Gauge invariant regularisation via $SU(N|N)$

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ABSTRACT: We construct a gauge invariant regularisation scheme for pure $SU(N)$ Yang-Mills theory in fixed dimension four or less (for $N = \infty$ in all dimensions), with a physical cutoff scale $\Lambda$, by using covariant higher derivatives and spontaneously broken $SU(N|N)$ supergauge invariance. Providing their powers are within certain ranges, the covariant higher derivatives cure the superficial divergence of all but a set of one-loop graphs. The finiteness of these latter graphs is ensured by properties of the supergroup and gauge invariance. In the limit $\Lambda \to \infty$, all the regulator fields decouple and unitarity is recovered in the renormalized pure $SU(N)$ Yang-Mills theory. By demonstrating these properties, we prove that the regularisation works to all orders in perturbation theory.

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1. Introduction

In this paper we construct a Poincaré invariant and gauge invariant ultraviolet cutoff which works for pure $SU(N)$ Yang-Mills theory in fixed dimension $D \leq 4$. In the large $N$ limit the regularisation works for all dimensions. There are a number of reasons why such a regularisation may prove useful. Our primary motivation is that this regularisation may be incorporated into an elegant and potentially powerful manifestly gauge invariant exact renormalization group [1]–[6]. The key features required of such a regulator are that it is gauge invariant, introduces a mass scale, and that it makes sense non-perturbatively. Although we only prove here that the regularisation works to all orders in perturbation theory, it is natural to expect that it is valid non-perturbatively since it is based on a physical cutoff, i.e. a real ultraviolet cutoff scale $\Lambda$ [7, 8].\(^1\) Since the dimension remains fixed, it can thus be used to study chiral and/or topological effects such as instantons. Our regularisation should be compared in these respects to lattice regularisation, which of course is not formulated in the continuum thus breaking Poincaré invariance, and Slavnov’s higher derivative scheme [9, 10, 11].

We review this and other earlier work, and provide all the details, below. First we sketch the basic idea, which is very simple. (See also refs. [7, 8].) Covariant higher derivative regularisation of $SU(N)$ Yang-Mills does not work on its own because one-loop divergences slip through [12]. We can cure this problem by working instead with $SU(N|N)$ Yang-Mills which has sufficiently improved ultraviolet properties. $SU(N|N)$ contains $SU(N) \times SU(N)$ as a subgroup. Correspondingly the $SU(N|N)$ super-gauge field, $A$, contains two gauge fields transforming separately under each $SU(N)$, a normal one, $A^1$, which will be the physical gauge field, and a copy, $A^2$, with wrong sign action. $A$ also contains a complex fermionic gauge field $B$ that transforms as a fundamental under one $SU(N)$ and complex conjugate fundamental under the other.\(^2\) The remaining potential divergences are cancelled via the (linear representation of) supersymmetry in the fibres\(^3\) of the unbroken $SU(N|N)$: in these cases for every purely bosonic loop,

\(^1\)as opposed to \textit{e.g.} analytic behaviour of perturbative amplitudes as in dimensional regularisation

\(^2\)There can be also a decoupled $U(1)$ gauge field $A^0$, depending on how one represents $SU(N|N)$. We cover such subtleties later.

\(^3\)as opposed to spacetime supersymmetry formed by non-linear representation from the super-Poincaré over super-Lorentz coset space. This in turn leads to supermatrix valued fields as opposed to superfields valued on a superspace.
there is also a purely fermionic one which is an exact copy but which enters with the opposite sign. For mixed bosonic/fermionic loops the wrong sign $A^2$ gauge field propagators ensure cancellation. Of course neither $B$ nor $A^2$ is physically meaningful. By introducing a superscalar Higgs field which spontaneously breaks the supersymmetry in the fermionic directions, we can give arbitrarily high masses to the fermionic field. $B$ then behaves exactly like a massive Pauli-Villars field and it is only through this that the two $SU(N)$ gauge fields can interact. Since cancellation will still take place but now only at large loop momenta, in effect a new physical cutoff has been introduced that suppresses high momentum modes. Initially we verified these mechanisms by working explicitly in components, however in this paper the work is presented using the full superfields, so this Fermi-Bose interpretation will remain hidden just below the surface. The cancellations will arise through a supergroup theoretical analogue: the ‘supertrace mechanism’. In particular, quantum corrections that yield $\text{tr} \left( \frac{1}{2} \right) = N$ in $SU(N)$ Yang-Mills now yield $\text{str} \left( \frac{1}{2} \right) = 0$.

Note that apart from $A^1$, only the unphysical $A^2$ field remains massless. We need to verify that no effective interaction is left between $A^1$ and $A^2$ as the symmetry breaking scale $\Lambda$ is sent to infinity, so that we can simply ignore the non-unitary $A^2$ sector. But this is guaranteed by the Appelquist-Carazzone theorem providing the theory is renormalizable [13] (which is where the restriction to dimensions $D \leq 4$ comes in) since the lowest dimension gauge invariant effective interaction involves the square of the two field strengths:

$$\text{tr}(F^1)^2 \text{tr}(F^2)^2$$

and is thus irrelevant, vanishing as an inverse power of $\Lambda$.

The necessary double trace makes all such terms subleading in large $N$, thus decoupling takes place in any dimension in the large $N$ limit [14]. Furthermore, in this limit the supertrace mechanism ensures that the theory is finite in all dimensions (otherwise by the same token (1.1), or rather their supergroup version, is the lowest dimension term not suppressed by the supertrace mechanism at one-loop, resulting in a divergence in $D \geq 8$ dimensions). These are the reasons why the $N = \infty$ limit of $SU(N)$ Yang-Mills is regulated by our theory in all dimensions $D$.

Our regularisation scheme was initially inspired by earlier work of Slavnov and others, on covariant higher derivatives. As already remarked, it is a well established problem that higher covariant derivatives fail to cure ultra-violet divergences at one loop [12]. The one-loop divergences can be regularised by also introducing gauge invariant Pauli-Villars (PV) fields [9], the action being bilinear in these fields so that they provide, on integrating out, the missing one loop counterterms (plus other finite contributions). But further one-loop divergences then typically arise when the PV fields are external. Whilst these might be ignored on the grounds that the Pauli-Villars quanta are not to be regarded as external physical particles, the divergences reappear in internal subdiagrams as overlapping divergences at higher loops [15]. Despite some earlier controversy [16], these problems appear to have been cured by adding more PV fields.
and by judicious choices for their actions [10, 11]. Even so, there are problems: the solution is unwieldy, and inappropriate for incorporation in the exact renormalization group framework since it is not possible to preserve the property that the PV fields appear only bilinearly at the level of the effective action [2, 4]. Instead, we needed and here furnish, a framework in which the gauge fields and gauge invariant PV fields are treated on the same footing. From the start the regularisation applies to all fields simultaneously, and thus the above problem of “overlapping divergences” never arises.

An earlier $N = \infty$ version of the present framework was constructed by adding such PV fields by hand but insisting that the resulting regularisation respected the exact renormalization group flow [6], which in particular meant that higher order interactions for the PV fields had to be added [2]. The result regularised only the one-loop diagrams without external PV fields [4], i.e. still suffered the above problem of “overlapping divergences”. Nevertheless, it was realised that this version could be understood at a deeper level as arising from a spontaneously broken $SU(N|N)$ gauge theory, albeit in a form of unitary gauge and with some small differences, and this led to the suggestion that higher derivative regularisation based exactly on spontaneously broken $SU(N|N)$ may work to all loop orders and also at finite $N$ [4].

In implementing these ideas we uncovered numerous novel features. In sec. 2, we meet one of the causes. $SU(N|N)$ is reducible but indecomposable: it has a bosonic $U(1)$ subalgebra in the II direction, which thus commutes with everything, but which cannot be discarded because it is itself generated by fermionic elements of the algebra. As we see in sec. 3, for the gauge theory this means that there is a $U(1)$ gauge field $A^0$ with no kinetic term and which interacts with nothing, but is nevertheless necessary to ensure gauge invariance! We show in sec. 3 how to construct two equivalent representations in one of which the $A^0$ is ‘projected out’. In fact, we step back to consider also $U(N|N)$ which does not factor, even locally, into $SU(N|N) \times U(1)$. We find that a gauge theory built on $U(N|N)$ automatically contracts itself to $SU(N|N)$! We also carefully consider the implications of these and similar peculiarities for the superscalar sector. We finish this section discussing another novelty: the ghost degrees of freedom do not (anti)commute sensibly with the supergroup directions leading to a breakdown of global $SU(N|N)$ invariance and failure of at least naïve implementations of BRST. We furnish an elegant solution by going beyond the simple distinction of fermion or boson and introducing two separate gradings for ghost and supergroup degrees of freedom.

In sec. 4 we present our higher derivative regularised theory, its form in the appropriate ’t Hooft gauge and the resulting ghost action. In sec. 5, we ignore the special cancellations provided by the supergroup and find necessary and sufficient conditions on the powers of the higher derivatives (in fact ranks of polynomials of these) to regularise as many diagrams as possible. Whilst furnishing sufficient conditions and isolating a set of ‘One-loop Remainder Diagrams’ which still need further regularisation, follows quickly from standard power counting methods, finding the minimal set of sufficient
conditions requires more cunning. In sec. 6 we prove that two and three-point One-loop Remainder Diagrams are regularised by the supertrace mechanism. We also show that all but the unbroken parts of the One-loop Remainder Diagrams are superficially finite by covariant higher derivatives alone. We finish by noting that in the large $N$ limit the symmetric phase has no quantum corrections at all, and thus in this limit the theory is finite in all dimensions $D$. In sec. 7 we turn to the remaining One-loop Remainder Diagrams at finite $N$, which are ultraviolet finite only after gauge invariance is taken into account, by a combination of the higher derivatives and the supertrace mechanism, in all dimensions $D < 8$. Actually since we are dealing with the gauge fixed theory these arguments need to be phrased in terms of Ward identities and BRST invariance which we develop here. With this structure in place we complete the proof of finiteness to all orders in perturbation theory of covariant higher derivative regularised spontaneously broken $SU(N|N)$ in all dimensions $D < 8$. In order for this to act as a regulator for $SU(N)$ Yang-Mills we are left to show that the low energy sector is given by $SU(N)$ Yang-Mills. There is a case to answer because the wrong-sign $A^2$ field remains massless. In sec. 8 we first confirm that the wrong sign leads to negative probabilities and a non-unitary S-matrix for this sector and then prove that the two sectors decouple in $D \leq 4$ dimensions, or at $N = \infty$ in all dimensions, as already sketched above. Finally in sec. 9 we summarise and draw our conclusions.

2. $SU(N|N)$

We start with an elementary exposition of the $SU(N|N)$ superalgebra [17] and its invariants, covering the notation and key formulae needed later on. The defining representation of the graded Lie algebra of $U(N|M)$ is constructed from commutators of Hermitian $(N+M) \times (N+M)$ matrices of the form:

$$\mathcal{H} = \begin{pmatrix} H_N & \theta \\ \theta^t & H_M \end{pmatrix},$$

(2.1)

where $H_N$ ($H_M$) is an $N \times N$ ($M \times M$) Hermitian matrix with complex bosonic elements and $\theta$ is an $N \times M$ matrix composed of complex Grassmann numbers. $\mathcal{H}$ is thus a Hermitian supermatrix. The supertrace replaces the trace as the natural invariant for supermatrices:

$$\text{str}(\mathcal{H}) = \text{tr}(\sigma_3 \mathcal{H}) = \text{tr}(H_N) - \text{tr}(H_M),$$

(2.2)

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(2.3)

with $1_N$ ($1_M$) being the $N \times N$ ($M \times M$) identity matrix. This is because only the supertrace is cyclically invariant (compensating the signs picked up by anticommuting Grassmann components):

$$\text{str} XY = \text{str} YX$$

(2.4)
(where $X$ and $Y$ are two general supermatrices), ensuring the supertrace of a commutator vanishes, and thus in turn ensuring invariance under adjoint action of the group.

We will define the generators of the group to be Hermitian matrices with only complex number entries, the Grassmann character being carried as appropriate by the coefficients (the superangles). In terms of the generators then, we obtain a superalgebra with commutators or anticommutators as appropriate.

To be elements of the algebra of $SU(N|M)$ we require that $H$ be supertraceless, i.e. $\text{str} H = 0$. With the traceless parts of $H_N$ and $H_M$ corresponding to $SU(N)$ and $SU(M)$ respectively and the orthogonal traceful part giving rise to a $U(1)$, we see that the bosonic sector of the $SU(N|N)$ algebra forms a $SU(N) \times SU(M) \times U(1)$ subalgebra.

We now specialise to $M = N$. In this case the algebra is reducible because the bosonic $U(1)$ subalgebra is generated by the unit matrix $1_{2N}$ which thus commutes with all the other generators, forming an Abelian ideal (invariant subspace). In contrast to compact bosonic Lie algebras however, we cannot then decompose $SU(N|N)$ into a direct product of smaller algebras, because $1_{2N}$ is itself generated by fermionic elements of the superalgebra, for example

\[
\{\sigma_1, \sigma_1\} = 21_{2N},
\]

where
\[
\sigma_1 = \begin{pmatrix}
0 & 1_N \\
1_N & 0
\end{pmatrix}.
\] (2.5)

Ref. [17] excludes $1_{2N}$ by redefining the Lie bracket in this representation. As we explain in sec. 3, it will turn out that in constructing our action, we cannot exactly exclude $1_{2N}$ in this way, and thus our definition of $SU(N|N)$ is different from that of ref. [17]. We do note though that the unit matrix has a special role to play and for this reason we separate it from the other generators.

