Classifying orientifolds by flat $n$-gerbes

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ABSTRACT: The discrete tensorial charges carried by orientifold planes define $n$-gerbes in space-time. The simplest way to ensure a consistent string compactification is to require these gerbes to be flat. This results in expressions for the local gerbe-holonomies around each orientifold plane, describing its charges. Inverting the procedure and considering all flat gerbes leads to a classification of orientifold configurations. Requiring that the tadpole is cancelled by adding D-branes, we classify all supersymmetric orientifolds on $T^k/\mathbb{Z}_2$ with $2^k \cdot O(9-k)$ planes at the fixed points, for $k \leq 6$. For $k = 6$ these theories organize in orbits of the $SL(2,\mathbb{Z})$ S-duality symmetry of $\mathcal{N} = 4$ supersymmetric gauge theories.

KEYWORDS: Orientifolds, gerbes.
1. Introduction

An orientifold is a construction in string theory, which involves the gauging of a symmetry of the theory which is a combination of a symmetry of the target space together with the orientation reversal of the string world sheet \([26, 27]\). If the target space symmetry has a fixed point set, its connected components are called orientifold fixed planes, or just orientifold planes for short (also orientifold lines and points, for sufficiently low dimension of the orientifold plane).

In this article we will restrict slightly further to the best studied category of orientifold planes, the ones that appear in type II string theories and locally look like \(\mathbb{R}^{p,1} \times (\mathbb{R}^{9-p})/\mathbb{Z}_2\), where the \(\mathbb{Z}_2\) inverts the coordinates on \(\mathbb{R}^{9-p}\) (we will review the full action in the next section). These planes preserve 16 of the 32 supersymmetries of the type II theories.

Orientifold planes have various interesting properties, that are intrinsically related. As stack of \(n\) D-branes has generically a \(U(n)\) gauge theory propagating on its world-volume; when the stack of branes is coinciding with an orientifold plane, the group is projected to \(O(n)\) or \(Sp(n/2)\) depending on the kind of orientifold plane. Whether the group is projected to \(O(n)\) or \(Sp(n/2)\) is correlated with a discrete charge carried by the orientifold plane. Another charge distinguishes between \(O(2n)\) and \(O(2n + 1)\) symmetry. Although in principle more charges are possible \([8]\), it is not clear what the effect of such charges is on string perturbation theory (if such an effect can be treated in perturbation theory at all), and we will restrict throughout this paper to the two charges mentioned.

The charges are tensorial in nature; they result in fluxes of certain tensorfields. The spectrum of charges can be studied in a suitable formalism such as cohomology or K-theory \([2]\). In this note we wish to study another aspect of the presence of such charges. In the same sense as that electric and magnetic charges give rise to electromagnetic fields, which can in turn be thought of in terms of a connection on and curvature of bundles, the tensorial charges carried by orientifold planes define tensorfields over space-time. As, in form notation, the one form and higher form tensorfields are very similar, it is tempting to also give a description of higher form tensorfields as connections on “something”. A particular formalism that is suited for this is the idea of (Abelian) gerbes \([9]\) (see also \([20]\) for a discussion with applications closer to the one in this
Gerbes are a generalization of bundles. As a matter of fact, there is a whole tower of mathematical objects, called $n$-gerbes.

The discrete tensorial charges of the orientifold planes give rise to $n$-gerbes. For a consistent string background, we have to satisfy the equations of motion. The easiest way to do so is to require the gerbes to be flat, and this is the problem we will study in this paper. A single orientifold plane does not give us any problems, its discrete charges are derived from a cohomology analysis that implicitly tells us that the curvature of the gerbe is at most torsion. For a composition of multiple orientifold planes in a certain geometry this is no longer necessarily true. The analysis in terms of a flux determined by the cohomology of the surrounding space at infinity is no longer valid; the discrete identifications that come with multiple orientifold planes modify the topology of the space at infinity.

