Embeddings in Non–Vacuum Spacetimes

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Abstract

A scheme is discussed for embedding any $n$–dimensional, Riemannian manifold in an $(n + 1)$–dimensional Einstein space. Criteria for embedding a given manifold in a spacetime that represents a solution to Einstein’s equations sourced by a massless scalar field are also discussed. The embedding procedures are illustrated with a number of examples.

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1 Introduction

The possibility that our universe contains hidden, spatial dimensions has attracted considerable attention over recent years. In particular, advances in our understanding of the non-perturbative limits of superstring theory indicate that spacetime may be eleven-dimensional [1]. A further important development has been the realization that these extra dimensions need not have finite volume. Indeed, four-dimensional gravity can be recovered if the observable universe is represented by a co-dimension 1 brane embedded in a higher-dimensional space with a non-factorizable geometry [2, 3, 4, 5, 6].

Embedding theorems of differential geometry provide a natural framework for relating higher- and lower-dimensional theories of gravity and it is important to study such theorems in the light of the above developments. A particularly powerful theorem, due to Campbell, states that any $n$-dimensional, Riemannian manifold can be locally and isometrically embedded in an $(n+1)$-dimensional, Riemannian space, where the Ricci tensor of the latter vanishes [7]. This theorem was discussed by Romero, Tavakol and Zalaletdinov [8] within the context of the non-compactified approach to Kaluza–Klein gravity [9]. (For earlier work, see Refs. [10]). Further embeddings into Ricci-flat spaces were also established for a wide class of superstring backgrounds [11].

The purpose of the present paper is to develop extensions of Campbell’s scheme, where the Ricci tensor of the higher-dimensional spacetime is non-trivial. Specifically, we consider the case where the embedding manifold is an Einstein space with a covariantly constant energy–momentum tensor. Such embeddings are directly relevant to the second Randall–Sundrum braneworld scenario, where the bulk corresponds to pure Einstein gravity sourced only by a negative cosmological constant [6]. The embedding of the four-dimensional, isotropic and homogeneous radiation universe into a Schwarzschild–Anti de Sitter space was recently investigated [12]. Einstein spaces with a positive cosmological constant have also become the focus of attention [13, 14] and cosmological solutions with such a term represent one of the simplest manifestations of the inflationary scenario [15].

We also consider embeddings into spacetimes sourced by a massless, minimally coupled scalar field. Such a field represents one of the simplest forms of matter and can be identified, after suitable field redefinitions, with the dilaton field that arises in the string effective action [16]. A massless field also parametrizes the volume of an internal, Ricci-flat space in conventional Kaluza–Klein compactification. (For a review, see, e.g., [17]).

The paper is organized as follows. We develop the embedding schemes in Section 2 and proceed in Section 3 to illustrate these techniques by establishing embeddings for the general class of Einstein and plane wave spacetimes. We conclude with a discussion of further applications in Section 4.
2 Embeddings in Higher Dimensions

2.1 Einstein Spaces

We consider the local and isometric embedding of the $n$-dimensional, Riemannian
manifold, $(M, g_{\alpha \beta})$, with line element
\[ ds^2 = g_{\alpha \beta}(x^\mu)dx^\alpha dx^\beta \] (2.1)
in the $(n + 1)$-dimensional manifold, $(\hat{M}, \hat{h}_{AB})$, defined by the metric
\[ d\hat{s}^2 = h_{\alpha \beta}dx^\alpha dx^\beta + \epsilon \phi^2 dy^2, \] (2.2)
where $h_{\alpha \beta} = h_{\alpha \beta}(x^\mu, y)$ and $\phi = \phi(x^\mu, y)$ are analytic functions of the $(n + 1)$ variables $\{x^\mu, y\}^1$. The constant $\epsilon = \pm 1$ and we therefore allow for the possibility that the extra dimension is either spacelike or timelike. Although there are well known problems with introducing an additional timelike dimension, it has been argued that the duality symmetries of string/M theory compactified on Lorentzian tori result in extra time dimensions in the strong coupling limit [18]. Braneworld scenarios where the transverse dimension is timelike have also been proposed [19].

