i(v_i + V_i)
\]

\[
-\frac{1}{2} m^2 \theta_{jk} \left( \frac{\partial_j V_i}{m} \right)(v_i + V_i)(v_k + V_k) - \frac{q^2}{m^2} A_i A_k.
\]

Written in this way it is instructive to show the structure of the approximate noncommutative Lagrangian for a generic case.

Using a three-dimensional notation, one can further rewrite the first order expression compactly, as

\[
L_{\theta}^1 = -\frac{q m}{4 \hbar^2} \bar{\theta} \cdot \left( \vec{v} + \vec{V} \right) \times \nabla A_i \left( v_i + V_i + \frac{q}{m} A_i \right)
\]

\[
-\frac{m^2}{4 \hbar^2} \bar{\theta} \cdot \left( \vec{v} + \vec{V} + 2 \frac{q}{m} \vec{A} \right) \times \nabla V_i \left( v_i + V_i \right)
\]

\[
-\frac{q^2}{4 \hbar^2} \bar{\theta} \cdot \left( \vec{A} \times \nabla V_i \right) A_i.
\]

where we have defined \( \bar{\theta} = \epsilon_{ijk} \theta_{jk} \). One can easily see that eq.(30) coincides with the approximate (first order in \( \theta \)) result derived in [15] for the special case of \( V = 0 \). It should be stressed, however, that eq.(28) provides an exact form for the Lagrangian to be considered in the transition amplitude \( Z \) for the quantum noncommutative model.

The quantum interference device

We are now in conditions to discuss the quantum dynamics of charged particles in a rotating disk, in the presence of a gauge field background. In this way, by studying the interference pattern of the particles when a two slit device is put on a rotating disk, we shall be able to determine noncommutative effects in connection both with the gauge field and with the non-inertial frame. As we shall see, both effects interfere each other and provide a \( \theta \) shift which can be accurately calculated.

Let us start by observing that the phase shift \( \Delta \Phi \) between two electrons reaching a detector through different paths can be computed from the formula

\[
\Delta \Phi = \Delta \int_{t_i}^{t_f} dt L_{eff}
\]

where \( \Delta \) indicates subtraction between both integrals computed in the interval \( (t_i, t_f) \) that the particle takes from the source to the detector. For the particular case of a constant
the full Lagrangian eq.(28) takes the simple form

\[ L_{\text{eff}} = \alpha v_i^2 + \beta x_i^2 + \gamma v_i \epsilon_{ij} x_j \]  

(32)

with

\[
\alpha = \frac{m}{2f_\theta} \\
\beta = \frac{1}{2m} \left( \frac{g_\theta^2}{f_\theta} - q^2 B^2 \right) \\
\gamma = -\frac{g_\theta}{f_\theta} \\
f_\theta = 1 + \theta(m \omega + q B) + \theta^2 \frac{q^2 B^2}{4} \\
g_\theta = m \omega + q B + \theta \frac{q^2 B^2}{2}.
\]  

(33)

Now, the result of the integration in eq.(31) in terms of the de Broglie wavelength \( \lambda = 2\pi/p \) associated with the particle is

\[
\Delta \Phi = \frac{2\pi}{\lambda f_\theta} \Delta(d) + \frac{\lambda}{4\pi} \left( \frac{g_\theta^2}{f_\theta} - q^2 B^2 \right) \Delta(d^3) + \oint \vec{\gamma} \cdot d\vec{x}
\]  

(34)

where \( d \) is the distance from the source to the detector (thus, \( \Delta(d) \) represents the difference between the two paths to the same point in the detector) and \( \gamma_i = \gamma \epsilon_{ij} x_j \). While the first two terms do not depend on the flux of the fields, the last one does. Indeed, the \( \gamma \) factor in the third term depends on the area where the flux of magnetic and \( \lambda \) fields are nonzero. Suppose that one confines the \( B \) flux into a solenoid in the center of the rotating disk. Then the \( B \) part of the curl \( \vec{\gamma} \) flux would be multiplied by the area of the solenoid while the \( \omega \) part by the area defined by the path difference. In this way we would have an Aharonov-Bohm effect combined with a rotational effect. If the solenoid is very thin, and the magnetic field not too strong, then the relevant phase shift will depend on the angular velocity. Nevertheless, it must be noted that these effects are not only summed but also multiplied each other, (see for example eq.(30)).