In this paper we will study configurations of orientifold planes on the space $\mathbb{R}^{p,1} \times (T^{9-p})/\mathbb{Z}_2$. The orientifold planes at the fixed points are allowed to carry two kinds of discrete charges. We will review some properties of the relevant orientifold planes in section 2. Using some technology developed in a previous paper [3], it is actually not hard to write down the possible flat gerbes. We will do so in section 3.

The constraints we will obtain are not very restrictive for large $p$, but will become increasingly so for smaller values of $p$. We conjecture that, at least for the configurations we are studying, there is only one more criterion needed for a consistent compactification. This is the requirement that the RR $(p + 1)$-form tadpole be cancelled. This can always be done by adding a suitable number of D$p$- or anti D$p$-branes. The most non-suspect backgrounds are the supersymmetric ones, and these require the addition of D$p$-branes. We will restrict mostly to these supersymmetric orientifolds, although we will occasionally comment on their non-supersymmetric cousins.

With the restrictions made, we first study orientifolds with two kinds of planes and classify these up to geometrical symmetries. This will be done in section 4. The more elaborate case where we no longer restrict to a subset of the possible kinds of planes will be taken up in section 5. The results of these sections, a complete classification for $p \geq 3$, can be found in our tables 1, 2, 3 and 4. Due to a technical problem we cannot proceed beyond 4 noncompact dimensions. In 4 dimensions, the orientifolds we find lead to $\mathcal{N} = 4$ supersymmetric gauge theories. We compare our data with the requirement of the S-duality of these theories in section 6. Finally we summarize and conclude in section 7.

2. Aspects of orientifold planes

The orientifold planes we are interested in locally look like $\mathbb{R}^{p,1} \times (\mathbb{R}^{9-p})/\mathbb{Z}_2$. The generator of the relevant $\mathbb{Z}_2$ transformation is a product of three factors

$$I \cdot \Omega \cdot J$$
where \( I \) acts as inversion on the \( 9-p \) spatial coordinates of \( \mathbb{R}^{9-p} \) torus, and \( \Omega \) is the worldsheet parity operator. Finally, \( J \) is either the identity operator for \( p = 0, 1 \mod 4 \), or \((-)^{F_L}\) for \( p = 2, 3 \mod 4 \), with \( F_L \) the left moving fermion number. For \( p \) even resp. odd, this \( \mathbb{Z}_2 \) is a symmetry of IIA resp. IIB theory. As a consequence, in these theories one can find orientifold planes of spatial dimension \( p \), subsequently denoted as \( O_p \) planes.

For a single \( O_p \) plane, the space at infinity is \( S^{8-p} / \mathbb{Z}_2 = \mathbb{RP}^{8-p} \). Possible tensorial fluxes coming from the orientifold plane are traditionally classified by the cohomologies\(^1\) of \( \mathbb{RP}^{8-p} \). We will be interested in orientifold planes with a flux from the NS-NS \( B \)-field. The operators \( I \) and \( J \) acts trivially on this form, but it receives a minus sign from the operator \( \Omega \). It is therefore classified by

\[
[ dB ] = [ H ] \in H^3( \mathbb{RP}^{8-p}, \mathbb{Z}) = \mathbb{Z}_2
\]  

(2.1)

This only makes sense for \( p \leq 6 \), but for higher values of \( p \) one may work with equivariant cohomology \([3, 22]\), which also results in a \( \mathbb{Z}_2 \) spectrum. The two elements of \( \mathbb{Z}_2 \) correspond to different signs for a closed string diagram. As is well known, this is correlated with the action of the orientifold projection on the Chan-Paton matrices of the open string. The trivial element of \( \mathbb{Z}_2 \) results in an \( O(n) \) gauge group\(^2\) on \( Dp \)-branes, and the non-trivial one in an \( Sp(n/2) \) gauge group. In the first case, we will denote the plane as \( O_p^- \), in the second case as \( O_p^+ \).