When evaluated on an arbitrary hypersurface, $dy = 0$, the components of the Ricci tensor calculated from the metric (2.2) take the general form
\[ \hat{R}_{\alpha \beta} = R_{\alpha \beta} - \nabla_\alpha \nabla_\beta \phi + \frac{1}{2\epsilon \phi^2} \left( \frac{\phi^*}{\phi} h^*_{\alpha \beta} - h_{\alpha \beta}^{**} - \frac{1}{2} h_{\gamma \delta}^\gamma h_{\gamma \delta}^\delta h_{\alpha \beta}^* + h_{\gamma \delta}^\gamma h_{\alpha \gamma}^* h_{\beta \delta}^* \right) \] (2.3)
\[ \hat{R}_{\alpha y} = \phi \frac{\phi^*}{2} \nabla_\beta P_\alpha^\beta \] (2.4)
\[ \hat{R}_{yy} = -\epsilon \phi \nabla^2 \phi - \frac{1}{2} h_{\gamma \delta}^\gamma h_{\gamma \delta}^{**} - \frac{1}{2} \left( h_{\gamma \delta}^\gamma \right)^* h_{\gamma \delta}^\delta + \frac{1}{2} h_{\gamma \delta}^\gamma h_{\gamma \delta}^\delta \frac{\phi^*}{\phi} - \frac{1}{4} h_{\gamma \delta}^\gamma h_{\delta \alpha}^\delta h_{\gamma \alpha}^* h_{\beta \delta}^* \] (2.5)
where the $n$-dimensional Ricci tensor, $R_{\alpha \beta}$, and covariant derivative operator, $\nabla_{\alpha \beta} = \nabla_\beta \nabla_\alpha$, are calculated from $h_{\alpha \beta}$, a star denotes a partial derivative with respect to $y$, $\partial / \partial y |_{y = \text{constant}}$, evaluated on the hypersurface $y = \text{constant}$, $\nabla^2 \equiv h^{\alpha \beta} \nabla_{\alpha \beta}$ and the quantity, $P_\alpha^\beta$, is defined by [20]
\[ P_\alpha^\beta \equiv \frac{1}{\phi} \left( h_{\beta \gamma}^\gamma h_{\alpha \alpha}^* - \delta_\alpha^\alpha h_{\gamma \delta}^\gamma h_{\gamma \delta}^\delta \right). \] (2.6)

$^1$Greek and Latin indices run from $(0, 1, \ldots, n-1)$ and $(0, 1, \ldots, n)$, respectively, and the coordinate of the $(n + 1)$th dimension is denoted by $y$. All curvature tensors relevant to the $(n + 1)$-dimensional metric, $\hat{h}_{AB}$, are represented with a circumflex accent and those constructed from the hypersurface metric, $h_{\alpha \beta}$, have no accent. We employ Wald’s conventions with signature $(-, +, +, \ldots)$ for the $n$-dimensional spacetime, $g_{\alpha \beta}$ [21]. In all cases, the embeddings considered in this paper are local and isometric and do not refer to any aspects of the global topology of the spaces.
If we now define the functions $\Omega_{\alpha\beta}(x^\mu, y)$ [7, 8]:

$$\frac{\partial h_{\alpha\beta}}{\partial y} \equiv -2\phi \Omega_{\alpha\beta},$$

(2.7)

it follows that Eqs. (2.3)–(2.5) simplify to

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \nabla_\alpha \phi \nabla_\beta \phi - 2\epsilon \nabla_\alpha \phi - \epsilon \phi R_{\alpha\beta} - \frac{2\epsilon \Lambda \phi}{n-1} h_{\alpha\beta},$$

(2.8)

$$\hat{R}_{\alpha y} = \phi \nabla_\beta \left[ \delta_\beta^\alpha h_{\gamma\delta} - h_{\gamma\delta} \Omega_{\gamma\delta} \right],$$

(2.9)

$$\hat{R}_{yy} = -\phi h_{\gamma\delta} \left[ \epsilon \nabla_\gamma \phi - \Omega_{\gamma\delta} - \phi h_{\alpha\delta} \Omega_{\alpha\gamma} \Omega_{\gamma\delta} \right].$$

(2.10)

In this subsection, we show that the metric (2.1) can be embedded in an Einstein space of the form (2.2), where the constraint equations

$$\hat{R}_{AB} = \frac{2\Lambda}{1-n} \hat{h}_{AB}$$

(2.11)

are satisfied and $\Lambda$ is a spacetime constant. That such an embedding is possible was stated by Campbell [7], but the proof was not given. The proof proceeds iteratively by first assuming that the equations (2.11) are valid on a specific hypersurface $y = y_0$, where $y_0$ is arbitrary, and then verifying that they are also valid for any $y$ in the neighbourhood of this hypersurface.

To proceed, we substitute Eq. (2.11) into Eqs. (2.8)–(2.10):

$$\Omega^*_{\alpha\beta} = h^{\lambda\mu} \left( \Omega_{\alpha\beta} \Omega_{\lambda\mu} - 2\Omega_{\alpha\lambda} \Omega_{\beta\mu} \right) \phi + \epsilon \nabla_{\alpha\beta} \phi - \epsilon \phi R_{\alpha\beta} - \frac{2\epsilon \Lambda \phi}{n-1} h_{\alpha\beta},$$

(2.12)

$$h^{\mu\nu} \left( \nabla_\mu \Omega_{\alpha\nu} - \nabla_\alpha \Omega_{\mu\nu} \right) = 0,$$

