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We study the large mass asymptotics of the Dirac operator with a nondegenerate mass matrix\( m = \text{diag}(m_1, m_2, m_3) \) in the presence of scalar and pseudoscalar background fields taking values in the Lie algebra of the \( U(3) \) group. The corresponding one-loop effective action is regularized by the Schwinger’s proper-time technique. Using a well-known operator identity, we obtain a series representation for the heat kernel which differs from the standard proper-time expansion, if \( m_1 \neq m_2 \neq m_3 \). After integrating over the proper-time we use a new algorithm to resum the series. The invariant coefficients which define the asymptotics of the effective action are calculated up to the fourth order and compared with the related Seeley-DeWitt coefficients for the particular case of a degenerate mass matrix with \( m_1 = m_2 = m_3 \).

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The effective action plays the central role in lagrangian quantum field theory. Dividing fields into a classical background and quantum fluctuations and integrating out the last ones one obtains the effective action which properly accumulates the short distance dynamics of quantum fields [1,2]. In this letter we study the real part of the one-loop effective action with virtual heavy fermions of different masses using the Schwinger’s proper-time technique [3]. In QCD the nondegenerate mass matrix of heavy (constituent) quarks results from the spontaneous breakdown of chiral symmetry, as it is the case, for instance, in the Nambu – Jona-Lasinio model [4], or from the specially added invariant term to the QCD Lagrangian which regulates the infrared behaviour of the effective action [5]. We do not consider here the manifest chiral symmetry breaking effect on the effective action [6]. A careful analysis of this problem would lead us too far away from the subject, leaving the present result without changes. It might be well to emphasize, however, that a nondegenerate mass matrix of heavy fermions in theories with spontaneous breakdown of chiral symmetry appears as a consequence of manifest symmetry breaking by the nondegenerate mass matrix of light fermions. The mathematical formalism being presented in this letter is a necessary element of the approach which faithfully mirrors the vacuum structure of such a theory. This approach was formulated for the theory with an explicit and spontaneous breakdown of the global \( SU(2) \times SU(2) \) chiral symmetry in [6] and generalized to all orders of the asymptotic expansion in [7]. Here we extend this scheme to the \( SU(3) \times SU(3) \) chiral theory where the mass matrix has a form \( m = \text{diag}(m_1, m_2, m_3) \).

The explicit forms of dominant effective local vertices induced by virtual heavy fermions in general spontaneously broken gauge theories have been obtained in [8,9]. However both the method of our evaluation of the heat kernel as well as the result, which is cast as an asymptotic series for the effective action with the order by order chiral invariant structure of the derived asymptotic coefficients, are different from the cited papers and new. Let us clarify: we do not use the equation of the Schrödinger type for the heat kernel in the first stage of calculations, preferring the direct expansion of the heat kernel in powers of background fields, like, for instance, it has been reviewed in [10]. Additionally, we suggest a new algorithm for resumptions. Starting from this place our calculations essentially deviate from the known ones. The resummation procedure in [8] finally leads to an asymptotic series where the \( n \)-th term gives the immediately breaks chiral symmetry at each order of \( n \). The most direct way to see this is to consider the theory with a linear realization of chiral symmetry. In the spontaneously broken phase the symmetry transformations of scalar and pseudoscalar fields include mass dependent parts. Therefore, the asymptotic coefficients must have terms with different powers of masses, to be chiral invariant. The resummation procedure must take this fact into account properly. In our approach resummations are organized in such a way that every coefficient of the asymptotic series is automatically chiral invariant, if the Lagrangian possesses this symmetry. We consider this property to be a crucial condition on any generalization of the Schwinger – DeWitt result. Moreover, the resummation procedure used in [8] is slightly misleading. Indeed, the terms like, for example, \([\phi, m^2]\) where \( \phi \) is a background field, are considered to be of \( m^2 \) order. However, it is clear that the commutator contributes as the difference of mass squares, i.e., as \( \sim (m_1^2 - m_2^2) \) (in the case of \( 2 \times 2 \) mass matrix \( m \)). This value can be small and as a result contribute to lower orders.