We will also distinguish an RR charge. Normally the effect of RR charges would be hard to describe in string perturbation theory. The exception to the rule was discovered first for \( O_3 \) planes \([23]\). The \( SL(2, \mathbb{Z}) \) duality of type IIB theories translates to \( S \)-duality of the \( N = 4 \) supersymmetric Yang-Mills theory that is found on the D3 brane. In the presence of an \( O_3^+ \) plane, this is an \( Sp(k) \) gauge theory. \( S \)-duality maps this to an \( SO(2k + 1) \) gauge theory \([6]\), and the NS-NS 2–form charge to an RR 2–form charge. By T-duality, this should extend to other \( O_p \) planes with \( p \neq 3 \).

Indeed, one can define a RR \((5-p)\)–form charge for \( O_p \) planes. The relevant cohomologies are given by

\[
\begin{align*}
[ dC^{5-p} ] &= [ G^{6-p} ] \in H^{6-p}( \mathbb{RP}^{8-p}, \mathbb{Z}) = \mathbb{Z}_2, & p \text{ even} \\
[ dC^{5-p} ] &= [ G^{6-p} ] \in H^{6-p}( \mathbb{RP}^{8-p}, \mathbb{Z}) = \mathbb{Z}_2, & p \text{ odd}
\end{align*}
\]  

(2.2)

This puts a natural upper bound at \( p = 5 \). Above this bound, one still may use T-duality, but the result is that the RR flux is no longer localized at the plane.

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\(^1\)According to current lore, one should use K-theory for the RR fluxes \([2]\). For the orientifold planes we will consider this gives the same result as cohomology.

\(^2\)Actually, due to subtle effects that are non-perturbative from the viewpoint of the type II description, the actual gauge group is not (necessarily) \( O(n) \). For \( O_9^- \), the actual gauge group is \( Spin(32)/\mathbb{Z}_2 \) \([24]\). For lower values of \( p \) in \( O_p^- \) there are in general multiple possibilities, among which \( O(n) \), \( SO(n) \), \( Spin(n) \). The topology of this group does not even have to be such that it can be embedded in \( Spin(32)/\mathbb{Z}_2 \) \([3]\). We will simply write down the group that is manifest in perturbation theory, which is \( O(n) \), and ignore symmetry breakings and/or extensions by non-perturbative effects.
There is a simple realization of a plane with such a flux: As $SO(2k+1)$ is the gauge group that is found on an $O_p^-$ plane with an odd number of $D_p$ branes on it, and as even number of branes can always be moved away from the plane, there is obviously a bound state of an $O_p^-$ with a single $D_p$-brane. We will reserve a special notation for this bound state, calling it $\tilde{O}_p^-$. The tilde is meant to indicate the non-trivial RR flux associated to the plane. The RR $(p+1)$–form charge of this plane is the sum of one unit of $D_p$ brane charge and the charge of the $O_p^-$ plane. This shift in $(p + 1)$–form charge can also be understood from K-theory [2]. The upper bound of $p = 5$ seems to suggest that $\tilde{O}_p^-$ planes are absent for $p > 5$. An attempt to evade this conclusion can be found in [11]. The approach of this paper requires the introduction of a discrete cosmological constant, which would break supersymmetry, and should modify the Einstein equation, excluding flat space (away from the orientifold plane) as a solution. It therefore falls outside the category of planes that we are interested in in this paper.

The final plane is the $\tilde{O}_p^+$. This carries both the NS-NS 2–form charge and the non-trivial RR $(5 - p)$–form charge. It is not possible to form a bound state of an $O_p^+$ with an odd number of $D_p$-branes. Therefore there is no reason to expect a shift in the RR $(p + 1)$–form charge, and also a K-theory computation does not indicate this [2]. It is actually not so easy to distinguish $O_p^+$ and $\tilde{O}_p^+$. For $p = 3$ some differences can be seen in the non-perturbative spectrum of monopoles and dyons in the $\mathcal{N} = 4$ theories that arise when a stack of D3 branes coincides with the O3 plane [8]. More on $\tilde{O}_p^+$ planes can be found in [5, 7, 8, 10, 16, 23].