(2.13)

$$h^{\lambda\beta} \left( \epsilon \nabla_{\lambda\beta} \phi - \Omega^*_{\lambda\beta} - \phi h^{\alpha\rho} \Omega_{\gamma\delta} \Omega_{\beta\delta} \right) - \frac{2\epsilon \Lambda \phi}{n-1} = 0.$$

(2.14)

Subtracting the trace of Eq. (2.12) from Eq. (2.14) then results in the contracted ‘Gauss’ equation:

$$\Omega^2 - \Omega^\mu_{\mu} \Omega^\alpha_{\mu} = \epsilon (R + 2\Lambda),$$

(2.15)

where $\Omega \equiv h^{\alpha\beta} \Omega_{\alpha\beta}$ and the covariant constancy of the metric in Eq. (2.13) yields the ‘Codazzi’ equation:

$$\nabla^\nu \Omega_{\alpha\nu} = \nabla_\alpha \Omega.$$

(2.16)

A crucial property of the higher–dimensional metric (2.2) is that it must simplify to the embedded metric (2.1) when on the hypersurface, $y = y_0$:

$$h_{\alpha\beta}(x^\mu, y_0) = g_{\alpha\beta}(x^\mu).$$

(2.17)

We then assume that the symmetric functions

$$\Omega_{\alpha\beta} = \Omega_{\beta\alpha},$$

(2.18)
can be found that satisfy the constraints (2.15) and (2.16) on this ‘initial’ hypersurface. Moreover, it is also assumed that these functions evolve according to Eq. (2.7) and the set of differential equations\(^2\)

\[
\frac{\partial \Omega^n}{\partial y} = \epsilon \nabla^\gamma \phi + \phi \left( \Omega \Omega_\beta^n - \epsilon R^n_\beta - \frac{2\epsilon \Lambda}{n-1} \delta^n_\beta \right),
\]

(2.19)

where the boundary conditions

\[
h^*_{\alpha\beta} = -2\phi(x^\mu, y_0)\Omega_{\alpha\beta}(x^\mu, y_0)
\]

(2.20)

are satisfied.

As shown in the appendix, if conditions (2.7), (2.15)–(2.20) are satisfied, it follows that

\[
(\nabla^\nu \Omega_{\alpha\nu} - \nabla_\alpha \Omega)^* = 0
\]

(2.21)

\[
\left( \Omega^2 - \Omega^\beta_\alpha \Omega^n_\beta - \epsilon (R + 2\Lambda) \right)^* = 0
\]

(2.22)

and Eqs. (2.21) and (2.22) then imply that Eqs. (2.15), (2.16) and (2.18) are valid for all hypersurfaces, \(dy = 0\), in the *neighbourhood* of \(y = y_0\). Given the validity of Eq. (2.19), therefore, we may further deduce that the Einstein conditions (2.11) are satisfied for all \(y\) in this neighbourhood. Consequently, the \((\alpha\beta)\)-components of the higher-dimensional metric \(\hat{h}_{AB}\) can be expanded as a Taylor series in \(y\) to first-order:

\[
\hat{h}_{\alpha\beta} = g_{\alpha\beta} - 2\phi(x^\mu, y_0)\Omega_{\alpha\beta}(x^\mu, y_0)y,
\]

(2.23)

where Eqs. (2.7) and (2.17) have been employed. Likewise, the value of \(\Omega_{\alpha\beta}\) in this vicinity can be determined from Eq. (2.19). Since the analysis is valid for an arbitrary hypersurface, this local extension can be repeated recursively and this establishes the embedding of the metric (2.1) in the Einstein space (2.2). An important feature of the above embedding scheme is that the consistency of the Gauss–Codazzi equations is ensured and this implies that the embedding is always possible in principle. We may conclude, therefore, that any \(n\)-dimensional, Riemannian manifold may be locally and isometrically embedded in an \((n+1)\)-dimensional Einstein space.

### 2.2 Massless Scalar Fields

It is also of interest to consider whether Campbell’s technique can be extended to include embeddings of the metric (2.1) in non–vacuum spacetimes, \((\hat{M}, \hat{h}_{AB})\). One possible source of matter is a minimally coupled scalar field, \(\chi\), that satisfies the Einstein field equations

\[
\hat{R}_{AB} = \frac{1}{2} \hat{\nabla}_A \hat{\nabla}_B \chi
\]

(2.24)

\[
\hat{h}^{AB} \hat{\nabla}_{AB} \chi = 0.
\]

(2.25)