We define the generators, $S_\alpha \equiv \{1, T_A\}$, where the $T_A$ are complex block diagonal and block off-diagonal Hermitian traceless matrices. $A$ runs over the $2(N^2 - 1)$ bosonic (a.k.a. block diagonal) generators and $2N^2$ fermionic (a.k.a. block off-diagonal) generators and $\alpha \equiv \{0, A\}$. An element of the $SU(N|N)$ algebra is then

\[
\mathcal{H} = \mathcal{H}^\alpha S_\alpha = \mathcal{H}^0 1 + \mathcal{H}^A T_A,
\]

\[
(\mathcal{H})^i_j = \mathcal{H}^0 \delta^i_j + \mathcal{H}^A (T_A)^i_j.
\] (2.7)

enabling us to identify the Killing super-metric

\[
h_{\alpha\beta} = 2\text{str}(S_\alpha S_\beta).
\] (2.8)

This is symmetric when $\alpha$ and $\beta$ are both bosonic and antisymmetric when both are fermionic, i.e.

\[
h_{\alpha\beta} = h_{\beta\alpha}(-1)^{f(\alpha)f(\beta)},
\] (2.9)
where \( f(\alpha) \) is 0 when \( \alpha \) is bosonic and 1 when it is fermionic. The generators are normalised such that

\[
h_{\alpha\beta} = \begin{pmatrix}
0 & 1 & 1 & \cdots & -1 & -1 & \cdots \\
-1 & 0 & 0 & \cdots & 0 & i & -i \\
i & 0 & 0 & \cdots & 0 & i & -i \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]  

(2.10)

Obviously, this has no inverse. However, we can define the restriction of \( h_{\alpha\beta} \) to the \( T_A \) space:

\[
g_{AB} = 2 \text{str}(T_AT_B) = h_{AB},
\]

(2.11)

with inverse determined by

\[
g_{AB}g^{BC} = g^{CB}g_{BA} = \delta^C_A.
\]

(2.12)

The metric can be used to raise and lower indices

\[
X_A := g_{AB}X^B \quad \Longrightarrow \quad X^A = X_Bg^{AB} \neq X_Bg^{BA}.
\]

(2.13)

Note that it is the second index of the metric that is summed over; from (2.9) it is clear that the ordering of the indices of the metric is important. Using (2.9) and the above relations one may readily verify that for superangles \( X^AY_A = Y^AX_A \neq X_AY^A \) in agreement with (2.4). We can add the dual relations for raising and lowering indices on generators:

\[
T^A := T_Bg^{BA} \quad \text{so} \quad X^AT_A = X_AT^A
\]

(2.14)

(with ordering of \( X \) and \( T \) of course unimportant). Finally, the \( T_A \) generators of \( SU(N|N) \) give rise to a completeness relation

\[
(T^A)^i_j(T_A)^k_l = \frac{1}{2} \delta^i_l (\sigma_3)^k_j - \frac{1}{4N} \left[ \delta^i_j (\sigma_3)^k_l + (\sigma_3)^i_j \delta^k_l \right],
\]

(2.15)

which is most usefully cast contracted into arbitrary supermatrices \( X \) and \( Y \):

\[
\text{str}(XT^A)\text{str}(T_AY) = \frac{1}{2} \text{str}XY - \frac{1}{4N} (\text{trXstrY} + \text{strXtrY}),
\]

(2.16)

\[
\text{str}(T^AXT_AY) = \frac{1}{2} \text{strXstrY} - \frac{1}{4N} \text{tr}(XY + YX).
\]

(2.17)
3. Alternatives and Otherwise

When treated as a gauge group, the $SU(N|N)$ algebra has some unusual features which means that a number of steps in constructing the action have to be rethought from the beginning. As we will see, the existence of fermionic directions present their own novelties particularly for the BRST algebra, but the main novelty is that, as mentioned in sec. 2, $SU(N|N)$ is an example of a superalgebra that is reducible but indecomposable.

Actually, even the step of taking $SU(N|N)$ rather than $U(N|N)$ needs to be carefully rethought: the reduction to $SU(N|N)$ is achieved by excluding the space of generators with nonvanishing supertrace [spanned by adding any representative e.g. $\sigma_3$, as defined in (2.3)], but there is no corresponding ideal. (Note for example that $\sigma_3$ does not commute with the fermionic generators.) In other words, it is not the case that the $U(N|N)$ group is even locally isomorphic to $SU(N|N) \times U(1)$!

Let us recall the reasons for treating separately parts of a reducible compact bosonic Lie group, for example $U(N)$. In this case we know that we can locally decompose it into $SU(N) \times U(1)$, but the precise reason we treat the two subgroups separately in this context is because the Lagrangian inevitably contains relevant pieces which are invariant separately under $SU(N)$ and $U(1)$. Since the $SU(N)$ piece and the $U(1)$ piece do not mix under the action of $U(N)$, they will receive different divergent contributions and must therefore have their own couplings – which renormalize separately. Thus we see that reducibility matters for the renormalization of a Lagrangian and should be understood in terms of the possible invariants. These issues are not directly relevant in the present case since we will be interested only in Lagrangians that result in a finite theory. Nevertheless, we will comment on the more general situation.

We start then by considering $U(N|N)$ and the pure gauge sector of the theory. Unless otherwise specified we will be working in $D$ Euclidean dimensions. Extending (2.7), we write an element of the Lie superalgebra of $U(N|N)$ by extending the index to include the label $\sigma$:

$$\mathcal{H} = \mathcal{H}^\sigma \sigma_3 + \mathcal{H}^0 \mathbbm{1} + \mathcal{H}^A T_A.$$  \hfill (3.1)

Using $\text{tr} T_A = 0$, the Killing supermetric (2.10) extends as follows:

$$h_{\sigma\sigma} \equiv 2\text{str}(\sigma_3^2) = 0,$$

$$h_{A\sigma} = h_{\sigma A} \equiv 2\text{str}(\sigma_3 T_A) = 0,$$

$$h_{0\sigma} = h_{\sigma 0} \equiv 2\text{str}(\sigma_3 \mathbbm{1}) = 4N.$$  \hfill (3.2)

Writing the covariant derivative as $\nabla_\mu = \partial_\mu - ig A_\mu$, where $A_\mu$ is a member of the Lie superalgebra as in (3.1), the field strength is

$$F_{\mu\nu} = \frac{i}{g}[\nabla_\mu, \nabla_\nu].$$  \hfill (3.3)
The Lagrangian density is then given by $\sim \text{str} \mathcal{F}_{\mu \nu}^2$ (plus higher order interactions through extra commutators of $\nabla$, providing the covariant higher derivative regularisation, cf. sec. 4).

Recall from sec. 2, that we need the supertrace for its invariance properties. Thus under gauge transformations

$$\delta \mathcal{A}_\mu = \frac{1}{\bar{g}} [\nabla_\mu, \omega]$$  \hspace{1cm} (3.4)

(where $\omega$ is valued in the Lie superalgebra), we see that

$$\delta \text{str} \mathcal{F}_{\mu \nu}^2 = -i \text{str} [\mathcal{F}_{\mu \nu}^2, \omega],$$  \hspace{1cm} (3.5)

with the r.h.s. (right hand side) vanishing after using cyclicity of the supertrace.

Actually, we can also construct the $U(1)$-like covariant derivative

$$\nabla^{(\sigma)}_\mu = \partial_\mu - i \bar{g}_\sigma \text{str} \mathcal{A}_\mu = \partial_\mu - i \bar{g}_\sigma \mathcal{A}^\sigma_\mu$$  \hspace{1cm} (3.6)

and add the corresponding field strength squared. \textit{A priori}, if we were dealing with a divergent theory or were interested in finite renormalizations within the $U(N|N)$ theory, we would not be allowed to exclude this term. This much is similar to the case of $U(N)$ versus $SU(N)$, but note that $\mathcal{A}^\sigma$ also appears in the kinetic term and interactions in $\text{str} \mathcal{F}_{\mu \nu}^2$, and contributes to the gauge transformations of the other components via (3.4). The latter would mean that there is only one wavefunction renormalization, all components being bound together via Ward identities. In fact we will see shortly that the dynamics forces $\mathcal{A}^\sigma$ to disappear from the spectrum so none of these curiosities need be pursued further here.

Since $\mathbb{1}$ commutes with everything, there are no $\mathcal{A}^0$ interactions. This is true even when we introduce adjoint matter fields. Its only appearance is in the kinetic term as

$$\sim -2N \mathcal{A}^0_\mu (\partial^2 \delta_{\mu \nu} - \partial_\mu \partial_\nu) \mathcal{A}^\sigma_\nu,$$  \hspace{1cm} (3.7)

as follows from (3.2). (When covariant higher derivative regularisation is included, an invertible polynomial $c^{-1}(-\partial^2/\Lambda^2)$ is also inserted.) Consequently $\mathcal{A}^0$ acts as a Lagrange multiplier field, and integrating over it enforces the constraint that $\mathcal{A}^\sigma$ is longitudinal \textit{i.e.} can be written

$$\mathcal{A}^\sigma_\mu = \partial_\mu \Omega^\sigma$$  \hspace{1cm} (3.8)

for some $\Omega^\sigma$. (We will not consider the possibility that spacetime has non-trivial cohomology.) But under $U(N|N)$ gauge transformations $\mathcal{A}^\sigma$ changes like a $U(1)$ field

$$\delta \mathcal{A}^\sigma_\mu = \frac{1}{\bar{g}} \partial_\mu \omega^\sigma,$$  \hspace{1cm} (3.9)

receiving no contribution from $\mathcal{A}$ and the other generators because all graded commutators in the superalgebra are supertraceless. Therefore (3.8) means that $\mathcal{A}^\sigma$ is constrained to be pure gauge. We may as well pick the gauge corresponding to $\Omega^\sigma = 0$
and thus get rid of $\mathcal{A}^\sigma_\mu$; the associated ghost Lagrangian is free and decoupled and can be ignored. We see that the net result is that even if we start with $U(N|N)$ Yang-Mills, it collapses to $SU(N|N)$ Yang-Mills!

Without loss of generality we can start with $SU(N|N)$ Yang-Mills, so $\mathcal{A}_\mu = \mathcal{A}^\sigma_\mu S_\sigma$ as in (2.7). Now $\mathcal{A}^0$ does not appear in the Lagrangian at all! Thus the $\mathcal{A}^0_\mu$ part of the partition function is a free functional integral, i.e. without even a Gaussian weight, contributing at most an infinite constant to the vacuum energy. But we cannot simply exclude it because, as we saw from (2.5), gauge transformations do appear in the $\mathbb{I}_{2N}$ direction, and $\mathcal{A}^0_\mu$ must be there to absorb them!

An alternative is to dispense with $\mathcal{A}^0$ by redefining the Lie bracket to remove $\mathbb{I}$. We define a “*bracket” [17]

$$[\ , \ ]^* = [\ , \ ] \pm - \frac{\mathbb{I}}{2N} \text{tr}[\ , \ ] \pm,$$

(3.10)

where $[\ , \ ] \pm$ is applied to the generators and is a commutator or anticommutator as appropriate (so that passing to supermatrices as in (2.7), they all become commutators). Note that the *bracket still satisfies the Jacobi identity. This follows because

$$[\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]^*] = [\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]]$$

(3.11)

(which in turn follows after noting that tr$[\mathcal{H}_2, \mathcal{H}_3]$ is always bosonic) and thus

$$[\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]^*]^* = [\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]]^*$$

$$= [\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]] - \frac{\mathbb{I}}{2N} \text{tr}[\mathcal{H}_1, [\mathcal{H}_2, \mathcal{H}_3]]$$

(3.12)

for any elements $\mathcal{H}_i$ of the algebra. Since the *bracket is bilinear and antisymmetric and satisfies the Jacobi identities, it may equally well represent the Lie product. Using this, members of the Lie algebra may be written $\omega^A T_A$, and thus $\mathcal{A} \equiv \mathcal{A}^A T_A$, and all Lie algebra commutators such as in (3.3) and (3.4), become *brackets.

The *bracket simply sets to zero the structure constants that generated $\mathbb{I}$, leaving all other structure constants alone, and because the Killing supermetric vanishes in the $\mathbb{I}$ directions, none of the interactions change, as can be seen directly from the first of the two relations in (3.12) and

$$\text{str} \mathcal{H}_1 [\mathcal{H}_2, \mathcal{H}_3]^* = \text{str} \mathcal{H}_1 [\mathcal{H}_2, \mathcal{H}_3],$$

(3.13)

which again holds for any elements $\mathcal{H}_i$ of the algebra. Since the Lagrangian is actually unchanged by the introduction of the *bracket we see that physically the former “free $\mathcal{A}^0$” representation and this latter *bracket representation are equivalent.

Of course the Lagrangian so far defined cannot represent an acceptable physical theory not the least because the fermionic vector $B_\mu$ violates the spin statistics theorem. But that is not our intention: instead we want to give $B_\mu$ a mass of order the cutoff $\Lambda$, breaking the fermionic gauge invariance of the theory. Providing at energies much
greater than \( \Lambda \), the theory behaves like unbroken \( SU(N|N) \), its finiteness properties will ensure that \( B_\mu \) acts like a Pauli-Villars field cutting off energies in the unbroken \( SU(N) \times SU(N) \) Yang-Mills\(^4\) above \( \Lambda \). Fortunately we know how to do this, we must break the theory \textit{spontaneously} in all and only the fermionic directions. The most general solution is to introduce a non-zero vacuum expectation value along a direction
\[
\sigma_3 + \alpha 1 \ll
\]
in the Lie superalgebra (where \( \alpha \) can be any real number).

Thus we introduce a superscalar field \( C \) which, since it must live in the Lie superalgebra containing \( \sigma_3 \), lies in the adjoint of \( U(N|N) \). Of course it is entirely consistent for \( C \) nevertheless only to transform locally under \( SU(N|N) \). It is the fact that \( U(N|N) \neq SU(N|N) \times U(1) \) that allows the theory to be nevertheless nontrivial. Under gauge transformations (3.4), \( C \) will transform as
\[
\delta C = -i [C, \omega].
\]

Now, in the *bracket representation this commutator cannot be replaced by a *bracket because the result would fail to be gauge invariant. To see this consider the supertrace of an \( n^{th} \) order monomial of adjoint representatives with \( n > 1 \). The \( C \) kinetic term, which is necessary to give \( B_\mu \) its mass, is an \( n = 2 \) example. Another example is \( \text{str} C^n \) which we will use to construct a potential. Clearly these are invariant under (3.15). But with \( \delta C = -i [C, \omega]^* \), we obtain
\[
\delta \text{str} C^n = \frac{in}{2N} \text{str} C^{n-1} \text{tr} [C, \omega].
\]
which is non-vanishing in general. (For \( n = 2 \) it is non-vanishing in general only if \( C \) contains \( \sigma_3 \), as indeed it does here.) The underlying problem is that, while the *bracket is a perfectly fine representative of the super-Lie product, we need it to be defined also in the universal enveloping algebra (effectively here, general products of adjoint fields). Gauge invariance then requires the Leibnitz rule
\[
[H_1, H_2 H_3] = [H_1, H_2] H_3 + H_2 [H_1, H_3],
\]
because it is this that implies \( \delta C^n = -i [C^n, \omega] \). But the Leibnitz identity fails for the *bracket.

We see that unlike the case for \( A \), we cannot exclude the \( 1 \ll \) direction from \( C \), which thus expands as
\[
C = C^\sigma \sigma_3 + C^0 \ll + C^A T_A.
\]
We can still dispense with \( A^0 \) however as follows: we use the *bracket for all pure gauge interactions as already described above, but commutators are required when \( \nabla \) acts on \( C \), \textit{e.g.} in the superscalar’s kinetic term
\[
\text{str} [\nabla_\mu, C]^2.
\]

\(^4\)Only if we include \( A^0 \), there is also a free functional integral over this, but since we have seen that it has no physical effect, we will not mention it further.
We cannot use a *bracket here because this time the non-1l interactions are altered and the result is not gauge invariant. This follows from the breakdown of (3.13) when $H_1$ contains $\sigma_3$, i.e. is an element of $U(N|N)$ (in turn the result of non-vanishing $h_{0\alpha}$). To summarise: in our *bracket representation $A$ is a representative of $SU(N|N)$ without $1l$; $C$ represents $U(N|N)$ containing $1l$. Under gauge transformations, $A$ transforms with a *bracket in (3.4), but $C$ transforms with a commutator as in (3.15). The result is consistent by relation (3.11) and the fact that $C$ transforms only into itself under (3.15). Trivially, the equality of the Lagrangian in the *bracket and free-$A_0$ representations carries through to this extension, and thus the two representations are still physically equivalent. We will pursue the free-$A_0$ representation in this paper since it is more elegant, using the existence of the equivalent *bracket representation to justify its consistency. (Let us also mention that we checked that just as with normal gauge transformations, these supergauge transformations leave the naïve functional measure invariant.)