The planes that are relevant to this paper, and their properties are summarized in the following table.

<table>
<thead>
<tr>
<th>$O_p$ plane</th>
<th>group</th>
<th>$R_{p+1}$</th>
<th>$\mathcal{B}$</th>
<th>$\mathcal{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_p^-$</td>
<td>$O(2k)$</td>
<td>$-2^{p-4}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$O_p^+$</td>
<td>$Sp(k)$</td>
<td>$+2^{p-4}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{O}_p^-$</td>
<td>$O(2k + 1)$</td>
<td>$1 - 2^{p-4}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{O}_p^+$</td>
<td>$Sp(k)$</td>
<td>$+2^{p-4}$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The last three columns denote the charges of a plane: in the third column the RR $(p + 1)$–form charge, denoted by $R_{p+1}$, in the fourth the discrete NS-NS 2–form charge, which we will denote by $\mathcal{B}$ in the remainder of this paper, while the last column denotes the discrete RR $(5 - p)$–form charge $\mathcal{C}$.

3. Gerbe holonomies and orientifold charges

For a single $O_p$ plane its charges can in principle be measured by surrounding it by a sufficiently large (hyper)surface. For example, it should be possible to measure $\mathcal{B}$ by computing $\int B_2$ over a suitably chosen 2-surface. For a single $O_p$-plane this causes no problems. For a configuration of multiple planes, the various fluxes of the planes have to be patched together. To achieve this, we turn to the formalism of (Abelian) gerbes.
Some of the material presented here appeared (in a slightly different form, and adapted to a different problem) in [3]. We will repeat some of the discussion because the formalism of gerbes has entered the physics literature rather recently, and to make the paper more self-contained.

Consider an \( n \)-gerbe\(^3 \) defined on a manifold \( M \), in the pragmatic sense as in [9] (see also [25]). This means that we define a gerbe by transition functions over patches, satisfying suitable cocycle identities. A connection on an \( n \)-gerbe is an \((n + 1)\)-form\(^4 \) \( C \), defined over the patches \( U_i \) of the manifold. On non-empty intersections \( U_{ij} = U_i \cap U_j \) the forms over the patches \( U_i \) and \( U_j \) are related by a gauge transformation \( C_i = C_j + dC_{ij} \). The \( C_{ij} \) can be viewed as connections on \((n - 1)\) gerbes defined at the overlap patches \( U_{ij} \). To see why, note that as only \( dC_{ij} \) is defined, the \( C_{ij} \) are only defined up to closed forms (with integer periods), and hence have their own gauge invariance. On triple intersections \( U_i \cap U_j \cap U_k \) there is the consistency condition \( d(C_{ij} + C_{jk} + C_{ki}) = 0 \) which is compatible with the ambiguity of adding closed forms. Therefore \( C_{ij} + C_{jk} + C_{ki} \) and \( C_{ij} + C_{jk} + C_{ki} + dC_{ijk} \) define the same cocycle on the triple overlap. The \( C_{ijk} \) can be thought of as connections on an \((n - 2)\)-gerbe, defined over the \( U_{ijk} = U_i \cap U_j \cap U_k \), which again have their own gauge invariance. This extends all the way until we arrive at zero-forms (functions), which can be thought of as transition functions for a 0-gerbe (that is, a line bundle).

When computing a specific holonomy, it is more convenient to reduce the overlaps to infinitesimal size. Hence we cut up \( M \) in pieces \( M_i \) by a partition of unity. If the pieces \( M_i \) and \( M_j \) have a common boundary we denote this \( M_{ij} \); a common boundary between \( M_i \), \( M_j \) and \( M_k \) is written as \( M_{ijk} \) etc. Define \((n + 1)\)-forms \( C \) over the pieces \( M_i \); one can think about the \( M_{ij} \) as the places where one “jumps” from one patch to another (one can embed the \( M_i \) in open patches \( U_i \), then the \( M_{ij} \) will be embedded in \( U_{ij} \), etc. which gives our previous description). Again on overlaps, the connections in \( M_i \) and \( M_j \) are related by the gauge transformations \( C_i = C_j + dC_{ij} \). \( C_{ij} \) itself is defined over the \( M_{ij} \), and has an intrinsic ambiguity by a closed form. One regards \( C_{ij} \) as a connection on an \((n - 1)\)-gerbe over \( M_{ij} \) with a \( U(1) \) gauge invariance etc.