\(^2\)The metric \(h_{\alpha\beta}(x^\mu, y)\) is employed in Eq. (2.19) to raise and lower indices and in calculating the curvature tensors and covariant derivatives.
In general, however, the coupling between the scalar field and gravity makes it far from clear that such an embedding exists for an arbitrary \( n \)-dimensional metric. For example, the right-hand side of Eq. (2.24) is no longer covariantly constant, as is the case for the Einstein spaces considered above. Moreover, the derivation of Eq. (2.22) depends crucially on the assumption that \( \Lambda \) is independent of \( y \). In view of this, we restrict the analysis of this subsection to the class of metrics (2.2) where the \((\alpha\beta)\)-components of the \((n+1)\)-dimensional Ricci tensor are \emph{independent} of the coordinate, \( y \), and can be expressed in terms of the derivative of a scalar function, \( \chi(x^\mu) \), such that
\[
\hat{R}_{\alpha\beta} = \frac{1}{2} \nabla_\alpha \chi \nabla_\beta \chi, \quad \frac{\partial \chi}{\partial y} = 0. \tag{2.26}
\]
The scalar field equation (2.25) then simplifies to
\[
h^{\alpha\beta} \nabla_\alpha \chi + h^{\alpha\beta} \nabla_\alpha \chi \nabla_\beta (\ln \phi) = 0. \tag{2.27}
\]
It is further assumed that all other components of \( \hat{R}_{AB} \) vanish.

It then follows that the components of the \((n+1)\)-dimensional Ricci tensor reduce to
\[
R_{\alpha\beta} = \frac{\nabla_\alpha \phi}{\phi} + \epsilon \left[ \frac{1}{\phi} \Omega^\alpha_{\alpha\beta} - h^{\gamma\delta} \Omega_{\gamma\delta} \Omega_{\alpha\beta} + 2 h^{\gamma\delta} \Omega_{\alpha\gamma} \Omega_{\beta\delta} \right] = \frac{1}{2} \nabla_\alpha \chi \nabla_\beta \chi \tag{2.28}
\]
\[
\nabla_\beta \left[ \delta^\gamma_\alpha h^{\gamma\delta} \Omega_{\gamma\delta} - h^{\gamma\delta} \Omega_{\gamma\alpha} \right] = 0 \tag{2.29}
\]
\[
-\phi h^{\gamma\beta} \left[ \epsilon \nabla_\gamma \phi - \Omega^\gamma_{\gamma\beta} h^{\alpha\delta} \Omega_{\alpha\gamma} \Omega_{\beta\delta} \right] = 0 \tag{2.30}
\]
on the hypersurface, \( y = y_0 \), where the symmetric functions, \( \Omega_{\alpha\beta} \), are defined, as before, in Eq. (2.7) and the boundary conditions (2.17) and (2.20) are also assumed to be valid.

The question that then arises is whether \( \Omega_{\alpha\beta} \) can be found that satisfy the conditions
\[
\Omega_{\alpha\beta} = \Omega_{\beta\alpha} \tag{2.31}
\]
\[
\nabla^\nu \Omega_{\alpha\nu} = \nabla_\alpha \Omega \tag{2.32}
\]
\[
\Omega^2 - \Omega_{\alpha\beta} \Omega^{\alpha\beta} = \epsilon \left[ R - \frac{1}{2} \nabla_\alpha \chi \nabla_\alpha \chi \right] \tag{2.33}
\]
on the hypersurface, \( y = y_0 \). If such functions exist and, furthermore, if they evolve according to
\[
\frac{\partial \Omega^\gamma_{\beta} }{\partial y} = \epsilon \nabla_\beta \phi - \epsilon \phi \left[ R^\gamma_{\beta} - \frac{1}{2} \nabla_\gamma \chi \nabla_\beta \chi \right] + \Omega \Omega^\gamma_{\beta} \phi, \tag{2.34}
\]
it can be shown, by following an argument similar to that presented in the Appendix, that the conditions
\[
\left( \nabla_\gamma \Omega^\gamma_{\beta} - \nabla_\beta \Omega \right)^* = \frac{\epsilon}{2} \nabla_\beta \left( \phi \nabla^2 \chi + \nabla_\gamma \phi \nabla^\gamma \chi \right) \tag{2.35}
\]
\[
\left( \Omega^\gamma_{\beta} \Omega^{\alpha\beta} - \Omega^2 + \epsilon \left[ R - \frac{1}{2} \nabla_\alpha \chi \nabla_\alpha \chi \right] \right)^* = 0 \tag{2.36}
\]
are satisfied.

Thus, Eq. (2.36) implies that Eq. (2.33) is also valid for all hypersurfaces in the neighbourhood of \( y = y_0 \). The right-hand side of Eq. (2.35) also vanishes since we have assumed \textit{a priori} that a scalar field can be found that satisfies\(^3\) Eq. (2.27). Hence, the field equations (2.24) are also valid for any \( y \) and Eq. (2.25) then follows from the contracted Bianchi identity.

To summarize, therefore, if we can find functions \( \{ \Omega_{\alpha\beta}, \phi, \chi \} \) satisfying Eqs. (2.27) and (2.31)–(2.33) on the hypersurface \( y = y_0 \), and evolving according to Eqs. (2.7), (2.25), (2.26) and (2.34), then the metric (2.1) can be embedded in the manifold (2.2), where the latter is a solution to the Einstein field equations (2.24) and (2.25) for a massless, minimally coupled scalar field.