Fortunately, in contrast to the case for $A$, both the $\sigma_3$ and $1l$ components of $C$ are dynamical since they both appear in the kinetic term and str $C^n$ interactions. In fact they propagate into each other: their only kinetic term being

$$2N\partial_\mu C^0 \partial_\mu C^\sigma.$$  \hspace{1cm} (3.20)

Similarly to the case of $A_\mu$ we have the option also to consider separately the invariant $C^\sigma = \text{str } C/2N$. Again similarly, under gauge transformations (3.15), $C^\sigma$ mixes into the other components, and thus all components of $C$ would have the same wavefunction renormalization if this were needed. Whilst this time the theory does not itself constrain $C^\sigma$, at first sight it appears that we are able to impose the linear gauge invariant constraint:

$$\text{str } C = 2N\Lambda^{D/2-1}.$$  \hspace{1cm} (3.21)

(Note again that by (3.14), $C^\sigma = \Lambda^{D/2-1}$ must be non-zero.) In contrast to a non-linear constraint we might expect this to leave the renormalizability or finiteness of the theory undisturbed.

In fact (3.21) spells trouble since the Lagrangian must also include a potential str $V(C)$. (Such a potential will be used to induce spontaneous symmetry breaking and give all remaining ‘Higgs’ a mass.) This is because (3.21) causes (3.20) to vanish. $C^0$ thus becomes a Lagrange multiplier imposing its equations of motion as a constraint:

$$\frac{\partial}{\partial C^0} \text{str } V(C) = 0.$$  \hspace{1cm} (3.22)

For a simple mass term $\sim \Lambda^2 \text{str } C^2$ (which would be needed to give the $C^A$ masses of order $\Lambda$) this constraint leads to the contradiction $C^\sigma = 0$. For the simplest allowable non-trivial $V$, i.e. of rank 4, $C^0$ appears as a cubic and (3.22) sets $C^0$ equal to the roots of a quadratic, with coefficients polynomial in $C^A$. As well as being messy this does not look promising for furnishing a finite theory. For these reasons we do not pursue this option further.
Finally, we discuss the form of the Faddeev-Popov ghosts and BRST algebra, which will appear upon gauge fixing. We write these super-ghosts as

\[ \eta = \begin{pmatrix} \eta^1 & \phi \\ \psi & \eta^2 \end{pmatrix}. \]  

(3.23)

When quantizing a bosonic gauge group we introduce fermionic ghosts so that the ghost action yields the Faddeev-Popov determinant and not its inverse. Naively here we would expect similarly to assign opposite grading to \( \eta \) so that \( \eta^i \) in (3.23) are bosons and \( \phi \) and \( \psi \) are fermions. However, full superfields are of indeterminate grading and the usual requirement of (anti)commutativity is here replaced (for supercoloured objects) by the (anti)cyclicity of the supertrace. One can readily check that with the above ghost assignments \( \text{str} \, \eta X = -\text{str} \, X \eta \) if \( X \) has odd ghost number, as required, but that \( \text{str} \, \eta X = \text{str} \, X \sigma_3 \eta \sigma_3 \) if \( X \) has even ghost number.

Even if such cyclicity breaking terms are excluded from the action (by e.g. being single supertrace and total ghost number zero), they can arise in multiple-supertrace terms at one loop and higher loops. This in turn leads to a breakdown of the (unfixed) global \( SU(N|N) \) invariance [because the supertrace of a commutator no longer vanishes as required cf. (2.4) or (3.5)] and thus presumably spurious \( \sigma_3 \) insertions appearing in the loop corrections. Further problems are uncovered when we try to construct the BRST invariance. Proceeding in standard fashion, we replace \( \omega \) by \( (\tilde{g} \times) \) the ghost in (3.4):

\[ \delta A_\mu = \tilde{\epsilon} [\nabla_\mu, \eta], \]  

(3.24)

and restore the grading by introducing the scalar fermionic parameter \( \tilde{\epsilon} \). One can readily check that the usual property that \( \tilde{\epsilon} \) commutes with gauge fields and anti-commutes with ghosts is used to prove BRST invariance. However, \( \tilde{\epsilon} \) has no simple (anti)commutation properties with the superfields. Indeed it matters whether we place \( \tilde{\epsilon} \) before or after \( \nabla_\mu \), and \( \eta \), and the expressions differ by more than just a sign.

There is an elegant solution: recall that it is actually a matter of convention whether different fermionic flavours commute or anticommute [18]. In other words, we are free to introduce a multiple grading. We will assign both a supergroup grading \( f \) and a ghost grading \( g \). All superfields including the ghosts have supergroup-odd block-off-diagonal entries (\( f = 1 \)) and supergroup-even block-diagonal entries (\( f = 0 \)). \( \tilde{\epsilon} \) is supergroup-even: \( f(\tilde{\epsilon}) = 0 \). \( A \) and \( C \) are ghost-even (\( g = 0 \)) and \( \eta, \bar{\eta} \) and BRST parameter \( \tilde{\epsilon} \) are ghost-odd (\( g = 1 \)). The algebra is completely determined by the requirement that elements commute up to a multiplicative extra sign whenever odd elements from the same grading are pushed past each other, i.e. for elements \( a \) and \( b \):

\[ ab = ba (-1)^{f(a)f(b)+g(a)g(b)}. \]  

(3.25)

One can readily check that (anti)cyclicity is now preserved \( \text{viz.} \)

\[ \text{str} \, \eta X = (-1)^{g(X)} \text{str} \, X \eta. \]  

(3.26)
\( \epsilon \) now simply (anti)commutes with (ghost-)superfields, and thus the usual form for the BRST algebra results. This can be used to prove all the usual properties of gauge fixing \( e.g. \) independence of the choice of gauge, transversality of on-shell Green functions and so on, and must thus yield the correct form for the Faddeev-Popov superdeterminant. In the next section we will give the explicit form of the gauge fixing function and ghost action that we will use, and in sec. 7 the explicit form of the corresponding BRST algebra.

4. Spontaneously Broken Action

Having settled the issues specific to the choice of \( SU(N|N) \), we now describe the full construction. Let the super-gauge field of \( SU(N|N) \) be denoted by \( A_\mu \equiv A^{\alpha}_\mu S_\alpha \). We can write this in supermatrix form with bosonic diagonal elements and fermionic off-diagonal elements:

\[
A_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix} + A^0_\mu 1. \tag{4.1}
\]

As discussed in sec. 3, \( A^0_\mu \) does not actually appear in the action, and we may either leave it in the theory where it has no effect (as we do here) or as shown in sec. 3, define it away entirely by modifying the Lie bracket selectively. Once again, our covariant derivative is taken\(^5\) to be \( \nabla_\mu = \partial_\mu - i\tilde{g} A_\mu \) with the field strength being \( F_{\mu\nu} := \frac{i}{\tilde{g}} [\nabla_\mu, \nabla_\nu] \). However it will prove helpful to make explicit the scale \( \Lambda \) hidden in the coupling constants when the number of (Euclidean) dimensions \( D \neq 4 \), by writing \( \tilde{g} = g \Lambda^{2-D/2} \), from now on. For the sake of generality, we introduce the covariant higher derivatives via functions \( W \). We introduce the convenient notation

\[
u \{ W \} v = v \{ W \} u = \text{str} \int d^Dx \ u(x) W(-\nabla^2/\Lambda^2) \cdot v(x), \tag{4.2}
\]

taking \( W \cdot v \) to mean that each \( \nabla_\mu \) acts via commutation. Let \( c^{-1} \) be a polynomial of rank \( r \). We can then write the pure Yang-Mills part of the action as

\[
S_{YM} = \frac{1}{2} F_{\mu\nu} \{ c^{-1} \} F_{\mu\nu}. \tag{4.3}
\]

Actually, these objects naturally arise and have a deeper meaning within the exact renormalization group [6, 2, 3, 4]. Thus \( c \) is a cutoff function, and there is actually a wide choice of the exact form of covariantization \( \{ W \} \). The formalism we present is independent of the choice (4.2) except for some particulars of the power counting proof in sec. 5.

The super-scalar field \( C \) is introduced:

\[
C = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix}, \tag{4.4}
\]

\(^5\)Unlike refs. [2, 4, 3] rescaling \( A_\mu \rightarrow A_\mu/\tilde{g} \) will be of no benefit here as we will fix the gauge.
with no restriction placed upon \( C \). Thus \( C \) can be expressed uniquely in terms of components as (3.18), or more simply just as an unconstrained Hermitian supermatrix field with components \( C^i_j \). We require that classically \( C \) picks up an expectation value \( \langle C \rangle = \sigma_3 \Lambda^{D/2-1} \), so that \( SU(N) \) is spontaneously broken to the bosonic subgroup \( SU(N) \times SU(N) \). (Again, the naively expected \( U(1) \) associated to \( A^0 \) has no effect or does not appear, cf. sec. 3.) The contribution of the \( C \) field to the action is chosen to be

\[
S_C = \frac{1}{2} \nabla_{\mu} \cdot C \{ \tilde{c}^{-1} \} \nabla_{\nu} \cdot C + \frac{\lambda}{4} \Lambda^{4-D} \text{str} \int d^D x \left( C^2 - \Lambda^{D-2} \right)^2. \tag{4.5}
\]

Note the introduction of a new cutoff function \( \tilde{c} \). Again it is convenient to choose \( \tilde{c}^{-1} \) to be a polynomial, this time of rank \( \tilde{r} \). By construction \( C = \sigma_3 \Lambda^{D/2-1} \) is a stationary point of the potential. Expanding about this \( (i.e. C \mapsto C + \sigma_3 \Lambda^{D/2-1}) \), the action (4.5) becomes

\[
S_C = -\frac{g^2}{2} \Lambda^2 [A_\mu, \sigma_3] \{ \tilde{c}^{-1} \} [A_\mu, \sigma_3] + ig \Lambda [A_\mu, \sigma_3] \{ \tilde{c}^{-1} \} \nabla_{\mu} \cdot C \\
+ \frac{1}{2} \nabla_{\mu} \cdot C \{ \tilde{c}^{-1} \} \nabla_{\nu} \cdot C + \frac{\lambda}{4} \Lambda^{4-D} \text{str} \int d^D x \left( \Lambda^{D/2-1} \{ \sigma_3, C \} + C^2 \right)^2. \tag{4.6}
\]

The first term of (4.6) gives a mass of order the effective cutoff to the fermionic part of \( A \). The bosonic part of \( C \) also gains a mass via the last part of (4.6).

To further investigate the properties of this action, we need to fix the gauge. To rid us of the part in the second term of (4.6) that mixes single powers of \( A \) and \( C \) fields, we follow ’t Hooft’s lead [19] and make the following choice of gauge fixing function:

\[
F = \partial_\mu A_\mu - ig \frac{\Lambda}{2\xi} \tilde{c} \{ \sigma_3, C \}, \tag{4.7}
\]

utilising another new cutoff function \( \tilde{c} \), \( \tilde{c}^{-1} \) being chosen polynomial of rank \( \tilde{r} \). Here the cutoff functions have argument \( (-\partial^2/\Lambda^2) \) because, being part of the gauge fixing, there is no need for covariantization in (4.7). After ’t Hooft averaging, the gauge fixing part of the action is

\[
S_{\text{Gauge}} = \xi F \cdot \tilde{c}^{-1} \cdot F \\
= \xi (\partial_\mu A_\mu) \cdot \tilde{c}^{-1} \cdot (\partial_\nu A_\nu) - ig \Lambda (\partial_\mu A_\mu) \cdot \tilde{c}^{-1} \cdot [\sigma_3, C] \\
- g^2 \Lambda^2 [\sigma_3, C] \cdot \tilde{c} \cdot [\sigma_3, C], \tag{4.8}
\]

using the notation \( u \cdot W \cdot v \equiv \text{str} \int d^D x \ u(x) W(-\partial^2/\Lambda^2) v(y) \). Introducing (4.8) into the action cancels the required term as well as providing a mass term for the fermionic part of \( C \).

Apart from the two \( SU(N) \) gauge fields (and the decoupled or missing \( A^0_\mu \)), all fields have masses of order the cutoff \( \Lambda \). Note that in the usual unitary gauge interpretation, the fermionic part of \( C \) is the would-be Goldstone mode which is eaten by the fermionic part of \( A \). \( A^1_\mu \) is the \( SU(N) \) field we set out to regulate. \( A^2_\mu \) is unphysical because the
sign of its action is the opposite of $A^1$ by the supertrace (2.2). As we explain in sec. 8, this leads to unitarity violations. Fortunately, since the two gauge fields belong to two different $SU(N)$ groups there is no bare interaction between them. Indeed any such interaction would have to involve products of gauge invariant supertraces. In sec. 8, we use this insight to show that the two sectors must decouple in the limit that the cutoff is removed.

The gauge fixing also introduces the Faddeev-Popov ghost super-fields (3.23). To tidy up the contribution to the action, we change antighost variables: $\bar{\eta} \rightarrow \hat{c}^{-1}\tilde{c}\bar{\eta}$. As we will see in sec. 5, this has the added benefit of ensuring that power counting arguments will be assigning the correct momentum behaviour to some of the ghost interaction vertices. Consequently, the ghosts appear in the action as

$$S_{\text{Ghost}} = -2\bar{\eta} \cdot \hat{c}^{-1} \tilde{c} \cdot \partial_\mu \nabla_\mu \eta - \frac{g^2}{\xi} \text{str} \int d^Dx [\sigma_3, \bar{\eta}] [\Lambda^2 \sigma_3 + C\Lambda^{3-D/2}, \eta].$$

(4.9)

As we saw in sec. 3, the introduction of a separate ghost grading ensures that simply an extra sign appears whenever two ghosts are moved passed each other.