The concrete holonomy formula is (note the alternating sign):

\[
\int_M C = \sum_i \int_{M_i} C_i - \sum_{ij} \int_{M_{ij}} C_{ij} + \sum_{ijk} \int_{M_{ijk}} C_{ijk} - \ldots \tag{3.1}
\]

This is invariant (for \( M \) without boundary) under

\[
\begin{align*}
C_i &\rightarrow C_i + dL_i \\
C_{ij} &\rightarrow C_{ij} + L_i + L_j + dL_{ij}
\end{align*}
\]

\(^3\)It is customary to call 1-gerbes simply gerbes. 0-gerbes are line bundles [9].
\(^4\)Although some may find the convention to label the gerbe by the number \( n \) while it has a \((n + 1)\)-form connection confusing, it is actually convenient for discussions in physics. It implies that \( n \)-gerbes should naturally appear in the description of objects extended in \( n \) spatial directions (\( n \)-branes), as these couple in a natural way to \((n + 1)\)-forms [25].
\[ C_{ijk} \rightarrow C_{ijk} + L_{ij} + L_{jk} + L_{ki} + dL_{ijk} \quad (3.2) \]

Two extreme cases of this formula are for a globally well-defined form \( C \), in which case the sum on the r.h.s. reduces to a single term, and the case when only the last sum of integrals in the expression contributes. These are analogues of the bundle case where physicists are used to either using well-defined connections over large patches, or to “putting the holonomy in the transition functions”. For the case of connections on gerbes, there is a much larger freedom to “put” the holonomy somewhere, due to the multiple gauge invariances in (3.2).

Note that, even if the dimension of \( M \) is smaller than the degree of the form \( C \), eq. (3.1) gives a perfectly sensible, and not necessarily trivial result. One sets forms of too high degree formally to zero, but this does not necessarily imply that transition gerbes/bundles/functions are trivial. This may seem a rather pathological situation, but will be very useful to us below, because it actually allows us to avoid some technicalities when \( M \) has a low dimension.

The above formula can be found in [3], and was more or less inspired by gauge invariance, and inductive reasoning starting with the much simpler case of line bundles (0-gerbes). The reader may also want to consult [17] and compare with eq.(8) and fig. 5 of that paper for the holonomy formula for 1-gerbes (or just gerbes for short).

In this paper we will use the above formula exclusively on \( M = T^k/\mathbb{Z}_2 \) (The fact that this \( \mathbb{Z}_2 \) defines an orientifold instead of an orbifold is relevant for the classification of charges, but not for the gerbe computations that follow). We choose coordinates \( x_i \ (i = 1, \ldots, k) \) in \((\mathbb{R}/2\mathbb{Z})^k\) on \( T^k \), and quotient by the \( \mathbb{Z}_2 \) reflecting all coordinates. We label the \( 2^k \) fixed points of the \( \mathbb{Z}_2 \) by \( x_i = p_i, \ p_i \in (\mathbb{Z}_2)^k \). The periodicity \( 2\mathbb{Z} \) was chosen to allow us to work with the additive representation of the field of two elements, which we will denote as \( \mathbb{Z}_2 \). With this convention we can use ordinary addition and multiplication, with reduction modulo 2 understood, in a number of future computations.