This concludes our discussion of the embedding schemes. In the following Section, we employ the procedures to embed Einstein and plane wave spacetimes in non-vacuum, higher-dimensional manifolds.

3 Applications of the Embedding Schemes

3.1 Embedding Einstein Spaces in Einstein Spaces

We first consider the embedding of an \( n \)-dimensional Einstein space

\[
R_{\alpha\beta}(g) = \frac{2\lambda}{2-n} g_{\alpha\beta}
\]

(3.1)

in the \((n + 1)\)-dimensional Einstein space (2.11) for arbitrary constants \( \{ \lambda, \Lambda \} \). The embedding is achieved by invoking the \textit{ansatz}

\[
\Omega_{\alpha\beta} \equiv Ch_{\alpha\beta}, \quad \phi = 1,
\]

(3.2)

where \( C = C(x^\mu, y) \) is a scalar function. Eq. (2.16) immediately implies that the prefactor, \( C \), may be a function only of \( y \). On the other hand, Eq. (2.15) implies that

\[
C^2 = \frac{\epsilon(R + 2\Lambda)}{n(n-1)}
\]

(3.3)

and, since \( \phi = 1 \), it follows that Eq. (2.7) may be formally integrated to yield

\[
h_{\alpha\beta} = a^2(y)g_{\alpha\beta},
\]

(3.4)

where the ‘warp factor’, \( a \), is defined by \( a \equiv \exp \left[ - \int y \, dy' C(y') \right] \). If the constant of integration is chosen such that \( a(y_0) = 1 \), the \( n \)-dimensional metric \( g_{\alpha\beta} \) may be interpreted as the embedded Einstein space satisfying Eq. (3.1). Indeed, only Eqs. "if the scalar field is independent of \( x^\mu \), the embedding spacetime is Ricci-flat."
(2.19) and (3.3) remain to be solved for the embedding to be determined and these equations reduce to

\[
\frac{1}{a} \frac{d^2 a}{dy^2} = \frac{2\epsilon \Lambda}{n(n-1)}
\]

\[
\left( \frac{da}{dy} \right)^2 = \frac{2\epsilon \Lambda}{n(n-1)} a^2 - \frac{2\epsilon \lambda}{(n-1)(n-2)},
\]

respectively. The general solution satisfying the boundary condition (2.20) is then given by

\[
a = \cosh \left( \sqrt{\frac{2\epsilon \Lambda}{n(n-1)}} (y - y_0) \right) + B \sinh \left( \sqrt{\frac{2\epsilon \Lambda}{n(n-1)}} (y - y_0) \right),
\]

where

\[
B^2 = 1 - \frac{n\lambda}{(n-2)\Lambda}
\]

and the embedding of the Einstein space (3.1) is therefore given by

\[
ds^2 = a^2(y) g_{\alpha\beta} dx^\alpha dx^\beta + dy^2,
\]

where Eqs. (3.7) and (3.8) are satisfied.

Eq. (3.9) generalizes the embedding of maximally symmetric, four–dimensional Einstein spaces in five dimensions [22] as well as the embedding found in Ref. [23] for \( \lambda < 0 \). When the embedded manifold is Ricci–flat (\( \lambda = 0 \)), the warp factor (3.7) is exponential and Eq. (3.9) reduces to the metric considered in Ref. [24].

One interesting consequence of the embedding (3.9) is that it provides the bulk solution for non–fine–tuned versions of the Randall–Sundrum–type braneworld scenarios, where the co–dimension 1 branes have a non–vanishing cosmological constant [22, 25, 26]. Since the embedded metric is arbitrary in our analysis, it may be viewed as a non–linear generalization of the graviton zero mode on the brane. Within this context, a specific example is given by the Siklos class of solutions to Eq. (3.1) representing gravitational waves propagating in anti–de Sitter spacetime [27, 28].

3.2 Plane Waves

We conclude this Section by considering the embedding of the plane wave backgrounds [29]

\[
ds^2 = -du dv + du^2 + f_{ij} dx^i dx^j
\]

in a manifold sourced by a massless scalar field, \( \chi \), following the approach outlined in Section 2.2 for \( \epsilon = 1 \). The function \( f_{ij} = f_{ij}(u) \) is symmetric and depends only on the light–cone coordinate, \( u \). The metric (3.10) admits a covariantly constant, null Killing vector field, \( \partial/\partial v \), that is orthogonal to the Riemann curvature tensor.
Consequently, all curvature invariants vanish and this implies that metrics of this form can represent perturbatively exact solutions to the string equations of motion when the dilaton and antisymmetric form fields satisfy appropriate conditions [30, 31]. The only non-trivial component of the Ricci tensor is $R_{uu}$ and is also a function only of $u$.