In order to clearly distinguish the noncommutative contributions from those already present in the ordinary case, let us analyse the complete first order approximation, given by

\[
\Delta \Phi^{0,1} = -\frac{2\pi \Delta d}{\lambda} + \frac{\lambda \Delta d^3}{4\pi} (m^2 \omega^2 + 2q B m \omega) + 2(m \omega S_w + q B S_B) + \theta \left( \frac{2\pi \Delta d}{\lambda} (m \omega + q B) - \frac{\lambda \Delta d^3}{4\pi} (m \omega + q B)(m^2 \omega^2 + 2q B m \omega) - 2m^2 \omega^2 S_w - 2q B m \omega (S_w + S_B) - q^2 B^2 S_B \right),
\]  

(35)

where \( S_B \) and \( S_w \) are respectively the areas where the magnetic flux and \( \omega \) flux are nonzero (it must be noted that since \( S_w \) is defined by the particle’s contour, then it is
always \( S_B < S_w \)). The last term in the first line represents the usual Aharonov-Bohm contribution, while the second one is the rotational analog. In the third line, we find the corresponding noncommutative shifts, and their interference becomes apparent. The other terms do not depend on the topology of the device but we can also see both the ordinary and their noncommutative counterparts. Our result shows that the device can be used to exhibit the noncommutative shift in the Aharonov-Bohm effect, and introduces a new physical effect due to the interference of the two potential fields \( \vec{A} \) and \( \vec{V} \).

At this order, the \( \vec{V} = 0 \) phase shift is simply

\[
\Delta \Phi^{0,1}_{\vec{V}=0} = -\frac{2\pi \Delta d}{\lambda} (1 - qB\theta) - qBS_B(2 - qB\theta).
\]  

(36)

The accelerated interferometer could be also realized as a rotating SQUID (superconducting quantum interference device). In this case, irrespective of the external field distribution, the particle in the SQUID would not see any \( B \) field as a result of the Meissner effect, and thus no magnetic force would be measured in the rotating frame. Nevertheless, there is still a magnetic flux through the center of the SQUID, together with that related to \( \omega \), this amounting to the effect just described.

If on the other hand one has not a strong magnetic field or it is confined to a thin center hole in the SQUID, then the \( \omega \) field still affects the particle resulting in a phase shift of the same nature, also depending on \( \theta \) as follows

\[
\Delta \Phi^{0,1} = \left( -\frac{2\pi \Delta d}{\lambda} + \frac{\lambda \Delta d^4}{4\pi} m^2 \omega^2 + 2m\omega S_w \right) (1 - m\omega \theta).
\]  

(37)

In order to have a measurable noncommutative effect, the first order contribution should be a measurable fraction of the \( \theta=0 \) result, say a 1%. In this case, from eq.(36), one should have a strong magnetic field satisfying

\[
qB \simeq 10^{-2}\theta^{-1}.
\]  

(38)

Similarly, in eq.(37) one needs the angular velocity as fast as

\[
m\omega \simeq 10^{-2}\theta^{-1}.
\]  

(39)

Using the current bound for the noncommutative parameter, \( \theta \leq (10 \text{TeV})^{-2} \) \cite{14,19}, one finds that both requirements are experimentally hard to realize. One could perhaps think about an experimental setting involving astronomic velocities and field-strengths, but it seems to us that it is still beyond the current possibilities.

**Summary**

We have presented a path-integral approach to noncommutative quantum mechanics in the plane and discussed how the physics of a rotating interferometer is affected by the fact that spatial coordinates do not commute. One advantage of this approach is that all noncommutative effects are encoded in an effective Lagrangian which can be used to
compute transition amplitudes within the usual framework provided by Feynman integral over trajectories. The transition amplitude, originally written as a path-integral over phase space, eq.(23), can be reduced to a path-integral over coordinates, eq.(27), with an effective Lagrangian which includes the effects of noncommutativity of space coordinates. Using this result, we have investigated a rotating interferometer device to see how noncommutativity will eventually modify the physics of a quantum particle in a gauge field background together with the effects of locating the system in a noninertial frame. From the associated quantum effective Lagrangian, which for the present gauge we calculated to all orders in $\theta$, we were able to analytically compute the phase shift for electrons reaching the detector through different paths in an exact way. Finally, we discussed the possible implicancies of noncommutativity on the phenomenology of a rotating SQUID showing that the current bounds on $\theta$ require an experimental setting which is far beyond the present possibilities.

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