Finally, in order to keep the high momentum behaviour of the $A$ propagator unchanged by the introduction of the $C$ field and gauge fixing, we require the ranks of our polynomial cutoff functions to be bounded as $\hat{r} \geq r > \tilde{r} - 1$. In the next section, in order to get proper bounds on these indices, it will be convenient to take $r, \tilde{r}, \hat{r}$ as general real numbers, the restriction to integers being taken at the end. As a matter of fact it is consistent to take these parameters to be real having in mind more general cutoff functions (analytic around the origin, $p = 0$, and with asymptotic behaviour $c^{-1} \sim \frac{p^{2r}}{\Lambda^r}$ etc.). In this case strictly speaking we should add the condition $\hat{r} > -1$ which is necessary to ensure that the high momentum behaviour of the $C$ field is unaffected by the spontaneous symmetry breaking mass term in (4.6). Thus the following conditions are required on the indices:

$$\hat{r} \geq r > \tilde{r} - 1 \quad \text{and} \quad \tilde{r} > -1.$$  

(4.10)

5. Power Counting

We now establish the finiteness of this theory, to all orders in $\hbar$. We start by computing an upper bound on the superficial degree of divergence of any one-particle-irreducible (1PI) diagram and show that this is negative in any dimension $D$, for all but a small number of one-loop diagrams, providing the indices $r, \tilde{r}$ and $\hat{r}$ satisfy the inequalities (5.16). Then in secs. 6 and 7, we establish that these one-loop exceptions are themselves finite in all dimensions $D < 8$, as a consequence of cancellations resulting from the supersymmetry of $SU(N|N)$ and gauge invariance. Since the superficial degree of divergence of any given diagram and all its connected proper sub-diagrams is thus shown to be negative, finiteness to all loops follows from the convergence theorem [20].
Using standard rules for calculating the superficial degree of divergence [20] of a 1PI diagram in $D$ space-time dimensions, we get

$$
\mathcal{D}_T = DL - (2r + 2) I_A - (2\bar{r} + 2) I_C - (2\bar{r} - 2\bar{r} + 2) I_\eta + \sum_{i=3}^{2r+4} (2r + 4 - i) V_{A^i} \\
+ \sum_{j=2}^{2\bar{r}+2} (2\bar{r} + 2 - j) V_{A^j C^j} + \sum_{k=1}^{2\bar{r}+2} (2\bar{r} + 2 - k) V_{A^k C^k} + (2\bar{r} - 2\bar{r} + 1) V_{\eta^2 A^2},
$$

(5.1)

where $L$ is the number of loops and $I_f$ and $V_f$ correspond to the number of internal lines and vertices of $f$-type respectively. In (5.1), inequalities (4.10) have already been assumed for the degree of divergence of the vector and $C$ propagators respectively to be counted properly.

As it stands, (5.1) does not account properly for 1PI diagrams with external anti-ghost lines. In fact, the whole momentum dependence of the $V_{\eta^2 A^2}$ vertex is counted as flowing in the loop(s), without taking into account the fact that such a dependence is actually only carried by $\bar{\eta}$ lines and, thus, that one has to check whether such lines are external or not. This results in a systematic overestimate of $\mathcal{D}_T$. In order to remedy this and, thus, improve our upper bound, $\mathcal{D}_T$, we add $-(2\bar{r} - 2\bar{r} + 1) E_{\bar{A}}^A$, with $E_{\bar{A}}^A$ being the number of external anti-ghost lines which enter $V_{\eta^2 A^2}$ vertices. Therefore, the improved formula for the superficial degree of divergence is

$$
\mathcal{D}_T = DL - (2r + 2) I_A - (2\bar{r} + 2) I_C - (2\bar{r} - 2\bar{r} + 2) I_\eta + \sum_{i=3}^{2r+4} (2r + 4 - i) V_{A^i} \\
+ \sum_{j=2}^{2\bar{r}+2} (2\bar{r} + 2 - j) V_{A^j C^j} + \sum_{k=1}^{2\bar{r}+2} (2\bar{r} + 2 - k) V_{A^k C^k} + (2\bar{r} - 2\bar{r} + 1) \left( V_{\eta^2 A^2} - E_{\bar{A}}^A \right). 
$$

(5.2)

The variables upon which $\mathcal{D}_T$ is dependent can be easily related to the number of external lines of each type, $E_f$, as

$$
L = 1 + I_A + I_C + I_\eta \\
- \sum_i V_{A^i} - \sum_j V_{A^j C^j} - \sum_k V_{A^k C^k} - V_{\eta^2 A^2} - V_{\eta^2 C^2} - V_{C^3} - V_{C^4},
$$

(5.3)

$$
E_A = -2I_A + \sum_i iV_{A^i} + \sum_j jV_{A^j C^j} + \sum_k kV_{A^k C^k} + V_{\eta^2 A^2},
$$

(5.4)

$$
E_C = -2I_C + \sum_j V_{A^j C^j} + 2 \sum_k V_{A^k C^k} + 3V_{C^3} + 4V_{C^4} + V_{\eta^2 C^2},
$$

(5.5)

$$
E_\eta = E_{\eta}^A + E_{\eta}^C + E_{\bar{\eta}}^A + E_{\bar{\eta}}^C = -2I_\eta + 2V_{\eta^2 A^2} + 2V_{\eta^2 C^2}.
$$

(5.6)

In the last of the above relations, to ensure consistency with previous notation we split external ghost lines according to the vertices they are attached to. Thus $E_{\bar{\eta}}^f$ ($E_\eta^f$), $f = A, C$, is the number of external (anti)ghost lines entering $V_{\eta^2 f}$ vertices; they satisfy the expected constraint $E_{\eta}^A + E_{\eta}^C = E_{\bar{\eta}}^A + E_{\bar{\eta}}^C$. We note for later that (5.6) may thus be written

$$
E_{\eta}^A + E_{\eta}^C = -I_\eta + V_{\eta^2 A^2} + V_{\eta^2 C^2},
$$

(5.7)
as can be most simply understood by deriving the equation directly as a count over external antighosts. (5.3) is valid for connected diagrams only, as the first term in the r.h.s. - representing the number of connected components - has been set to 1.

By making use of the above formulae, it is possible to rewrite $D_\Gamma$ in a more useful form, independent of internal lines,

$$D_\Gamma = (D - 2r - 4)(L - 2) - E_A - (r - \tilde{r} + 1) E_C - 2(r + \tilde{r} - \hat{r} + 1) E^C_{\eta} - (2r + 3) E^A_{\eta} - (r - \tilde{r} + 1) \sum_j V_{A/C} + (r - 3\tilde{r} - 1) V_{C3} + 2(r - 2\tilde{r}) V_{C4} + (r + \tilde{r} - 2\tilde{r} - 1) V_{\eta C} + 2(D - r - 2).$$

(5.8)

We now establish necessary and sufficient constraints on $r$, $\tilde{r}$ and $\hat{r}$, such that $D_\Gamma$ is negative for as many diagrams as possible. Not all diagrams can be regularised this way. For example, the superficial degree of divergence of the one-loop diagrams involving only $A$ fields is $D - E_A$, which is non-negative for $E_A \leq D$ independent of the parameters $r$, $\tilde{r}$ and $\hat{r}$. We will start with the proof of a proposition which will help us dispense with some of the constraints we find; then we will analyse diagrams with two or more loops and, after, we will return to one-loop diagrams.

Let us denote by $S$ the collection of triples $(r, \tilde{r}, \hat{r})$ such that $D_\Gamma$ is negative for any given set of 1PI diagrams and (4.10) holds.

**Proposition 1:** If $(r_0, \tilde{r}_0, \hat{r}_0) \in S$, then the subset $\{(r, \tilde{r}, \hat{r}) \text{ s.t. } r \geq r_0, \tilde{r} = \tilde{r}_0, \hat{r} \geq \hat{r}_0, \tilde{r}_0 - 1 < r \leq \hat{r}\} \subset S$.

**Proof:**

The proof is essentially based on the one-particle-irreducibility of diagrams.

The whole dependence of (5.8) on $\hat{r}$ amounts to $2\hat{r}(E^C_{\eta} - V_{\eta C})$, which is always non-positive as it is not possible to have more external anti-ghost lines entering $V_{\eta C}$ vertices than $V_{\eta C}$ vertices themselves. Thus, increasing $\hat{r}$ above $\hat{r}_0$ can only leave stationary or decrease an already negative $D_\Gamma$.

As far as $r$ is concerned, it enters (5.8) as

$$r\left(-2L + 2 - E_C - 2E^C_{\eta} - 2E^A_{\eta} - \sum_j V_{A/C} + V_{C3} + 2V_{C4} + V_{\eta C}\right) = 2r\left(\sum_i V_{Ai} - I_A\right),$$

(5.9)

where the last equality follows by using (5.3)–(5.6), or directly from (5.2). This contribution is always non-positive as we know that in a 1PI diagram every $V_{A}$ vertex must attach to at least two internal $A$ lines. Therefore increasing $r$ above $r_0$ can at most cause $D_\Gamma$ to decrease further. \qed

### 5.1. Multiloop graph analysis

In order for every possible $L \geq 2$ loop 1PI diagram to have a negative $D_\Gamma$, we can impose all coefficients in (5.8) to be negative and, thus, get sufficient conditions. This
amounts to the following relations
\[ r > D - 2, \quad 2r + 3 > 0, \quad r < 2\tilde{r}, \quad \tilde{r} < r + \hat{r} + 1, \quad (5.10) \]
together with (4.10). \textit{(N.B. In the case of polynomial cutoff functions, it is easy to see that there are integers } r, \tilde{r}, \hat{r} \text{ satisfying (4.10) and (5.10).)}

The conditions (5.10) imply a lower bound on \( \tilde{r}, \tilde{r} > \frac{1}{2} \max \left( D - 2, -\frac{3}{2} \right) \), as well. The \( D \)-dependent lower bounds on \( r \) and \( \tilde{r} \) may be expected to be also necessary, as the higher the space-time dimension, the more divergent the diagrams. However, physics does not provide any reasonable arguments to explain upper bounds on \( \hat{r} \) and \( r \), apart from \( r \leq \hat{r} \) [cf. (4.10)]. In fact they are not necessary, as one can easily appreciate by making use of Proposition 1: applying it to all triples \((r_0, \tilde{r}_0, \hat{r}_0)\) which satisfy (5.10) and (4.10), we see that the third and the fourth inequalities in (5.10) are not necessary, and we are thus left with the sufficient relations
\[ r > \max \left( D - 2, -\frac{3}{2} \right), \quad \tilde{r} > \max \left( \frac{D}{2} - 1, -\frac{3}{4} \right) \quad \text{and} \quad \hat{r} \geq r > \tilde{r} - 1. \quad (5.11) \]
If we only rely on power counting, these conditions are also necessary, for any \( D \geq \frac{1}{2}, \) as they ensure finiteness in the two two-loop vacuum diagrams with only \( A^3 \) and \( C^4 \) vertices respectively. Actually, both these diagrams vanish by the supertrace mechanism explained in sec. 6, but we will see that there are non-vanishing one-loop diagrams that require the same conditions.

Before moving to one-loop diagrams, it may be helpful to illustrate the use of Proposition 1 within a restricted class of multiloop graphs where it easier to see in detail what is going on. Let us focus on the subset of multiloop vacuum diagrams formed from only \( C^4 \) vertices. The superficial degree of divergence takes a very simple form, \( \mathcal{D}_\Gamma = (D - 4\tilde{r} - 4)(L - 2) + 2(r - 2\tilde{r})V_{C^4} + 2(D - r - 2), \) which is negative for every possible diagram in the \( L \geq 2 \) set provided the relations \( r > D - 2, r < 2\tilde{r} \) are imposed. Again, these conditions imply a lower bound on \( \tilde{r}, \tilde{r} > \frac{D}{2} - 1. \) As the number of \( C^4 \) vertices can be arbitrarily large, one can be misled and conclude that the relation \( r < 2\tilde{r} \) is also necessary as an asymptotic condition. However, increasing the number of \( C^4 \) interactions also increases the number of loops, as follows for this restricted set from eqs. (5.3)–(5.6). Indeed in this simple case, reexpressing \( L \) in terms of \( V_{C^4} \) yields \( \mathcal{D}_\Gamma = (D - 4\tilde{r} - 4)(V_{C^4} - 1) + 2(D - 2\tilde{r} - 2). \) \( \mathcal{D}_\Gamma \) is thus independent of \( r, \) hence increasing \( r \) does not change it, which is actually the essence of Proposition 1. One is then left with the (necessary) constraint \( D - 2\tilde{r} - 2 < 0. \) Whilst for a number of general classes of diagrams, we can similarly demonstrate that (5.11) provides sufficient conditions by reexpressing \( \mathcal{D}_\Gamma \) using (5.3)–(5.6), this is not possible for the full set of multiloop graphs. Fortunately Proposition 1 comes elegantly to the rescue.

\subsection*{5.2. One-loop diagram analysis}

As mentioned below (5.8), not all one-loop diagrams can be regularised by imposing constraints on the ranks of the cutoff functions. This is why several sub-cases are to be
analysed when dealing with one-loop graphs. Nonetheless, the strategy we are going to use is pretty much the same as in the previous subsection. We will first rewrite $D_\Gamma$ in terms of the proper, non-negative variables and, then, we will impose all its coefficients to be negative so as to get sufficient conditions. Finally, we will relax some of those conditions by means of Proposition 1, and prove the remainder to be necessary.

Let us start with specialising (5.8) to $L = 1$:

$$D_\Gamma^{1\text{-loop}} = D - E_A - (r - \tilde{r} + 1) E_C - 2(r + \tilde{r} - \hat{r} + 1) E_{\bar{\eta}} - (2r + 3) E_{\bar{\eta}}^A$$

$$-(r - \tilde{r} + 1) \sum_j V_{A/C} + (r - 3\tilde{r} - 1) V_{C^3} + 2(r - 2\tilde{r}) V_{C^4} + (r + \tilde{r} - 2\hat{r} - 1) V_{\bar{\eta}^C}.$$  

(5.12)

The first set of diagrams we are going to analyse consists of all the one-loop diagrams with at least $D + 1$ external $A$ lines - so that the combination $(E_A - D - 1)$ is always non-negative within the set - and any number of external $C$ and (anti-)ghost lines. Reexpressing (5.12) in terms of $(E_A - D - 1)$ amounts to replacing $D - E_A$ with $-(E_A - D - 1) - 1$ without altering the rest of the expression. Therefore, in order for all its coefficients to be negative, we have to impose

$$r < 2\tilde{r}, \quad 2r + 3 > 0, \quad \hat{r} < r + \tilde{r} + 1,$$  

(5.13)

together with (4.10).

The next set we will deal with is made up of all the one-loop diagrams with at least one external antighost line, $E_{\bar{\eta}}^A \geq 1$, and any number of external $A$’s and $C$’s.

In the case of an $A$-type antighost line, upon changing $E_{\bar{\eta}}^A$ to $(E_{\bar{\eta}}^A - 1)$ we get $D_\Gamma^{1\text{-loop}} = (D - 2r - 3) - (2r + 3) (E_{\bar{\eta}}^A - 1) + \cdots$, where the ellipsis stands for unchanged terms in (5.12). Demanding that all coefficients in this relation be negative results in the extra condition $D - 2r - 3 < 0$ [together with (5.13)].

In the case of a $C$-type antighost line, we get a different constraint. Introducing $(E_{\bar{\eta}}^C - 1)$ into (5.12) causes it to change to $D_\Gamma^{1\text{-loop}} = D - 2(r + \tilde{r} - \hat{r} + 1) - 2(r + \tilde{r} - \hat{r} + 1) (E_{\bar{\eta}}^C - 1) + \cdots$, where again unchanged terms have not been written down. Imposing the coefficients in the above expression to be negative yields the condition $D - 2(r + \tilde{r} - \hat{r} + 1) < 0$, which has to replace the upper bound on $\hat{r}$ in (5.13) for any $D \geq 0$.

Let us now analyse the set of one-loop graphs with at least two external $C$’s and any number of external $A$ and (anti-)ghost lines. Rewriting (5.12) as $D_\Gamma^{1\text{-loop}} = D - 2(r - \tilde{r} + 1) - (r - \tilde{r} + 1) (E_C - 2) + \cdots$ and demanding all its coefficients to be negative we find another constraint, $r - \tilde{r} > \frac{D}{2} - 1$, which will turn out to be necessary as well.