To proceed with the embedding, we assume the ansatz

$$\Omega_{\alpha\beta} = \begin{cases} \frac{y}{y_0^2} & \text{if } \alpha = \beta = u \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

and

$$\phi = -1. \tag{3.12}$$

On the hypersurface $y = y_0$, where indices are raised with $g^{\alpha\beta}$, the only non-trivial components of $\Omega^\alpha_{\beta}$ and $\Omega^{\alpha\beta}$ are $\Omega^{uu} = 2\Omega^u = 4\Omega_{uu}$. Thus, Eq. (2.32) is solved since the embedded metric (3.10) and $\Omega^\alpha_{\beta}$ are both independent of $v$. Eq. (2.33) is also satisfied if the scalar field is a function only of $u$, $\chi = \chi(u)$, and this latter condition also ensures that Eq. (2.27) holds when $y = y_0$. We may then solve the set of equations (2.7) to deduce that

$$h_{\alpha\beta} = \begin{cases} \left(\frac{y}{y_0}\right)^2 & \text{if } \alpha = \beta = u \\ g_{\alpha\beta} & \text{otherwise} \end{cases} \tag{3.13}$$

This implies that in the neighbourhood of the hypersurface, only $\Omega^u_{\alpha}$ and $\Omega^{uv}$ are non-trivial and, consequently, Eqs. (2.32) and (2.33) are solved for arbitrary $y$. Moreover, Eq. (2.25) is trivially satisfied, since the scalar field is null. Thus, only Eq. (2.34) is yet to be solved and this set of conditions reduces to the single constraint:

$$\left(\frac{d\chi}{du}\right)^2 = 2 \left[ R_{uu} - \frac{1}{y_0^2} \right], \tag{3.14}$$

where $R_{uu}$ is the $(uu)$--component of the Ricci tensor calculated from the embedded metric (3.10). We may conclude, therefore, that the $(n+1)$--dimensional embedding metric is given by

$$d\hat{s}^2 = -dudv + \left(\frac{y}{y_0}\right)^2 du^2 + f_{ij}dx^idx^j + dy^2, \tag{3.15}$$

where the scalar field is determined by the quadrature

$$\chi = \sqrt{2} \int^u du' \left[ R_{uu}(u') - \frac{1}{y_0^2} \right]^{1/2}. \tag{3.16}$$

An interesting example of this embedding arises for the four–dimensional backgrounds defined by $f_{ij} = f^2(u)\delta_{ij}$, where $\delta_{ij}$ is the two–dimensional Kronecker delta.
and \( f = f(u) \) is an arbitrary function that parametrizes the amplitude of the plane wave. The Ricci tensor for such a metric is given by \( R_{uu} = -2f^{-1}(d^2f/du^2) \) and Eq. (3.14) therefore has the form of a one–dimensional Helmholtz equation:

\[
\left[ \frac{d^2}{du^2} + V(u) \right] f = 0, \tag{3.17}
\]

where the effective potential, \( V(u) \), is determined by the kinetic energy of the scalar field:

\[
V \equiv \frac{1}{2} \left[ \frac{1}{2} \left( \frac{d\chi}{du} \right)^2 + \frac{1}{y_0^2} \right]. \tag{3.18}
\]

It follows that if a particular solution, \( f_1(u) \), to Eq. (3.17) can be found for a given choice of \( \chi(u) \), the general solution can be expressed directly in terms of this solution such that

\[
f_{\text{gen}} = f_1 \left( \kappa + \int^u du' \frac{du'}{f_1^2(u')} \right), \tag{3.19}
\]

where \( \kappa \) is an arbitrary constant. In general, this implies that there is not a one–to–one correspondence between the amplitude of the embedded metric and the functional form of the scalar field that generates the Ricci curvature of the embedding metric.

Finally, a second metric of interest is the Nappi–Witten WZW model

\[
ds^2 = -dudv + du^2 + dx^2 + 2 \cos udxdy + dy^2 \tag{3.20}
\]

that corresponds to a conformal field theory describing a homogeneous, monochromatic plane wave [32]. The Ricci tensor of this background is \( R_{uu} = 1/2 \), implying that the scalar field takes the particularly simple form \( \chi = [1 - (2/y_0^2)]^{1/2} u \).

4 Discussion

In this paper, we have developed a procedure, introduced by Campbell, to embed a given Riemannian manifold into an Einstein space with a non–trivial cosmological constant. Such an embedding has a number of applications.

Firstly, the scheme is iterative and does not depend on the dimensionality of the embedded space. Thus, if the embedding of a particular \( n \)–dimensional space, \( M \), in an \( (n + 1) \)–dimensional Einstein space, \( M_{\text{Ein}} \), can be determined, an embedding of the space \( M \) into an \( (n + 2) \)–dimensional Einstein space follows immediately by embedding \( M_{\text{Ein}} \) along the lines outlined in Section 3.1.