A further approach consists in studying the proper-time asymptotics of the heat kernel with arbitrary matrix-

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can be written in the following way

\[
W[Y] = -\ln |\det D| = \frac{1}{2} \int_0^\infty dt \rho(t, \Lambda^2) \text{Tr} \left( e^{-tD^4} \right).
\]  

We use here the Schwinger’s proper-time representation for the modulus of the functional fermion determinant. In the one-loop approximation the real part of the corresponding effective action is given by

\[
W[Y] = -\ln |\det D| = \frac{1}{2} \int_0^\infty dt \rho(t, \Lambda^2) e^{-t D^4} \text{Tr} \left( e^{-tD^4} \right).
\]  

We find it convenient to use the orthogonal basis, \( E_i \), in our analysis. The integral over \( t \) is divergent at the low limit and needs to be regularized. This can be done by inserting the regularizing kernel \( \rho(t, \Lambda^2) \) with the ultraviolet cutoff parameter \( \Lambda \). In the following we do not need an explicit form for \( \rho(t, \Lambda^2) \). Using the technique developed by Fujikawa one can obtain

\[
W[Y] = \frac{1}{2} \int d^4x \int \frac{d^4\rho}{(2\pi)^4} \int_0^\infty dt \rho(t, \Lambda^2) e^{-\rho^2} \text{tr} \left( e^{-t(m^2+A)} \right) \cdot 1,
\]  

where \( A = B - 2i\rho \). As soon as we have derivatives it is necessary to clarify the meaning of the trace in Eq.(3). The coordinate space \( \{x\} \) is assumed to have no boundary, so that one can integrate by parts rendering the functional trace cyclic. Nevertheless, it is useful to define the cyclically symmetrized trace of matrix-valued functions \( A_i \) by

\[
\text{str} (A_1, A_2, \ldots, A_n) = \sum_{\text{perm}} \frac{1}{n!} \text{tr} (A_1 A_2 \ldots A_n),
\]  

where \( \text{perm} \) means \( n \) possible cyclic permutations inside trace. For algebraic objects \( \text{str} \) is equivalent to \( \text{tr} \) provided the trace is cyclic.

Since we do not want to expand the mass-dependent part of the heat kernel in powers of the proper-time, we need the operator identity, which is well-known in quantum mechanics

\[
\text{tr} \left( e^{-t(m^2+A)} \right) = \text{tr} \left( e^{-tm^2} \left[ 1 + \sum_{n=1}^\infty (-1)^n f_n(t, A) \right] \right).
\]  

Here \( f_n(t, A) \) is equal to

\[
f_n(t, A) = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_n-1} ds_n A(s_1) A(s_2) \ldots A(s_n),
\]  

where \( A(s) = e^{sm^2} A e^{-sm^2} \). For our purpose we need the first five terms of the series. The \( n \)-th coefficient is equal to

\[
\text{tr} \left[ e^{-tm^2} f_n(t, A) \right] = \frac{t^n}{n!} \sum_{i_1, i_2, \ldots, i_n} c_{i_1 i_2 \ldots i_n(t)} \text{str} (A_{i_1}, A_{i_2}, \ldots, A_{i_n}),
\]  

where \( c_{i_1 i_2 \ldots i_n(t)} \) are the Seeley-DeWitt coefficients. In fact, we use the proper-time representation only to separate the field dependent part of the heat kernel from the mass dependent one. We do not expand the mass dependent piece of the heat kernel in powers of the proper-time; instead we manipulate all terms of the proper-time series at once. This is an essential part of our approach and it means that there is no way to obtain our result from the well known proper-time asymptotics.