Finally, let us take into account one-loop diagrams with just one external $C$ and no external $A$ nor (anti)ghost lines. We need not consider the more general case when any number of external $A$’s and (anti)ghosts is allowed because, within the ranges (5.13) and (4.10), they both contribute negatively to $D_\Gamma^{1\text{-loop}}$, irrespective of the choice of vertices. Despite containing three elements only, depending on the ‘flavour’ $A$, $\eta$ or
\( \mathcal{C} \), of the loop, this set of diagrams gives us two further conditions: \( r - \tilde{r} > \frac{D}{2} - 1 \) and \( \hat{r} - \tilde{r} > \frac{D}{2} - 1 \) for the \( \mathcal{A} \) and \( \eta \) loop flavours respectively. The latter, however, is always fulfilled if the former is, \( \hat{r} \) being greater than or equal to \( r \).

The results of one-loop diagram analysis can be therefore summarised as follows: all the \( L = 1 \) diagrams except those with up to \( D \) external \( \mathcal{A} \) legs and no \( \mathcal{C} \) or \( \bar{\eta} \) (and thus also \( \eta \)) external lines, are regulated by imposing the following constraints,

\[
2r + 3 > 0, \quad \hat{r} < r + \tilde{r} + 1, \quad D - 2r - 3 < 0, \quad r < 2\tilde{r}, \quad \hat{r} < r + \tilde{r} + 1 - \frac{D}{2}, \quad r - \tilde{r} > \frac{D}{2} - 1, \tag{5.14}
\]

together with (4.10).

By inspection of (5.12) it is possible to reduce further the set of diagrams that appear to remain unregularised: within the set of one-loop diagrams with up to \( D \) external \( \mathcal{A} \) legs and no \( \mathcal{C} \) or (anti)ghost external lines, only those formed from \( V_{\mathcal{A}^kC^2j} \), \( V_{\mathcal{A}''}, V_{\eta''\mathcal{A}} \) need to be analysed. (These vertices contribute nothing to \( D^1_{\text{1-loop}} \).) From topological considerations \( V_{\mathcal{C}^3} = V_{\mathcal{C}^4} = V_{\eta''\mathcal{C}} = 0 \) and either \( \sum_j V_{\mathcal{A}^jC} = 0 \) or \( \sum_j V_{\mathcal{A}^jC} \geq 2 \) in which case by (5.14) already \( D^1_{\text{1-loop}} < 0 \). It is helpful to give this set a name: the “One-loop Remainder Diagrams”.

Relations (5.14) imply lower bounds on \( \tilde{r}, \tilde{r} > \frac{1}{2} \max \left( D - 2, \frac{D - 3}{2}, -\frac{3}{2} \right) \), and on \( r, \quad r > \max \left( D - 2, \frac{D - 3}{2}, -\frac{3}{2} \right) \), as well. To get the former note that \( r < \hat{r} < r + \tilde{r} + 1 - \frac{D}{2} \) and use \( 2r + 3 > 0, D - 2r - 3 < 0 \); as for the latter, use \( r - \tilde{r} > \frac{D}{2} - 1 \) together with \( 2r + 3 > 0, D - 2r - 3 < 0 \).

By making use of Proposition 1 it is possible to rid us of the upper bounds on \( r, \hat{r}, \tilde{r} \), so that we are left with

\[
r > \max \left( D - 2, \frac{D - 3}{2}, -\frac{3}{2} \right), \quad \hat{r} > \frac{1}{2} \max \left( D - 2, \frac{D - 3}{2}, -\frac{3}{2} \right), \quad r - \tilde{r} > \frac{D}{2} - 1 \tag{5.15}
\]

and (4.10).

For any \( D \geq 1 \) the above set of solutions contains (5.11), together with one new relation, \( r - \tilde{r} > \frac{D}{2} - 1 \). We have already seen that these conditions are also necessary for regularisation purely by power counting, i.e. if we ignore cancellations arising from the supertrace mechanism. In fact there are one-loop diagrams for which these conditions are necessary even with the supergroup factors taken into account. To see this, we borrow a result from the next section, that group theory factors for unbroken one-loop corrections take the form of a product of two supertraces over the external fields (resulting in vanishing terms whenever one supertrace is empty or contains only a single \( \mathcal{A} \)). It follows that in the broken \( SU(N|N) \) theory, the results still take this form except that \( < \mathcal{C} > = \sigma_3 \Lambda^{D/2 - 1} \) terms may also appear in the supertraces. Now, the condition \( r - \tilde{r} > \frac{D}{2} - 1 \) arose from power counting the one-loop graph made by attaching an \( \mathcal{A} \) propagator to the \( \mathcal{C} \mathcal{A}^2 \) vertex [i.e. by inspection the vertex from \( -ig \Lambda [\mathcal{A}_\mu, \sigma_3] \{ \tilde{c}^{-1} \} \nabla_\mu \cdot \mathcal{C} \) of (4.6)]. Thus \( r - \tilde{r} > \frac{D}{2} - 1 \) is necessary for the contributions
with group theory factor $\text{str} \mathcal{C} \text{str} \sigma_3$ (which one can readily check are non-vanishing). The condition $\tilde{r} > D/2 - 1$ is necessary for finiteness of $(\text{str} \mathcal{C})^2$ contributions arising from attaching a $\mathcal{C}$ propagator to the $\text{str} \mathcal{C}^4$ vertex, or for the $\text{str} \mathcal{C} \text{str} \sigma_3$ term arising from attaching the $\mathcal{C}$ propagator to the $\mathcal{C}^2 \{\mathcal{C}, \sigma_3\}$ vertex. (Again one can confirm that these contributions are non-vanishing.) The final condition for any $D \geq 1$, namely $r > D - 2$, already follows from combining these two.

The analysis of one-loop diagrams can also be performed by adopting a completely different strategy, based on a form of *divide and conquer* algorithm. Here we will explain only the general idea, without going into details [8]. We start with cutting up diagrams into tadpole-like pieces, defined as the sub-diagrams which contain just one internal propagator attached to one vertex [4]. This can be done in two different ways, according to which propagator remains attached to the vertex being cut. We then compute the degree of divergence of every possible piece we can end up with, aiming to show that they all contribute negatively to the overall $D_\Gamma$. If this is the case - and it is indeed - what is left is just the analysis of the simplest possible one-loop graphs, as any other can be obtained by inserting tadpole-like pieces, which causes $D_\Gamma$ to decrease further. In other words, one can always bound from above the degree of divergence of a one-loop diagram by removing tadpole-like pieces one by one and joining together the rest of the diagram - hence increasing the overall $D_\Gamma$. Eventually one will be left with a very simple graph, usually a proper tadpole. The first part of the analysis, that is calculating $D_\Gamma$ for such “components”, is straightforward: by inspection of (5.1) it is easy to appreciate that all the possible sub-diagrams contribute negative terms within the bounds we have already set on $r$, $\tilde{r}$ and $\hat{r}$ (eqs (4.10) and (5.10)). However, this is not all that we need: when sewing back the diagram after a tadpole-like piece has been removed, it is possible to end up with a different vertex. This happens every time the sewn propagator is different from the one removed. Such effect should be taken into account as different vertices contribute differently to $D_\Gamma$. The second part of the analysis, i.e. showing that all the simplest possible one-loop diagrams\(^6\) can be regulated by a suitable choice of $r$, $\tilde{r}$ and $\hat{r}$, is quite long but straightforward as well. One further constraint, $r - \tilde{r} > D/2 - 1$, is obtained, precisely as before.

To summarise, all but the small set of One-loop Remainder Diagrams, as defined below (5.14), can be regulated simply by a suitable choice of the ranks of the cutoff functions; from (5.14) and (4.10) the allowed ranges for any $D \geq 1$ are

$$\hat{r} \geq r, \quad r - \tilde{r} > D/2 - 1, \quad \tilde{r} > D/2 - 1.$$  \hspace{1cm} (5.16)

It is interesting and comforting to note that these resulting relations for $r$ and $\tilde{r}$ are precisely the ones deduced for our earlier un-gauge-fixed but more limited regularisation scheme [4]. For the case that the inverse cutoff functions are polynomials, and $D$ is integer greater than or equal to 2, these inequalities imply $\hat{r} \geq r \geq D - 1$, $\tilde{r} \geq \left[\frac{D-2}{2}\right] + 1$ and $r - \tilde{r} \geq \left[\frac{D-2}{2}\right] + 1$, $[x]$ being the integer part of $x$.

\(^6\)except those with no external $\mathcal{C}$ or $\bar{\eta}$ lines and with up to $D$ external $\mathcal{A}$ legs.
6. Supertrace Mechanism

We have seen that the conditions (5.16) are necessary and sufficient to ensure that, in $D \geq 1$ dimensions, all diagrams are superficially finite already by power counting, with the exception of the One-loop Remainder Diagrams: those formed from only $C^2A^j$, $A^i$ and $\eta A\eta$ vertices, with $D$ or less external $A$ legs and no external $C$ or (anti)ghost legs. By the power counting (5.12), their superficial degree of divergence is $D_{\Gamma} = D - E_A$. Actually, these diagrams are also finite as a consequence of cancellations that are not incorporated in the power counting analysis. In this section we show that of these one-loop diagrams, the ones with three or less external $A$ legs are finite as a consequence of the supersymmetry of the unbroken $SU(N|N)$. This cancellation will be referred to as the supertrace mechanism. The One-loop Remainder Diagrams with $3 < E_A \leq D$, will be shown in sec. 7 to be finite in all dimensions $D < 8$ as a consequence of this mechanism plus constraints arising from gauge invariance. Actually, as a bonus, we will see in this section that in all the One-loop Remainder Diagrams, the parts arising from the spontaneous symmetry breaking are already finite by power counting. This observation will prove useful in simplifying the analysis in sec. 7.

Clearly One-loop Remainder Diagrams are formed in one of three ways: either we use only $C$ propagators joining $C^2A^j$ vertices together, or we use $A$ propagators to join just $A^i$ vertices together, or finally we can use just $\eta$ propagators to join just $\eta A\eta$ vertices together. It will prove useful to construct the one-loop diagrams in two steps: by first constructing a tree diagram and then closing the tree into a loop by attaching a further propagator. We discuss the $C$ case first.

6.1. One-loop Remainder Diagrams with $C$ propagators

For large momentum the $C$ propagator behaves as:

$$<C^i_j(p)C^k_l(-p)> = \frac{\tilde{c}}{p^2} \delta^i_j \langle \sigma_3 \rangle^k_l + O(p^{-4-2m}),$$

(6.1)

where $\tilde{c} \equiv \tilde{c}(p^2/\Lambda^2)$ and $m = \min(2\tilde{r}, \hat{r})$. The first term on the r.h.s comes from inverting the unbroken part of the kinetic term in (4.6), and at large momentum gives the behaviour already incorporated in the power-counting analysis, and the second term gives the asymptotic behaviour coming from the symmetry breaking terms (namely the mass term $\frac{1}{4}\Lambda^2\{\sigma_3, C\}^2$ in (4.6) and the last term in (4.8). The propagators may be computed in the usual way by adding a source term to the Lagrangian, and noting that the unbroken kinetic term has the form $\frac{1}{4}C^i_j[p^2\tilde{c}^{-1}\langle \sigma_3 \rangle^j_l\delta^i_k]C^k_l$. In sec. 7, it will prove convenient to introduce the source as $\text{str} JC$ where $J$ is a supermatrix field and thus $(\sigma_3 J)^T$ is the usual source.)

We see by (5.16) that parts of the one-loop integral involving the symmetry breaking mass term in (6.1), are already finite since their degree of divergence is bounded by $D_{\Gamma} \leq D - E_A - 2 - 2\min(\tilde{r}, \hat{r} - \tilde{r}) < 0$. Thus, since the $C^2A^j$ vertices come from
the unbroken part of (4.6), we note that the potentially divergent contribution has the same structure as the symmetric $SU(N|N)$ theory.

Diagrams are constructed by Wick contracting (\textit{i.e.} creating propagators) in expressions constructed out of supertraces (originating from the interactions). Ignoring the momentum dependence (since we are only interested here in the group theory factors) tree contributions formed from $\mathcal{C}$ propagators thus take the form:

$$\text{str}(XC) \text{str}(CY) = \text{str}(XY) + \cdots.$$  \hspace{1cm} (6.2)

Here $X$ and $Y$ are superfields or products of superfields. We have used the freedom to cycle the two supertraces containing $\mathcal{C}$, and combined them with (6.1). The ellipsis corresponds to the neglected terms in (6.1).

These are closed into a $\mathcal{C}$ flavour loop by a further Wick contraction. The resulting terms have either already been shown to be finite (since they come from the symmetry broken part) or else without loss of generality the group theory part takes the form:

$$\text{str}(CXY) = \text{str}X \text{str}Y + \cdots,$$ \hspace{1cm} (6.3)

where again we use (6.1), $X$ and $Y$ are (products of) remaining superfields, and the ellipsis is the neglected finite term generated by the second term in (6.1).

The one-loop diagrams we are presently interested in are thus given by a sum of contributions which are either already shown to be finite or have the group theory structure $\text{str}X \text{str}Y$, where $X$ and $Y$ may contain in total up to three gauge fields. But such a term vanishes trivially, since either $X$ or $Y$ must have one or less gauge field and thus yield $\text{str}\mathcal{A} = 0$ or $\text{str}\mathcal{I} = 0$. Thus we see that $\mathcal{C}$ flavour One-loop Remainder Diagrams with $E_A \leq 3$, are finite as a consequence of power counting and these supertrace identities.

### 6.2. One-loop Remainder Diagrams with $\mathcal{A}$ propagators

The analysis for $\mathcal{A}$-type loops proceeds similarly, however this time with an extra twist because only $\mathcal{A}^4$ propagates \textit{(cf. sec. 3)}. For large momentum the $\mathcal{A}$ propagator behaves as

$$<\mathcal{A}^4(p)\mathcal{A}^B(-p)> = \frac{c}{p^2} g^{AB} \left[ \delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \left( \frac{\hat{c}}{\xi c} - 1 \right) \right] + O(p^{2\gamma-4\gamma-4}),$$ \hspace{1cm} (6.4)

where we have used (2.11) and (2.12), and again we suppress the $p^2/\Lambda^2$ dependence of the cutoff functions. The first term on the right hand side comes from (4.3) and (4.8), has the same form as in the unbroken theory and gives the behaviour already accounted for in the power-counting analysis, and the second term is the asymptotic behaviour coming from the regularised symmetry breaking mass term in (4.6).

Once more we see by (5.16) that parts of the one-loop integral involving symmetry breaking terms, are already finite: to form the One-loop Remainder Diagrams we need
to use pure $\mathcal{A}^i$ vertices and these are either the unbroken ones from (4.3), giving the index of divergence $2r + 4 - i$ as ascribed in the power counting analysis, or again from the regularised mass term in (4.6) with index $2\tilde{r} + 2 - i$. Thus if we use the symmetry breaking part of the propagator in (6.4) and/or the symmetry breaking vertices the degree of divergence of the resulting integral is bounded by $D_G \leq D - E_A - 2(r - \tilde{r} + 1) < 0$. Once again these are the terms that will be indicated only by the ellipsis.