The \( B \)-field, which is a 2–form, can be regarded as a connection on a (1-)gerbe. We will require this gerbe to be flat, which means that the field strength \( H = dB = 0 \). A non-zero \( H \) would couple to gravity, and warp the background. We will compute the \( B \)-holonomy around an orientifold plane. Because the gerbe is flat, this is independent of the 2–surface we use to compute the holonomy (as long as it has the point \( p \) in its interior, and the only possible source for \( B \) is at \( p \)). For two different surfaces that can be deformed into eachother, the difference in holonomy is expressible as the integral of \( H \) over the 3-surface that is swept out in the deformation, which has the two surfaces as boundary, and as \( H \) is zero, this difference is 0. With the conventions used, it is possible to write down relatively simple expressions for gerbe holonomy defined over hypersurfaces around the \( \mathbb{Z}_2 \) fixed points.

We pick a fixed point which we will denote by \( p \), with coordinates \( p_i \). We define a box around this point by using the coordinate ranges \( p_{n-2} - \epsilon < x_{n-2} < p_{n-2} + \epsilon, \quad p_i - \epsilon < x_i < p_i + \epsilon \).
\[ x_i < p_i + \epsilon \quad \text{for} \quad i = (n - 1), n, \text{and localizing the box in the remaining directions by setting} \quad x_i = p_i \quad \text{for} \quad i < (n - 2). \]

The parameter \( \epsilon \) is not necessarily small; we only require that the box contains at most one possible source, and therefore \( 0 < \epsilon < 1 \). The surface of this box is a 2-surface; we will compute the \( B \)-field holonomy over it. This seems to assume that \( n \geq 3 \), but as noted before, this is not strictly necessary. For \( n < 3 \), we can still evaluate (3.1), but one truncates the computation by setting forms of too high degree, and the integrals with them formally to 0. We will not attempt to prove in all rigor that this procedure makes sense, but just note that this pragmatic approach reproduces known results.

Let there be a 2-form field

\[ \sum_{i<j<k} b_{ij} dx_i dx_j / 4 \]

in the bulk. We can assume that this is a constant 2-form; one can always use gauge transformations to set the 2-form locally to a constant. Another viewpoint is that this 2-form is inherited from \( T^k \), as it is invariant under the \( \mathbb{Z}_2 \) action that we wish to quotient out, and hence survives the quotient. Because we can take it to be constant on \( T^4 \), we can take it constant on the quotient. This fixes the first of the gauge invariances in (3.2), but it is not hard to see that the result of our computation will be gauge independent. We will integrate this over a surface that has the topology of \( S^2 / \mathbb{Z}_2 = \mathbb{RP}^2 \).

From now on we will always assume that the summed over indices are ordered. In the equations below, this will actually present a choice of gauge that is convenient for the choice of surface that we made. The first integral in the holonomy formula is over the faces of the cube; there are 5 of these, of which 4 pair up as opposite faces. The opposite faces do not contribute to the integral, because their contributions cancel (and the form is constant). The only contribution comes from the face at \( x_{n-2} = p_{n-2} + \epsilon \).

This gives

\[ \sum_i \int_{M_i} B_i = b_{n-1,n} \epsilon^2 \]

(3.3)

We next consider the transition functions: these are defined at \( x_i = p_i \) for \( i \leq (n-2) \), \( p_{n-1} < x_{n-1} < p_{n-1} - \epsilon, p_n - \epsilon < x_n < p_n + \epsilon \). The constant 2-form jumps upon traversing the plane at \( x_{n-2} = p_{n-2} \) by an amount \( b_{ij} dx_i dx_j / 2 \), and we should write this as the derivative of something. This is inherently ambiguous (by (3.2)), so we fix the gauge and write

\[ b_{ij} dx_i dx_j = d \left( \sum_{i<j} b_{ij} x_i dx_j + b_i dx_i \right). \]

(3.4)

Here the second term is a closed 1-form that does not contribute to the transition function, but will contribute to the holonomy. Independence of the gerbe from local definitions requires that the \( b_i \) are independent of the patch we choose.