After these general remarks, to be more specific, let us consider an euclidean quantum field theory with a fermion propagator \( D^{-1} \) depending on the background fields collected in \( Y \). In the one-loop approximation the real part of the mass-dependent part of the heat kernel is the cutoff parameter \( \Lambda \). In the following we do not need an explicit form for \( \rho(t, \Lambda^2) \). This can be done by inserting the regularizing kernel \( \rho(t, \Lambda^2) \) with the ultraviolet cutoff parameter \( \Lambda \). In the following we do not need an explicit form for \( \rho(t, \Lambda^2) \). Using the technique developed by Fujikawa one can obtain

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\]  

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If the indices are all equal, one can obtain that $c_i = e^{-tm_i^2}$, with the definition $\Delta_{ij}$ determined by the recursion formulas (7) on the basis of the novel algorithm for resummations inside the starting expansion (10). The resummations are correct here. The main problem here is to find the algorithm which automatically yields a chiral invariant grouping natural requirement selects from the infinite number of possibilities for rearrangements in Eq.(10) only one as the correct answer. Thus one can obtain the standard inverse mass expansion. In the nondegenerate case for the first five terms in (5) we find
\[
\int \frac{d^4x}{32\pi^2} \int_0^\infty \frac{dt}{t^3} \rho(t, \Lambda^2) \left\{ \sum_{i=1}^3 c_i tr(E_i) - t \sum_{i=1}^3 c_i tr(A_i) + t^2 \left[ \frac{3}{2} \sum_{i=1}^3 c_i tr(A_i^2) + \sum_{i<j}^3 c_{ij} tr(A_i A_j) \right] \right\} + \ldots.
\]
Substituting this expression in Eq.(3), putting there $A = Y - \partial^2 - 2ip\partial/\sqrt{7}$ and integrating over the four-momenta $p_\mu$ we obtain
\[
W[Y] = \int \frac{d^4x}{32\pi^2} \int_0^\infty \frac{dt}{t^3} \rho(t, \Lambda^2) \left\{ \sum_{i=1}^3 c_i tr(E_i) - t \sum_{i=1}^3 c_i tr(Y_i) + t^2 \left[ \frac{3}{2} \sum_{i=1}^3 c_i tr(Y_i^2) + 2 \sum_{i<j}^3 c_{ij} tr(Y_i Y_j) \right] \right\} + \ldots.
\]
Here we have used $Y_i \equiv E_i Y$ and considered only terms up to and including $t^3$ order (we count powers of $t$ as like it would be in the limit $\Delta_{ij} \to 0$). Let us remind that the effective action is defined up to total derivatives, which can be omitted. Let us also stress that the above expression is not a proper-time expansion, since the coefficients $c_{i_1i_2\ldots i_n}$ are functions of $t$.

The integrals over the proper-time $t$ can be simply evaluated and reduced to combinations of some set of elementary integrals $J_n(m_i^2)$
\[
J_n(m^2) = \int_0^\infty \frac{dt}{t^{2-n}} e^{-tm^2} \rho(t, \Lambda^2),
\]
where $n$ is integer. However in some sense the result of such integrations does not mean much by itself. One has to put the series in a form in which every term of the asymptotic expansion is chiral invariant if $W[Y]$ does. This natural requirement selects from the infinite number of possibilities for rearrangements in Eq.(10) only one as the correct answer. The main problem here is to find the algorithm which automatically yields a chiral invariant grouping for the background fields as well as the mass dependent factors before them. This problem has been solved recently [7] on the basis of the novel algorithm for resummations inside the starting expansion (10). The resummations are determined by the recursion formulas.
\[ J_i(m_j^2) - J_i(m_j^2) = \sum_{n=1}^{\infty} \frac{\Delta_i^m}{2\pi n!} [J_{i+n}(m_j^2) - (-1)^n J_{i+n}(m_j^2)]. \]  

(12)