From (2.16) and (6.4), the group theory part of tree contributions take the form:

$$\text{str}(X \mathcal{A}) \text{str}(\mathcal{A}Y) = \frac{1}{2} \text{str}(XY) + \cdots. \quad (6.5)$$

At first sight we should also add the terms

$$-\frac{1}{4N} (\text{tr}X \text{str}Y + \text{str}X \text{tr}Y), \quad (6.6)$$

coming from (2.16). These terms express the fact that only the parts which are both traceless and supertraceless, viz. $X_R \equiv X - \frac{\sigma_3}{2N} \text{str}X - \frac{1}{2N} \text{tr}X$, couple to $\mathcal{A}^4$. Indeed (6.6) may be absorbed into (6.5) to give $\frac{1}{2} \text{str}(X_R Y_R) + \cdots$. Nevertheless if (6.6) really remained, it would imply that the propagation of only $\mathcal{A}^4$ is inconsistent since these terms arise in the unbroken theory but $\text{tr}X = \text{str} \sigma_3 X$ (and ditto $Y$) breaks $SU(N|N)$. Actually since all $\mathcal{A}$ interactions are through commutators, by rearrangement we can always express $X$ and $Y$ themselves as commutators, whence $\text{str}X$, $\text{str}Y$ and (6.6) actually vanishes.

By these arguments and (6.5), such tree diagrams themselves are supertraces of $\mathcal{A}$ times nested commutators (in the free-$\mathcal{A}^0$ representation) and thus for any given pair of (external) $\mathcal{A}$s in such a tree diagram, the group theory part may be expressed as a sum of contributions of the form

$$\text{str}([\mathcal{A}, Z_1]Z_2[\mathcal{A}, Z_3]Z_4) \quad \text{and/or} \quad \text{str}(\mathcal{A}[\mathcal{A}, Z_1]) \quad (6.7)$$

where the $Z_i$ are of course supermatrices. $Z_2$ and $Z_4$ could be $\mathbb{I}$ or non-trivial (in which case in fact further commutators can be made). Closing the resulting trees into an $\mathcal{A}$ flavour loop, by using (6.4) and (2.17), the group theory part generically may be written:

$$\text{str}(\mathcal{A}X \mathcal{A}Y) = \frac{1}{2} \text{str}X \text{str}Y + \cdots. \quad (6.8)$$

Here we have used the fact that only the block-diagonal, i.e. super-group even, part of $X$ contributes (otherwise the Wick contraction connects $B$s to $A$s in (4.1) and trivially vanishes). Again at first sight we ought to be including some unexpected $SU(N|N)$ breaking terms:

$$-\frac{1}{4N} \text{tr}(XY + YX), \quad (6.9)$$

\footnotetext[7]{i.e. without also $\mathcal{A}^0$ and/or $\mathcal{A}^\sigma$}

\footnotetext[8]{in the free-$\mathcal{A}^0$ representation or else extra interactions arise from the *bracket, cf. sec. 3}
coming from (2.17), however expanding the actual structures (6.7) and summing over (6.9) with the resulting $X$ and $Y$, one readily finds that these terms vanish.

The net result is the same as the $C$ loop: One-loop Remainder Diagrams are given by a sum of contributions that either include spontaneous symmetry breaking terms in which case they are finite, or take the form of the unbroken theory in which case they are the product of two supertraces, which vanishes for $E_A \leq 3$.

6.3. One-loop Remainder Diagrams with $\eta$ propagators

The analysis of this case is virtually the same as for $A$ above, with the same conclusions, unsurprisingly since $\eta$ is by BRST intimately related to gauge transformations (see sec. 7). The symmetry breaking mass term in (4.9) yields asymptotic contributions whose $D_T$ is bounded above by $D - E_A - 2(\tilde{r} - \tilde{r} + 1)$ and is thus already finite by (5.16). Only the components $\eta^A$ and $\tilde{\eta}^B$ propagate and thus individual tree contributions may result in terms of form (6.6), but these vanish once we collect the interactions into their commutator form. The same comments apply to loops and (6.9).

To summarise, we have seen that whatever flavour is involved in the loop, those parts of the One-loop Remainder Diagrams associated with the spontaneous breaking of $SU(N|N)$ are finite by (5.16). (Note that this includes the second term in the gauge fixing function (4.7), and the corresponding terms in (4.8) and (4.9).) Although we have concentrated on the One-loop Remainder Diagrams, it is clear from the preceding analysis that all one-loop unbroken $SU(N|N)$ contributions, have double supertrace form $\text{str}X\text{str}Y$, where $X$ and $Y$ are (products) of the external $\cdot$As, and thus vanish for $E_A \leq 3$. Consequently One-loop Remainder Diagrams with three or less external gauge fields are finite by power counting and the supertrace mechanism.

6.4. Large $N$ limit

It is appropriate to note here that in the large $N$ limit all these unbroken $SU(N|N)$ contributions vanish. The large $N$ limit for Yang-Mills is achieved by rescaling $g^2$ to $g^2/N$. As a loop-counting parameter this is balanced by those terms which contribute an extra $\text{tr}\mathbb{1} = N$ at each new loop order resulting in a non-trivial limit [14]. In our case, these double (super)trace terms are down by a factor $1/N$, unless one of them is empty but then the result vanishes by $\text{str}\mathbb{1} = 0$. It follows in particular that all the One-loop Remainder Diagrams are thus finite in the large $N$ limit, and thus we have proved that in the $N = \infty$ limit, the theory is finite in all dimensions $D$.

In fact in this way there are no factors of $N$ coming from loops to balance the rescaled $g^2/N$, since $\text{tr}\mathbb{1} = N$ has been replaced by $\text{str}\mathbb{1} = 0$. Thus as a consequence of the supertrace mechanism, in the large $N$ limit the symmetric phase $SU(N|N)$ theory has no quantum corrections at all [2][3][4].
7. Ward Identities

At finite \( N \), we have not ruled out the possibility of divergent one-loop contributions with \( 3 < E_A \leq D \) external \( A \)s (and no external \( C \)s or ghosts) originating from the unbroken theory, however we have yet to use the constraints of gauge invariance, which limit the possible divergences (just as they do for quantum electrodynamics and ordinary Yang-Mills, \( e.g. \) in the finiteness of the four-photon vertex, and the Slavnov-Taylor identities respectively). We will show that these remaining diagrams are in fact finite in all dimensions \( D < 8 \). Since all other corrections have already been shown to be superficially finite in any dimension,\(^9\) the finiteness to all orders in perturbation theory of the full theory is then proved for all dimensions \( D < 8 \).

Since the contributions in question arise from the unbroken parts of the Lagrangian only [including only those generated by the first term of (4.7)], we may as well work with unbroken Ward identities: this simplifies the arguments, and the broken Ward identities only introduce further terms which as we have already seen in sec. 6, are finite, given (5.16), already by power counting.

Working from now on in this section with the unbroken phase, we remind that the one-loop two and three-point pure \( A \) vertices actually vanish by the supertrace mechanism (\( cf. \) sec. 6). In \( D = 4 \) dimensions for example, considerations of renormalizability would make it very surprising if the one-loop four-point vertex then turned out to diverge!

Indeed, gauge transformations equate any longitudinal part of the four-point vertex to the sum of three-point vertices. (See for example the discussion of such identities in refs. [3, 4]. The explicit equation is that given by just the first three terms in (7.17).) Since the three-point vertex vanishes, the four-point vertex must be transverse on all four legs. This is only possible if the diagram works out to have a tensor structure involving at least four external momenta.\(^{10}\) This means that there are four less powers of loop momentum available, so the superficial degree of divergence of the one-loop four-point vertex drops from \( D_\Gamma = D - 4 \) to \( D_\Gamma = D - 8 \). In other words the one-loop four-point pure \( A \) vertex is finite in all dimensions \( D < 8 \).

Proceeding in this way, we can show the finiteness of all the remaining vertices. Thus any longitudinal part of the five-point pure \( A \) vertex is equal to the sum of four-point vertices, and thus is finite for all \( D < 8 \). All that remains is a totally transverse part which by the arguments above actually has \( D_\Gamma = D - 5 - 5 \), and thus is finite for all \( D < 10 \). By iteration, we see that for all dimensions \( D < 8 \) the remaining \( 3 < E_A \leq D \) One-loop Remainder diagrams are finite as a consequence of power counting, the supertrace mechanism and gauge invariance.

However, the above arguments are only strictly valid in a scheme such as developed in ref. [4], in which manifest gauge invariance is maintained at all stages. In

\(^{9}\) for suitable choice of ranks \( r, \tilde{r}, \check{r} \), \( cf. \) (5.16)

\(^{10}\) An \( A_\mu(p) \) external line must have tensor structure \( \delta_{\mu \alpha} p_\alpha - \delta_{\mu \beta} p_\beta \) where \( \alpha \) and \( \beta \) are other external indices or contract into indices in the rest of the diagram. This is the structure of \( F_{\alpha \beta} \).
order for the arguments to be rigorous in this more traditional gauge fixed context, we must demonstrate the existence of the corresponding BRST invariance and develop the appropriate Lee-Zinn-Justin identities.

One can readily check that with the multiple grading assignments of sec. 3, the usual BRST algebra:

\[
\begin{align*}
\delta A_\mu &= \epsilon \Lambda D^{2-2}\nabla_\mu, \eta \\
\delta C &= -i g \epsilon [C, \eta] \\
\delta \eta &= i g \epsilon \eta^2 \\
\delta \bar{\eta} &= \epsilon \xi \Lambda D^{2-2} \bar{c}^{-1} F_{symm}
\end{align*}
\] (7.1)
is an invariance of the naïve functional measure and the unbroken action \( S_{YM} + S_C + S_{Gauge} + S_{Ghost} \), where \( S_{YM} \) is given in (4.3), \( S_C \) is the unbroken version (4.5), \( S_{Gauge} \) utilises only \( F_{symm} = \partial_\mu A_\mu \) which is the gauge fixing function (4.7) but discarding the second part referring to breaking, and similarly the ghost action refers only to the first part of (4.9). (\( \epsilon \) has been defined dimensionless. Of course it is straightforward to write the BRST algebra and so forth for the broken Ward identities and/or more general gauge fixing functions, but not helpful in the present context.)

The derivation of the Lee-Zinn-Justin identities proceeds in standard fashion \cite{21}.\footnote{except for care with sources (fields) that do not couple to \( \sigma_3 \) (I), and commutation}

As usual we add to the action source terms for the fields and the non-linear BRST transformations, however it is helpful to express them as supermatrices and contract using the supertrace:

\[
S_{Sources} = - \text{str} \int d^Dx \left( J_\mu A_\mu + J C + \bar{\zeta} \eta + \bar{\eta} \zeta + \Lambda D^{2-2} \kappa_\mu \nabla_\mu \cdot \eta - i g H[C, \eta] + i g L \eta^2 \right)
\] (7.2)

The sources thus live in the dual space as determined by the Killing metric (2.10) and (3.2). Therefore

\[
J = \begin{pmatrix} J^1 & K \\ \bar{K} & J^2 \end{pmatrix}
\] (7.3)
is an unconstrained superfield, but \( J_\mu \) (distinguished from \( J \) by the Lorentz index) expands only over \( T_A \) and \( \sigma_3 \) (or just over \( T_A \) for the *bracket formalism of sec. 3):

\[
J_\mu = 2 J_\mu^A T_A + \frac{1}{2N} J_\mu^\sigma \sigma_3 \quad \text{so} \quad \text{str} J_\mu A_\mu = J_\mu^A A_\mu + J_\mu^\sigma A_\mu^0
\] (7.4)

the same constraints applying for all the other sources: \( \zeta, \bar{\zeta}, \kappa, H \) and \( L \). We define the functional differentials of source or field so as to extract the conjugate from under the supertrace thus [4]

\[
\frac{\delta}{\delta J} := \begin{pmatrix} \delta/\delta J^1 & -\delta/\delta \bar{K} \\ \delta/\delta \bar{K} & -\delta/\delta J^2 \end{pmatrix}
\] (7.5)

so

\[
\frac{\delta}{\delta J} \text{str} \int d^Dx J C = C,
\] (7.6)
with a similar definition for $\delta/\delta C$, whilst

$$\frac{\delta}{\delta J_\mu} := T_A \frac{\delta}{\delta J_{\mu A}} + \frac{\delta}{\delta J^a_{\mu}}$$  \hspace{1cm} (7.7)$$

has the same effect on the $J_\mu A_\mu$ term ($\delta/\delta J_{\mu A} = g^{AB} \delta/\delta J^B_{\mu}$), the other source and field differentials being defined similarly, for example

$$\frac{\delta}{\delta A_\mu} := 2 T_A \frac{\delta}{\delta A_{\mu A}} + \frac{\sigma_3}{2N} \frac{\delta}{\delta A^0_{\mu}}.$$  \hspace{1cm} (7.8)$$

First order variation over sources (the chain rule) is then simply given by

$$\text{str} \int d^D x \left( \delta J_\mu \frac{\delta}{\delta J_\mu} + \delta J_\mu \frac{\delta}{\delta J^a} + \delta \zeta \frac{\delta}{\delta \zeta} + \delta \bar{\zeta} \frac{\delta}{\delta \bar{\zeta}} \right),$$  \hspace{1cm} (7.9)$$

with of course a similar expression for the fields.