We have to integrate this 1-form over the edge of \( x_i = p_i \) (\( i \leq (n - 2) \)), \( p_{n-1} < x_{n-1} < p_{n-1} + \epsilon, p_n - \epsilon < x_n < p_n + \epsilon \). Again only one side contributes (the one with
\( x_{n-2} = p_{n-2}, \ x_{n-1} = p_{n-1} + \epsilon \), resulting in

\[
\left( \sum_i b_{in}p_i + b_n \right) \epsilon + b_{n-1,n}\epsilon^2
\]  \( (3.5) \)

The last contribution comes from the point at \( x_i = p_i \) \((i \leq (n-1))\), \( x_n = p_n + \epsilon \).

The transition function is

\[
\left( \sum_{i<j} b_{ij}x_ix_j + \sum_i b_{i}x_i + b \right)/2.
\]  \( (3.6) \)

Again the value of \( b \) cannot depend on the patch we are in. Inserting values for the coordinates gives

\[
\sum_{i<j} b_{ij}p_ip_j + \sum_i b_{i}p_i + b + \left( \sum_i b_{in}p_i + b_n \right) \epsilon.
\]  \( (3.7) \)

To find the total holonomy we add the contributions (3.3), (3.5), and (3.7). Because the gerbe is flat, we can actually replace our specific surface by any surface \( M_p \) with the point \( p \) (and no other source) in its interior. The final result is then:

\[
\mathcal{B}(\{p_i\}) \equiv \int_{M_p} B = \sum_{i<j} b_{ij}p_ip_j + \sum_i b_{i}p_i + b.
\]  \( (3.8) \)

This formula is to be computed modulo 2; all coefficients \( b_{ij}, b_i, b \) and \( p_i \) are elements of \( \mathbb{Z}_2 \). The actual holonomy is

\[
\exp i\pi \mathcal{B}(\{p_i\})
\]  \( (3.9) \)

and converts the additive representation of \( \mathbb{Z}_2 \) into its multiplicative one. The holonomy is 1 around an \( Op^- \) or \( \tilde{O}p^- \), and \(-1\) around a \( Op^+ \) or \( \tilde{O}p^+ \). We see that, for sufficiently large \( k \), the holonomy around each of the orientifold planes is completely specified by \( k(k-1)/2 \) coefficients of the antisymmetric tensor, \( k \) coefficients \( b_i \) and 1 coefficient \( b \).

This is all the freedom one has, and as the NS charges are classified by the equivariant cohomology \( H^2_{\mathbb{Z}_2} (T^k, \mathbb{Z}_2) \) \([3]\) one should have

\[
H^2_{\mathbb{Z}_2} (T^k/\mathbb{Z}_2) = \mathbb{Z}_2^{k^2+k+2} \hspace{1cm} k \geq 2
\]  \( (3.10) \)

For \( k = 1 \) we cannot define a 2–form, and one should have \( H^2_{\mathbb{Z}_2} (S^1/\mathbb{Z}_2) = \mathbb{Z}_2^2 \) because we can only choose \( b_1 \) and \( b \). All this is in agreement with computations from \([3]\).

The explicit generators in the appendix D of that paper correspond to a basis for the polynomials (3.8) (which form a vector space over the field \( \mathbb{Z}_2 \)), by simply replacing the generator giving particular values at particular points by the polynomial associating the same values to those points. Finally, for \( k = 0 \) one considers the equivariant cohomology of \( S^0/\mathbb{Z}_2 \). As \( S^0 \) is two points, \( S^0/\mathbb{Z}_2 \) is a single point, and one has the
trivial result $H^2_{\mathbb{Z}_2}(S^0/\mathbb{Z}_2) = \mathbb{Z}_2$, which is correctly captured by formula (3.8) which in this case consists of the single coefficient $b$.