This provides a method for generating and classifying exact solutions to higher–dimensional theories of gravity. For example, the infra–red limit of M–theory is eleven–dimensional supergravity, with a bosonic sector consisting of the graviton and a three–form antisymmetric potential [1]. Recently, it was shown that the field equations for this theory can be written in such a way that only ten–dimensional Poincare
invariance is manifest [33]. This is equivalent to performing a generalized Scherk–Schwarz dimensional reduction to ten dimensions, where the fields are allowed to depend specifically on the compactifying coordinate [34]. The resulting ten–dimensional theory represents a ‘massive’ extension of type IIA supergravity. It was further shown that if the gauge fields are then frozen out, the ten–dimensional equations of motion reduce to the single equation [33]

\[ \hat{R}_{AB} = m^2 \hat{h}_{AB}, \]

where \( m^2 \) represents a cosmological constant. Thus, the embeddings that we have discussed in this paper may be employed to generate solutions to the massive type IIA supergravity and eleven–dimensional supergravity theories.

Embeddings in Einstein spaces are also relevant to Wesson’s ‘spacetime–matter’ (STM) theory, where the matter on any (3 + 1)–dimensional hypersurface is encoded at a classical level purely in terms of five-dimensional vacuum geometries [9, 35]. As discussed in Ref. [8], this interpretation is closely linked to that of Campbell’s theorem [7]. Thus, embeddings in Einstein spaces would be related to a generalisation of the STM theory, although such a generalization could only be achieved at the price of introducing a curvature length scale. It would be of interest to investigate the relationship between four–dimensional matter and the geometry of five–dimensional Einstein spaces further. Moreover, such a generalization would enable direct comparisons to be made between the STM theory and braneworld models. In particular, both approaches attempt to attach physical significance to the fifth coordinate [35, 36] and these attempts should share some common obstacles and insights.

In establishing the embedding of Einstein spaces we invoked the ansatz (3.2). This restriction could be relaxed by allowing \( \Omega_{\alpha\beta} \) to have more degrees of freedom. One possibility is to specify \( \Omega_{\alpha\beta} = Q_\gamma^\alpha h_{\gamma\beta} \), where \( Q_\gamma^\alpha \) has the block–diagonal form

\[ Q_\gamma^\alpha = \text{diag} \left[ C(x^A), \ldots, C(x^A), D(x^A), \ldots, D(x^A) \right] \]

for some scalar functions \( \{C, D\} \). It would be natural to consider such an ansatz when embedding an Einstein space that itself is the product of two or more lower–dimensional Einstein spaces. An alternative approach – relevant to spatially homogeneous cosmologies – is to first embed the \((n – 1)\)–dimensional spacelike hypersurface in a space with an extra spatial dimension and to then view the embedding to \((n + 1)\) dimensions as an initial value problem [21].

The embedding of manifolds in Einstein gravity with a massless scalar field can also provide the seed for generating new, higher–dimensional solutions to the string equations of motion. In the case where the embedded metric admits an Abelian isometry associated with a Killing vector, \( \partial/\partial z \), a conformal transformation to the string frame, followed by a T–duality transformation, may be performed. This symmetry transformation inverts the string–frame metric coefficient associated with \( z \) and results in a new dilaton field. Solutions with non–trivial form fields may also be found by employing further duality transformations [17, 31].
Finally, we remark that since the cosmological constant and scalar field considered in Section 2 were uncoupled, it follows that embeddings in manifolds sourced by both degrees of freedom can in principle be found by extending the above analyses. In particular, we may deduce immediately that if a solution, \( \{ g_{\alpha\beta}, \chi \} \), to the \( n \)-dimensional field equations

\[
R_{\alpha\beta}(g) = \frac{1}{2} \nabla_\alpha \chi \nabla_\beta \chi + \frac{2\lambda}{2 - n} g_{\alpha\beta}
\]

(4.23)

\[
g^{\alpha\beta} \nabla_\alpha \nabla_\beta \chi = 0
\]

(4.24)

is known, the metric

\[
d\hat{s}^2 = a^2(y) g_{\alpha\beta} dx^\alpha dx^\beta + dy^2
\]

(4.25)

represents a solution of the equations of motion derived from the \((n + 1)\)-dimensional action

\[
S = \int d^{n+1}x \sqrt{-h} \left[ \hat{R} - \frac{1}{2} \left( \hat{\nabla} \chi \right)^2 - 2\Lambda \right],
\]

(4.26)

where the warp factor, \( a = a(y) \), is given by Eqs. (3.7) and (3.8), and the functional form of the scalar field is unaltered. Scalar field spacetimes satisfying Eqs. (4.23) and (4.24) were recently studied within the context of the AdS/CFT correspondence [37]. This embedding generalizes the embedding for \( \lambda = 0 \) found recently by Feinstein, Kunze and Vazquez–Mozo [38].