Under the BRST transformations (7.1), the generator of connected diagrams $W = \ln Z$ then satisfies

$$\xi A^{D/2-2} \zeta \cdot \bar{c}^{-1} \cdot \partial_\mu \frac{\delta W}{\delta J_\mu} + \text{str} \int d^D x \left( J_\mu \frac{\delta W}{\delta K_\mu} + J \frac{\delta W}{\delta H} - \bar{\eta} \frac{\delta W}{\delta L} \right) = 0.$$  \hspace{1cm} (7.10)$$

Legendre transforming to the generator of 1PI diagrams:

$$\Gamma + \xi \partial_\mu A_\mu \cdot \bar{c}^{-1} \cdot \partial_\nu A_\nu = -W + \text{str} \int d^D x \left( J_\mu A_\mu + J C + \bar{\zeta} \eta + \bar{\eta} \zeta \right),$$  \hspace{1cm} (7.11)$$

where $A_\mu$, $C$ and $\eta$ are now classical fields. We have extracted the gauge fixing term, so that on using the antighost Dyson-Schwinger equation\textsuperscript{12}

$$\text{str} \int d^D x \left( \frac{\delta \Gamma}{\delta \bar{\eta}} A^{D/2-2} - 2 \bar{c}^{-1} \bar{c} \partial_\mu \frac{\delta \Gamma}{\delta K_\mu} \right) = 0,$$

we obtain the simplified Lee-Zinn-Justin identities:

$$\text{str} \int d^D x \left( \frac{\delta \Gamma}{\delta A_\mu} \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta \Gamma}{\delta C} \frac{\delta \Gamma}{\delta C} + \frac{\delta \Gamma}{\delta \eta} \frac{\delta \Gamma}{\delta \eta} \right) = 0.$$  \hspace{1cm} (7.13)$$

Before we can use these to establish finiteness of the remaining One-loop Remainder Diagrams, we have to investigate the finiteness of the new diagrams (in the full broken theory) involving interactions introduced by the BRST sources $K_\mu$, $H$ and $L$ in (7.2). It turns out that since these interactions do not involve the higher derivatives, all such diagrams are superficially finite by power counting. It is straightforward to adapt the arguments of sec. 5 to prove this. We sketch the alterations. We note that (5.2) is unchanged, however eqns (5.4)–(5.6) pick up corrections from the BRST source vertices. The ghost equation in the desired form (5.7) is however unchanged, as can be most

\textsuperscript{12}the restriction to the dual of $S_\alpha$ arising from (2.7), i.e. $\delta \bar{\eta} = \delta \bar{\eta}^\alpha S_\alpha$.
simply understood again by deriving the equation directly as a count over external antighosts.\textsuperscript{13} The result is that (for 1PI diagrams) $\mathcal{D}_\Gamma$ in form (5.8) picks up the new terms

\[ -(2r + 3)E_K - (r + \tilde{r} + 3)E_H - (2r + 4)E_L, \]

whilst Proposition 1 holds unchanged. We obtain the same sufficient conditions (5.10) and (5.14) since corrections (7.14) are negative with these, which as before regulate all but a small set of diagrams. But before refinement, this latter set now contains diagrams with external BRST sources, since the constraint that there be no external antighosts no longer implies that there are no external ghosts. However, by conditions (5.14), (5.8) and (7.14), all diagrams containing $K, H$ or $L$ already result in $\mathcal{D}_\Gamma^{1\text{-loop}} < 0$. Thus under the earlier necessary and sufficient conditions on $r, \tilde{r}$ and $\hat{r}$, viz. (5.15) equivalently (5.16), all but the same set of One-loop Remainder Diagrams have $\mathcal{D}_\Gamma < 0$, and in particular all the diagrams involving BRST sources are superficially finite in any dimension $D$.

Finally, working to one-loop and writing $\Gamma$ in terms of its classical and one-loop parts, $\Gamma = \Gamma^0 + \hbar \Gamma^1$, keeping the $O(\hbar)$ terms of (7.13), and extracting from that those terms with one $\eta$ and otherwise only $\mathcal{A}$s, we obtain, up to unimportant corrections which are finite in all dimensions, precisely the Ward identities used at the beginning of this section to prove finiteness in all dimensions $D < 8$.

To be explicit, we write in the unbroken theory the one-loop pure $\mathcal{A}$ vertices as

\[ \frac{1}{2!} \sum_{m,n=2}^{\infty} \frac{1}{nm} \int d^Dx_1 \cdots d^Dx_n \, d^Dy_1 \cdots d^Dy_m \, \Gamma^1_{\mu_1 \cdots \mu_n, \nu_1 \cdots \nu_m} (x_1, \cdots, x_n; y_1, \cdots, y_m) \]

\[ \str \mathcal{A}_{\mu_1}(x_1) \cdots \mathcal{A}_{\mu_n}(x_n) \str \mathcal{A}_{\nu_1}(y_1) \cdots \mathcal{A}_{\nu_m}(y_m), \]

using the conclusions of sec. 6. The supertrace structure implies that the vertices are cyclic on the $x_i^{\mu_i}$ arguments and $y_j^{\nu_j}$ arguments separately, and symmetric under exchanging the two sets of arguments (see also [3][4]), and that the vertices $\Gamma^1$ may be defined to vanish identically for $n$ or $m$ less than 2. The $O(\hbar)$ terms in (7.13) with one $\eta$ and otherwise only $\mathcal{A}$s, only come from the terms

\[ \str \int d^Dx \left( \frac{\delta \Gamma^1}{\delta \mathcal{A}_\mu} \frac{\delta \Gamma^0}{\delta K_\mu} + \frac{\delta \Gamma^0}{\delta \mathcal{A}_\mu} \frac{\delta \Gamma^1}{\delta K_\mu} \right) , \]

and thus

\[ P^{\mu_1}_{\mu_2 \cdots \mu_n, \nu_1 \cdots \nu_m} (p_1, \cdots, p_n; q_1, \cdots, q_m) = \Gamma^1_{\mu_2 \cdots \mu_n, \nu_1 \cdots \nu_m} (p_1 + p_2, p_3, \cdots, p_n; q_1, \cdots, q_m) \]

\[ -\Gamma^1_{\mu_2 \cdots \mu_n, \nu_1 \cdots \nu_m} (p_2, \cdots, p_{n-1}, p_n + p_1; q_1, \cdots, q_m) + \text{finite}, \]

Ward identities for the other arguments following from cyclicity and exchange symmetry. The explicit terms are precisely those of the Ward identities [3, 4] we already

\textsuperscript{13} Alternatively incorporate $K, H$ and $L$ in the ghost-number conservation equation and note that these sources always appear as many times as the corresponding vertices.
referred to and used at the beginning of this section and follow from the first term in (7.16). The only change is the addition of the term “finite” which comes from the second term in (7.16); this is finite by (7.14) and the arguments below, as a consequence of the fact that only the terms in $\Gamma^1$ containing at least one $K$ contribute. We thus see that any longitudinal part of the four ($n = m = 2$) point vertex is finite in any dimension, leaving only a totally transverse part which is finite in all dimensions $D < 8$ by the arguments at the beginning of this section.\textsuperscript{14} The rest of the arguments from the beginning of this section follow through similarly.

8. Unitarity

The $A^2_\mu$ and $C^2$ fields of (4.1) and (4.4) have wrong sign actions as a consequence of the supertrace. Naively these functional integrals in the partition function do not make sense, however the correct prescription is to analytically continue these functional integrals whilst respecting $SU(N|N)$. Equivalently we may define the system through exact renormalization group methods \cite{3}\cite{4}\cite{6} or operator methods, neither of which suffer difficulties of definition. Actually, there is a choice of Fock vacuum (viz. annihilators) violating $SU(N|N)$, and resulting in an unbounded Hamiltonian. (This in turn would signal an unstable theory.) But covariant quantization leads to a bounded Hamiltonian. The problems of wrong sign action then show up in the appearance of negative norm states, which are thus unphysical and lead to a non-unitary S matrix. Below, we demonstrate these points on a simple quantum mechanics example. In the continuum limit $\Lambda \to \infty$, all fields apart from the $A^i_\mu$ become infinitely massive and, as we will see in the second subsection, for $N = \infty$ and any dimension $D$, or for finite $N$ but providing $D \leq 4$ dimensions, the unphysical $A^2_\mu$ field completely decouples from the physical $A^1_\mu$. In this way, a unitary $SU(N)$ Yang-Mills theory is recovered in the limit $\Lambda \to \infty$.

8.1. $U(1|1)$ quantum mechanics example and negative norms

Defining the Hermitian super-position $\mathcal{X}$ as

\begin{equation}
\mathcal{X} = \begin{pmatrix} x^1 & \theta \\ \bar{\theta} & x^2 \end{pmatrix},
\end{equation}

we consider the Minkowski type Lagrangian of a simple harmonic potential: $L = \frac{1}{2} \text{str} \dot{\mathcal{X}}^2 - \frac{1}{2} \text{str} \mathcal{X}^2$. Classically this Lagrangian is invariant under $SU(1|1)$ transformations $\delta \mathcal{X} = i [\omega, \mathcal{X}]$, however we buy for free invariance under the full $U(1|1)$. By Noether’s theorem these are generated by the triplet of charges (a.k.a. angular momenta)

\begin{equation}
\mathcal{Q} = i [\mathcal{Y}, \dot{\mathcal{X}}],
\end{equation}

\textsuperscript{14}Out of interest we note from (7.15) that it is the coefficient of $(\text{str} F^2_{\mu \nu})^2$ that diverges in $D = 8$.\textsuperscript{30}
through the Poisson bracket with $\text{str}\omega Q$. Note that the charge for $\omega \sim 1$, vanishes, reflecting its trivial action on $\mathcal{X}$. Defining a super-covariant derivative as in (7.5), the supermomentum is

$$\mathcal{P} := \frac{\partial}{\partial \dot{X}} L = \dot{\mathcal{X}},$$

(8.3)

and differs by some convenient signs from the usual definitions:

$$p_i = \frac{\partial L}{\partial \dot{x}^i}, \quad p_{\dot{\vartheta}} = \frac{\partial L}{\partial \dot{\vartheta}}, \quad p_{\bar{\vartheta}} = \frac{\partial L}{\partial \dot{\bar{\vartheta}}}.$$  

(8.4)

The Hamiltonian is then given as

$$H = \text{str} \mathcal{P} \dot{\mathcal{X}} - L,$$

(8.5)

and quantization is via the graded commutator:

$$[(\mathcal{X})^a_b, (\mathcal{P})^c_d]_\pm = i(\sigma_3)^a_d \delta^c_b.$$  

(8.6)

This is the form that respects $U(1|1)$, as can most easily be seen by writing it contracted with arbitrary constant supermatrices $U$ and $V$:

$$[\text{str}U\mathcal{X}, \text{str}V\mathcal{P}] = i\text{str} UV,$$

(8.7)

and actually corresponds to the usual relations for the usual definitions of momenta (8.4). However, as often happens, we have to be careful with operator ordering since the naive ordering implied by (8.2), on quantization no longer leaves $Q$ supertraceless. We can cure this by subtracting the supertrace which as we will see, for a sensibly defined vacuum, corresponds to normal ordering:

$$Q = \text{str} \mathcal{P} \dot{\mathcal{X}} - L,$$

(8.5)

and quantization is via the graded commutator:

$$[(\mathcal{X})^a_b, (\mathcal{P})^c_d]_\pm = i(\sigma_3)^a_d \delta^c_b.$$  

(8.6)

This is the form that respects $U(1|1)$, as can most easily be seen by writing it contracted with arbitrary constant supermatrices $U$ and $V$:

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$$Q = i [\mathcal{X}, \mathcal{P}] - \frac{i}{2} \sigma_3 \text{str}[\mathcal{X}, \mathcal{P}] = i [\mathcal{X}, \mathcal{P}] + 2\sigma_3.$$  

(8.8)

The definition of the vacuum follows from the choice of annihilation and creation operators

$$A = (\mathcal{X} + i\mathcal{P})/\sqrt{2}, \quad A^\dagger = (\mathcal{X} - i\mathcal{P})/\sqrt{2},$$

(8.9)

with normalised vacuum such that $A|0\rangle = 0$. The $A$s have the now expected graded commutation relations,

$$[(A)^a_b, (A^\dagger)^c_d]_\pm = (\sigma_3)^a_d \delta^c_b.$$  

(8.10)

It is straightforward to check that the vacuum respects $U(1|1)$: $Q|0\rangle = 0$. As advertised, the supercharges (8.8) alternatively may be written $Q = [A^\dagger, A]$.

Writing (8.9) in terms of components and using the usual definitions of momenta (8.4), $x^1$ has the usual form for an annihilation operator, $a^1 = (x^1 + ip^1)/\sqrt{2}$, but $x^2$ has an annihilation operator containing the wrong sign: $a^2 = (x^2 - ip^2)/\sqrt{2}$. These components therefore have the wrong sign commutation relations $[a^2, a^2] = -1$, as is easily seen from (8.10). This is just as needed to compensate the wrong sign for $a^2 a^2$.
in the Hamiltonian, \( H = \text{str} \ A^\dagger \ A + 2 \), which is thus bounded below. However, negative norms appear in the ‘2’ sector:

\[
|n_2\rangle = \frac{1}{\sqrt{n_2!}} (a^{2\dagger})^{n_2} |0\rangle, \quad \langle n_2|n_2\rangle = (-1)^{n_2}.
\]  

(8.11)

It is straightforward to verify that any attempt to repair this by keeping \( a^1 \) as it is, but changing the sign in \( a^2 \), results in an unbounded Hamiltonian and a vacuum that violates both \( U(1|1) \) and \( SU(1|1) \): \( Q|0\rangle \neq 0 \).

(Let us note that this situation is very similar to the Gupta-Bleuler quantisation procedure [20] which has to deal with a wrong sign action for time-like photons. The same choice of vacua exists, but Lorentz covariant quantization picks out the one with negative norm states. In our case however, we have no equivalent Gupta-Bleuler condition for excluding such unphysical states. Instead, the unphysical sector decouples, as we described in the introduction to this section, and now show.)

### 8.2. Recovery of unitarity in the \( A^1 \) sector

We have seen that our covariant higher derivative spontaneously broken \( SU(N|N) \) theory is finite in all dimensions \( D < 8 \). However, this is not enough to show that it acts as a regulator for \( SU(N) \) Yang-Mills theory. For this to be the case, we must show that for renormalized variables in the continuum limit \( \Lambda \to \infty \), \( SU(N) \) Yang-Mills theory is recovered.

All fields but the \( SU(N) \times SU(N) \) gauge fields \( A^i_{\mu} \) (and when gauge fixed their respective ghosts \( \eta^i \)), become infinitely massive in this limit and thus drop out of the spectrum. The issue then is to show that there are no remaining effective interactions between these two gauge fields: the wrong sign \( A^2 \) sector can then just be ignored. From above, it is also necessary to establish that unitarity is recovered in this limit. But this is the same question, since a non-unitary amplitude in the \( A^1 \) sector can only arise in the \( \Lambda \to \infty \) limit, from contributions with internal \( A^2 \)'s. Cutkosky cutting such an amplitude must then result in a non-vanishing amplitude connecting \( A^1 \)'s and \( A^2 \)'s [20]. Therefore providing that we can establish that there are no such effective interactions between \( A^1 \) and \( A^2 \), we can safely ignore the sick \( A^2 \) sector and recover a unitary continuum limit for the \( SU(N) \) Yang-Mills theory.

Actually, in the large \( N \) limit, there is nothing further to do: since only single trace interactions survive [14] (see also [2][3][4] and subsec. 6.4) and any interaction between \( A^1 \) and \( A^2 \) (or their ghosts) requires two traces, one for each \( SU(N) \), the separation of the two sectors is automatic. As shown in sec. 6, in this limit the theory is finite in all dimensions. Therefore in the \( N = \infty \) limit, our regularisation works for \( SU(N) \) Yang-Mills in any dimension \( D \).

For finite \( N \), we appeal to the Appelquist-Carazzone decoupling theorem [13] to show that the regularisation works for any dimension \( D \leq 4 \). The theorem states that for a renormalizable theory, as the mass scale of the heavy sector tends to infinity,
the (bare) effective Lagrangian is given by a renormalizable Lagrangian for the light fields with irrelevant corrections vanishing by inverse powers of the heavy scale. This scale is identified with the overall cutoff for the effective theory. This theorem for example justifies the assumption that a spontaneously broken GUT (Grand Unified Theory) is equivalent to the Standard Model \( SU(3) \times SU(2) \times U(1) \) at energies where the GUT scale can be neglected, and our case is closely analogous with the \( SU(N|N) \) theory playing the rôle of the GUT. Just as there would be no interactions between the \( SU(3) \), \( SU(2) \) and \( U(1) \) gauge fields of the Standard Model if it were not for the matter fields, as we confirm below there are no interactions between the two \( SU(N) \) gauge fields in our effective theory.