In the case under consideration one has to factorize the mass dependent factors \( I_i \), manipulating formula (12). As a result we arrive at the asymptotic series

\[ W[Y] = \int \frac{d^4x}{32\pi^2} \sum_{i=0}^{\infty} I_{i-1} \text{tr}(a_i), \quad I_i \equiv \frac{1}{3} \sum_{j=1}^{n} J_i(m_j^2), \]  

(13)

with the invariant coefficients \( a_i \). The first four of them are equal to

\[ a_0 = 1, \quad a_1 = -Y, \quad a_2 = \frac{Y^2}{2} + \frac{\Delta_{12}}{2} \lambda_3 Y + \frac{1}{2\sqrt{3}} (\Delta_{13} + \Delta_{23}) \lambda_8 Y, \]

\[ a_3 = -\frac{Y^3}{3!} - \frac{1}{12} \Delta_{12} (\Delta_{31} + \Delta_{32}) \lambda_3 Y + \frac{1}{12\sqrt{3}} (\Delta_{13}(\Delta_{21} + \Delta_{23}) + \Delta_{23}(\Delta_{12} + \Delta_{13})) \lambda_8 Y \]

\[ + \frac{1}{4\sqrt{3}} (\Delta_{31} + \Delta_{32}) \lambda_8 Y^2 + \frac{1}{4} \Delta_{21} \lambda_8 Y^2 - \frac{1}{12} (\partial Y)^2, \]  

(14)

To obtain this result we used relations between \( E_i \) matrices and \( U(3) \) hermitian generators, \( \lambda_0, \lambda_3 \) and \( \lambda_8 \). One can consider the resulting series an inverse mass expansion, since \( I_{i+1} \sim m_i^{-2l} \) for \( l \geq 1 \). However one should remember that the asymptotic coefficients \( a_i \) depend on mass differences. This dependence is completely fixed by the symmetry requirements. Indeed, it can be verified that if the operator \( D_i^j D^i \) is defined to transform in the adjoint representation \( \delta(D^i D^j) = i[\omega, D^i D^j] \), the coefficient functions \( a_i \) are invariant under the global infinitesimal chiral transformations with parameters \( \omega = \alpha + \gamma_5 \beta \). The present result is in agreement with the standard Schwinger-DeWitt inverse mass expansion, when \( m_1 = m_2 = m_3 \). For the case with \( \Delta_{ij} \neq 0 \) our formula (13) is a new asymptotic series which can be used to construct the low-energy EFT action when the local vectors are induced by one-loop diagrams involving heavy particles with different masses. In the low energy QCD, for instance, our approach can be used for the Nambu – Jona-Lasinio type models with \( SU(3) \times SU(3) \) chiral symmetry [5], or for heat kernel one-loop renormalizations in the framework of the chiral expansion [5]. The difference in the masses of the nonstrange and strange constituent quarks is large enough, \( m_u, m_d \sim 330 \text{ MeV} \) and \( m_s \sim 510 \text{ MeV} \), so that one may expect essential numerical deviations from the results obtained by other methods.

We conclude by pointing out that the method allows for extensions in a rather straightforward fashion. Special care has been taken to formulate the \( SU(3) \times SU(3) \) case in a basis (2) which is suitable for an extension to a general \( n \times n \) mass matrix. Futhermore we have shown that the algorithm for resummations (12) remains unchanged in going from the \( SU(2) \times SU(2) \) to the \( SU(3) \times SU(3) \) case. This is a central and non-trivial result, since it was a priori not evident that with three different masses the pairwise grouping of differences of \( J_n(m_i^2) \) integrals would prevail. All the more so, since the algorithm per se does not tell which combination of \( J_n(m_i^2) \) integrals should be factorized in Eq. (13) to get invariant coefficients \( a_i \). Therefore it was also not clear that this combination would turn out to be the mean value \( \bar{I}_i \). Finally the method can also be generalized to accommodate minimal couplings of the loop particles to gauge fields. Work along these lines is in progress.

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