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In this appendix we derive the conditions (2.21) and (2.22). In doing so, we employ the expressions

\[ \nabla_\mu \nabla_\nu \phi = \nabla_\lambda \left( \nabla_\nu^\lambda \phi \right) - R_\mu^\lambda \nabla_\lambda \phi \quad \text{(A.1)} \]

\[ (\nabla_\alpha \beta - \nabla_\beta \alpha) T^\gamma_\delta = -R_{\alpha \beta \epsilon}^\gamma T^\epsilon_\delta + R_{\alpha \beta \epsilon}^\gamma T^\epsilon_\delta \quad \text{(A.2)} \]

for a scalar field and a tensor field \( T^\gamma_\delta \) derived from the Ricci lemma \[21\]. We also require expressions for the \( y \)–derivative of the Christoffel matrices \[7\]:

\[ \Omega^\rho_\eta \left( \Gamma^\rho_\eta \right)^\kappa_* = - \left( \phi \nabla_\mu \Omega^\rho_\eta + \Omega^\rho_\eta \nabla_\rho \phi \right) \Omega^\rho_\eta \quad \text{(A.3)} \]

\[ \left( \Gamma^\kappa_\eta \right)^\mu_* = - \nabla_\eta \left( \phi \Omega \right). \quad \text{(A.4)} \]

We first consider Eq. (2.21). Differentiating Eq. (2.19) with respect to \( x^\gamma \), and employing Eqs. (A.1), (A.3) and (A.4) implies that

\[ (\nabla_\kappa \Omega^\kappa_\mu)^* = \Omega^\eta_\kappa \left( \phi \Omega^\eta_\mu + \Omega^\eta_\mu \nabla_\rho \phi \right) \Omega^\rho_\eta - \Omega^\rho_\mu \left( \Omega^\kappa_\mu \phi + \phi \nabla_\kappa \Omega \right) + \nabla_\mu (\nabla^2 \phi) + 2 \Lambda \frac{\epsilon}{n - 1} \nabla_\mu \phi + \phi \left( \Omega^\kappa_\mu \nabla_\kappa \Omega + \Omega \nabla_\kappa \Omega^\kappa_\mu - \epsilon \nabla_\kappa R^\kappa_\mu \right). \quad \text{(A.5)} \]

The trace of Eq. (2.19), on the other hand, is given by

\[ \Omega^* = \epsilon \nabla^2 \phi - \epsilon \phi \left( R + \frac{2n \Lambda}{n - 1} \right) + \phi \Omega^2 \quad \text{(A.6)} \]

and combining Eq. (A.5) with the covariant derivative of Eq. (A.6) with respect to \( x^\mu \) then implies that

\[ \left( \nabla_\lambda \Omega^\lambda_\mu - \nabla_\mu \Omega \right)^* = \nabla_\mu \phi \left( \epsilon \left( R + 2 \Lambda \right) \Omega^\eta_\mu \Omega^\rho_\eta - \Omega^2 \right) + \phi \left( \Omega^\kappa_\mu \nabla_\kappa \Omega^\rho_\eta + \Omega \nabla_\kappa \Omega^\rho_\eta - 2 \Omega \nabla_\rho \Omega + \epsilon \nabla_\rho \Omega \right). \quad \text{(A.7)} \]

The first bracketed term on the right hand side of Eq. (A.7) vanishes due to Eq. (2.15). The second bracketed term vanishes due to the covariant derivative of Eq. (2.15) with respect to \( x^\mu \) and the contracted Bianchi identity

\[ \nabla_\lambda R^\lambda_\mu = \frac{1}{2} \nabla_\mu R. \quad \text{(A.8)} \]

Hence, Eq. (2.21) is valid and Eq. (2.16) propagates.

In establishing the validity of Eq. (2.22), it is necessary to calculate \( R^* \). Since \( R \) is a scalar, its derivative can be evaluated in normal coordinates \[7\]. Employing Eqs. (2.16), (A.2) and (A.4) implies that

\[ R^* = 2 \Omega \nabla^2 \phi + 2 R_{\alpha \beta} \Omega^\alpha \beta \phi - 2 \Omega^\beta_\gamma \nabla^\beta_\gamma \phi. \quad \text{(A.9)} \]
However, it follows from Eq. (2.19) and (A.6) that

\[(\Omega^\mu_\lambda \Omega^\lambda_\mu - \Omega^2)^* = 2\Omega \phi \left( \Omega^\mu_\lambda \Omega^\lambda_\mu - \Omega^2 + \epsilon (R + 2\Lambda) \right) \]
\[-2\epsilon \left( \phi \Omega^\mu_\lambda R^\lambda_\mu + \Omega \nabla^2 \phi - \Omega^\mu_\lambda \nabla^\lambda \phi \right). \quad (A.10)\]

Thus, substitution of Eqs. (2.15) and (A.9) into Eq. (A.10) implies that Eq. (2.22) is valid.