Note that the Appelquist-Carazzone theorem applies only to an initially renormalizable theory. The spontaneously broken \( SU(N|N) \) theory without the higher derivatives is renormalizable in \( D \leq 4 \) dimensions (because the standard analysis does not care that we are dealing with a supergroup). The higher derivatives are a regularisation for this theory. In point of fact, our situation is simpler than the cases considered for the original proofs where the essential difficulty arises from the exchange of limits of the heavy scale tending to infinity and the overall cutoff tending to infinity [13]. In our case the two scales are identified and by construction in sec. 4, the only scale in the theory is \( \Lambda \).

Thus in \( D \leq 4 \) dimensions, the effective \( SU(N) \times SU(N) \) theory can be described by an effective bare Lagrangian containing only these fields, their own Yang-Mills couplings \( g_i \) (no longer equal to \( g \)) and further interactions weighted by appropriate powers of \( \Lambda \) as determined by dimensions. All of these other interactions are however irrelevant and vanish in the limit \( \Lambda \rightarrow \infty \). In particular, the lowest dimension interaction between the two fields comes from a group theory structure \( \text{tr} \ A^1_{\mu} A^1_{\nu} \text{tr} \ A^2_{\lambda} A^2_{\sigma} \), (with Lorentz indices contracted in some way). Since such an interaction must also be gauge invariant under \( SU(N) \times SU(N) \), the minimal dimension bare interaction actually takes the form

\[
\Lambda^{-D} \text{tr} \ F^1_{\mu\alpha} F^1_{\nu\beta} \text{tr} \ F^2_{\lambda\gamma} F^2_{\sigma\delta}, \tag{8.12}
\]

which is irrelevant in any dimension. (Here \( F^i_{\alpha\beta} \) is the field strength for \( A^i_{\mu} \), the Lorentz indices are again contracted in some fashion, and the \( \Lambda \) dependence is displayed up to \( \ln \Lambda \) multiplicative corrections.)

As in the initial arguments of sec. 7, we have assumed gauge invariance, whereas we have been working within a traditional gauge fixed approach. The extra details coming from ghosts and BRST do not change the conclusions and have already been treated in earlier work on the decoupling theorem [22].

Finally, we have seen that at finite \( N \), \( D \leq 4 \) is a sufficient condition for decoupling. It is also necessary, since in \( D > 4 \) dimensions the couplings \( g_i \) are non-renormalizable, and thus clearly all higher order interactions will be unsuppressed.
9. Conclusions

We first recall from sec. 3 the main issues encountered in building a spontaneously broken \( SU(N|N) \) gauge theory. Then we summarise the steps of the proof that, together with covariant higher derivatives, this provides a regularisation for \( SU(N) \) Yang-Mills. Finally we draw our conclusions noting in particular further properties and extensions.

We first noted that \( U(N|N) \) is not reducible to a product \( SU(N|N) \times U(1) \), removing the usual \textit{a priori} reason for not considering such a group. The enlarged group \( U(N|N) \) requires an extra gauge field \( \mathcal{A}^0 \). However if we build a gauge theory on \( U(N|N) \), then \( \mathcal{A}^0 \) acts as a Lagrange multiplier constraining \( \mathcal{A}^\sigma \) to be pure gauge. Effectively, \( U(N|N) \) contracts dynamically to \( SU(N|N) \). Working directly with \( SU(N|N) \), the gauge field \( \mathcal{A}^0 \), being associated with the \( \Pi \) generator, appears nowhere in the Lagrangian since the theory contains only adjoint fields. The partition function thus contains a free functional integral over \( \mathcal{A}^0 \) (\textit{i.e.} without even a Gaussian weight). We cannot simply delete the \( \mathcal{A}^0 \) field however because it is needed to absorb gauge transformations in the \( \Pi \) direction. This is the gauge theory reflection of the fact that \( SU(N|N) \) is reducible but indecomposable. Although we can follow Bars suggestion \cite{17}, modifying the algebra in the gauge field sector to remove this feature, and thus also \( \mathcal{A}^0 \), we cannot do so without destroying the Leibnitz property of the usual representation of the super Lie bracket and thus gauge invariance in the superscalar sector. Nevertheless the end result is that there are equivalent representations: the free-\( \mathcal{A}^0 \) representation already described (and the one we choose to work in because it is more elegant) and a \( * \)bracket representation in which the \( \mathcal{A}^0 \) is ‘projected out’. As we have seen, an alternative way of making sense of the \( \mathcal{A}^0 \) integral is to work instead with \( U(N|N) \). In all cases the bottom line is that only the information in the supertraceless and traceless part of the \( SU(N|N) \) superalgebra is actually propagated by the gauge fields \( \mathcal{A}_\mu \).

The superscalar, \( \mathcal{C} \), introduced to break the theory along all and only fermionic directions must be a representation of \( U(N|N) \) since it has to break along \( \sigma_3 \). At first sight we are able to impose a gauge invariant linear constraint on the coefficient field \( \mathcal{C}^\sigma \), but we note that this leads either to inconsistency or results in further constraints, which this time are non-linear. As we note in sec. 2, the \( SU(N|N) \) invariance of the theory is built on the cyclicity property of the supertrace for supermatrices. (We have taken care to make this manifest throughout.) The introduction of superghosts with opposite statistics breaks this cyclicity property. An elegant solution is to introduce a separate ghost grading, recalling that it is actually a matter of choice whether different fermionic flavours commute or anticommute. This also allows the usual form of the BRST transformations to be a symmetry and thus ensures that the usual required properties of gauge fixing (gauge independence, transversality of on-shell Green functions etc.) hold.

Another point of principle, dealt with in sec. 8, is the meaning of the wrong sign action that appears as a consequence of the supertrace for both \( A^2 \) and \( C^2 \). This does not signal an instability of the theory since both kinetic and interaction terms have
the wrong sign. Guided by invariance under the supergroup we show in sec. 8 that the result is mathematically well defined, however requiring an indefinite metric Hilbert space. Fortunately, in the continuum limit $\Lambda \to \infty$, $C^2$ becomes infinitely heavy, and $A^2$ decouples in the way described in sec. 8.

As a matter of fact in the most interesting case of $D = 4$ dimensions, the wrong sign in $A^2$ sector implies that its $\beta$ function is that of a trivial, rather than asymptotically free, theory. In the continuum limit in which the bare coupling $g$ is sent in the usual way logarithmically to zero as $\Lambda \to \infty$ (in order to achieve a finite interacting $A^1$ theory), the $A^2$ sector loses all interactions and becomes a free theory [4].

In sec. 4, the full spontaneously broken action is introduced, regularised with polynomials of covariant higher derivatives, ranks $r$ and $\tilde{r}$, for the $A$ and $C$ parts respectively. The superghosts are regulated by a polynomial of higher derivatives (not covariant) of rank $\tilde{r}$ introduced through the gauge fixing function.

The proof that the result is ultraviolet finite starts in sec. 5. Here we establish the necessary and sufficient constraints required on the ranks of the polynomials, such that the maximum number of Feynman diagrams are superficially finite simply by power counting. Furthermore, we show by finding one-loop examples that they are necessary even after taking into account cancellations resulting from supersymmetry. The constraints are given for the dimensions of interest, in fact for all $D \geq 1$, by (5.16) and agree precisely with the relevant inequalities proved in the manifestly gauge invariant but incomplete formulation of ref. [4].

In this way, we are left to consider only a set of One-loop Remainder Diagrams which are not finite purely by power counting, namely those formed from only $C^2A^j$, $A^j$ and $\bar{\eta}A\eta$ vertices, with $D$ or less external $A$ legs and no external $C$ or (anti)ghost legs. In sec. 6 we establish that for one-particle irreducible one-loop diagrams all the contributions associated with the spontaneous symmetry breaking are already finite by power counting. All the remaining contributions, equal to those in the symmetric phase, appear as the product of two supertraces over the external fields. This is shown by first demonstrating that for the symmetric phase, all the tree contributions appear as a single supertrace, and then showing that on closure into one-loop diagrams, the single supertrace always splits into two supertraces. At first sight this pattern is violated by gauge-sector (i.e. gauge and ghost) propagators, which also introduce ordinary trace terms as a consequence of completeness relations over the associated both-supertraceless-and-traceless generators. However, this is just another symptom of the strange rôle of $A^0$, and once we take into account properly that gauge-sector fields interact only through commutators, all the ordinary trace terms cancel out. Since $\text{str}A = 0$ and $\text{str}1 = 0$, we thus immediately find that for $E_A \leq 3$ external $A$s, the symmetric phase contributions vanish, and thus the full broken phase contributions are finite. The fact that, consistent with the structure of the supergroup, only the supertrace (and never the trace) appears in the final result, means that in the large $N$ limit the symmetric phase has no quantum corrections at all. In particular this means
that all One-loop Remainder Diagrams are finite in the large $N$ limit, and thus the full spontaneously broken theory is finite, in any dimension $D$.

Returning to finite $N$, in sec. 7 we tackle the remaining contributions not already shown to be superficially finite. These are symmetric phase one-loop vertices with $3 < E_A \leq D$ external gauge field legs. The key here is to take properly into account the gauge invariance of the theory. Thus gauge invariance tells us that any longitudinal part of the four-point vertex vanishes, since it is given by a sum over three-point vertices. By power-counting the remaining fully transverse four-point vertex is finite in all dimensions $D < 8$. Iterating to higher point vertices we thus establish finiteness of all these remaining contributions, and thus also the full theory, in all dimensions $D < 8$. However this argument, based as it is on an exact implementation of gauge invariance (as in ref. [3, 4]), is not rigorous in the present context: we have to worry that new divergences may occur in terms including ghosts. To check this, we develop in standard fashion, the full Lee-Zinn-Justin identities and check that the required BRST sources introduce no divergences. Apart from some unimportant finite corrections the argument above may then be repeated, with the same conclusion: at finite $N$, the full theory is ultraviolet finite in all dimensions $D < 8$.

Finally, in sec. 8, we turn to the other crucial requirement of a regulator: that in the limit $\Lambda \to \infty$ we are indeed left with the theory we set out to regulate, namely $SU(N)$ Yang-Mills theory carried by $A^1$. Actually in this limit we find $SU(N) \times SU(N)$, with the second $SU(N)$ being carried by $A^2$. We have to show then that $A^1$ and $A^2$ decouple, so that we can simply ignore the $A^2$ sector. In the large $N$ limit, no interaction is possible since only single trace interactions are allowed. At finite $N$, we show by a rather straightforward application of the Appelquist-Carazzone theorem that decoupling takes place providing that the theory is renormalizable, i.e. providing the dimension $D \leq 4$. We also note that decoupling fails at finite $N$ in $D > 4$ dimensions. This completes the proof that our spontaneously broken $SU(N|N)$ theory acts as a regulator for $SU(N)$ Yang-Mills, at least to all orders in perturbation theory, for any dimension $D \leq 4$, and in the large $N$ limit for all dimensions $D$.

We now draw some further conclusions. Despite much effort [14, 23], four dimensional large $N$ Yang-Mills theory has evaded solution. However we saw in sec. 6, that in the large $N$ limit the symmetric phase of this $SU(N|N)$ theory has no quantum corrections at all. It is therefore trivially exactly soluble. (This is of course equally true of the pure $SU(N|N)$ Yang-Mills theory, i.e. without the superscalar sector.) Surely, this ought to help understand the large $N$ limit for $SU(N)$ Yang-Mills, which as we have seen is recovered at energies much less than $\Lambda$ in the spontaneously broken theory? Note however that the large $N$ limit of the spontaneously broken theory is not the same as implementing spontaneous symmetry breaking after the large $N$ limit has been taken. The two procedures do not commute: as can be confirmed in the appropriate large $N$ variables,\footnote{which in particular requires $\Lambda^{D-2}$ to be replaced by $N\Lambda^{D-2}$ in (4.5)} whereas extra supertrace factors of strings of $n$ superscalars $\text{str}(C \cdots C)$,
formally counts as order one and thus subleading in the $1/N$ expansion, replacing these by their expectation values $\Lambda \sigma_3$ results for odd $n$, in a leading contribution $\sim N$.

Furthermore we can note that, trivially, the large $N$ limit of symmetric phase of this $SU(N|N)$ theory (or the pure case without superscalars) is finite without the introduction of covariant higher derivatives. This is not the case for the spontaneously broken theory as follows from the large $N$ limit of two of the one-loop examples below (5.15) used to prove the necessity of inequalities (5.16). At finite $N$, we see from the $C^4$ one-loop example, that the symmetric phase also needs covariant higher derivatives (or some other regularisation) to be ultraviolet finite. Likewise, from the analysis of the two-loop graphs and higher, we also expect that at finite $N$, pure $SU(N|N)$ Yang-Mills needs further regularisation e.g. the covariant higher derivatives, to make it ultraviolet finite.

Since in common with other Pauli-Villars approaches, finiteness is achieved only after subtracting separately divergent contributions, the answers are well defined only after applying and removing a ‘preregulator’ [15]. The obvious choice of preregulator is in effect to use dimensional regularisation by keeping the dimension $D$ general, taking the limit to the actual spacetime dimension only at the end of the calculation [4]. It is important to understand that the preregulator is not used to compute divergences (there are none) or to renormalize the theory, therefore the usual issues of defining what dimensional regularisation means non-perturbatively do not arise. Alternatively, at the diagrammatic level it may be possible to define the momentum routing so that it uniquely picks the right way to add the separately divergent contributions, as used in ref. [10]. For example in our case, a choice that preserves the (broken) Ward identities.

Finally, let us note that pure $SU(N|N)$ Yang-Mills and the symmetric phase of the theory has in common with ref. [4], a duality under

\[
\begin{align*}
\bar{h} &\mapsto -\bar{h} \\
A_\mu &\mapsto \sigma_1 A_\mu \sigma_1, \\
C &\mapsto \sigma_1 C \sigma_1
\end{align*}
\]

which exchanges the role of $A^1$ and $A^2$ (and similarly $C^1$ and $C^2$; $\sigma_1$ is defined in (2.6), and of course the last equation is ignored for pure $SU(N|N)$ Yang-Mills.) By the usual changes of variables on the fields to bring $g$ outside the action, the change of sign on the loop counting parameter $\bar{h}$ becomes $g^2 \rightarrow -g^2$ [4]. Unlike the version in ref. [4], this theory-space symmetry is broken once $C$ picks up an expectation value. However as in ref. [4], the duality is also broken by the differing renormalization required for the $A^1$ and $A^2$ sector (as noted earlier).

Having established that this regularisation framework really works, at finite $N$, to all orders in perturbation theory, and presumably thus also non-perturbatively, the stage is now set to generalise the manifestly gauge invariant exact renormalization group methods of refs. [2, 3, 4], to computations in higher order perturbation theory (e.g. to
check consistency) and non-perturbatively, allowing for the first time such continuum computations to be performed without gauge fixing [1].

Acknowledgments


References