We note that for $k \geq 2$ we have $\frac{k^2+k+2}{2}$ parameters specifying the holonomies around $2^k$ points. For $k > 2$, the former quantity is smaller, and hence there will be relations among the planes. These can be described in terms of constraints. For $k = 3$ one has 7 parameters describing 8 planes. The single constraint one has is easily derived from

$$\sum_{p_1=0}^{1} \sum_{p_2=0}^{1} \sum_{p_3=0}^{1} \left( \sum_{i<j} b_{ij} p_i p_j + \sum_i b_i p_i + b \right) = 0. \quad (3.11)$$

This tells us that summing the $\mathbb{Z}_2$ NS charges of all points gives zero, from which we conclude that the total number of NS charges is even. For $k = 4$ one has 11 parameters for 16 planes, and one has 5 constraints. One of these tells again that the number of planes with NS charge is even. Furthermore, one may also fix a single coordinate and sum over the remaining ones, to see that the number of planes in every $T^3/\mathbb{Z}_2$ is even. There are 4 independent choices for the fixed coordinate, and hence in total 5 constraints. In lower dimensions, these constraints become increasingly difficult to analyze, and instead, we will use below a method which stays closer to the original formula’s.

Similar formula’s can be derived for the RR $(5-p)$–form holonomy around an orientifold plane. In this case however, we have a different result for each dimension. We start with $T^k/\mathbb{Z}_2$ with $k = 4$. The fixed points will be O5 planes, which can carry a RR 0–form charge. A zero-form is just a function, flatness requires this to be a constant function so we simply have

$$C(\{p_i\}) = \int_{M_p} C = c \quad (3.12)$$

For $k = 5$ one has a 1–form RR charge. The relevant computation can be found in [3] (where it was computed for the orbifold K3 $T^4/\mathbb{Z}_2$, but the computation here is completely analogous), and one finds

$$C(\{p_i\}) = \int_{M_p} C = \sum_i c_i p_i + c \quad (3.13)$$

For $k = 6$ one has 2–forms, and we can copy our computation for the NS charges.

$$C(\{p_i\}) = \int_{M_p} C = \sum_{i<j} c_{ij} p_i p_j + \sum_i c_i p_i + c \quad (3.14)$$

For $k = 7$ again the relevant formula can be found in [3].

$$C(\{p_i\}) = \int_{M_p} C = \sum_{i<j<k} c_{ijk} p_i p_j p_k + \sum_{i<j} c_{ij} p_i p_j + \sum_i c_i p_i + c \quad (3.15)$$

The pattern will be clear by now. The holonomy around the orientifold plane is 1 around an $Op^-$ or $Op^+$, and $-1$ around a $\tilde{Op}^-$ or $\tilde{Op}^+$. 

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Therefore, the pair \((B(p_i), C(p_i))\) completely determines the identity of the orientifold plane. We define

\[
\begin{align*}
n_- &= \{ \#p : B(p_i) = 0, C(p_i) = 0 \} \quad (3.16) \\
n_+ &= \{ \#p : B(p_i) = 1, C(p_i) = 0 \} \quad (3.17) \\
\tilde{n}_- &= \{ \#p : B(p_i) = 0, C(p_i) = 1 \} \quad (3.18) \\
\tilde{n}_+ &= \{ \#p : B(p_i) = 1, C(p_i) = 1 \} \quad (3.19)
\end{align*}
\]

The notation “\(\#p\)” should be read as “the number of points \(p\) with \(\ldots\).”

The numbers \(n_-, n_+, \tilde{n}_-\) and \(\tilde{n}_+\) count the numbers of \(O_p^-\), \(O_p^+\), \(\tilde{O}_p^-\) and \(\tilde{O}_p^+\) respectively.

Another crucial number is represented by the RR tadpole. We represent this by the number \(r\), defined as

\[
r = 16 \frac{(n_- + \tilde{n}_-) - (n_+ + \tilde{n}_+)}{(n_- + \tilde{n}_-) + (n_+ + \tilde{n}_+)} - \frac{\tilde{n}_-}{2} \quad (3.20)
\]

This formula expresses tadpole cancellation, where the number of Dp-brane pairs is \(r\). In sufficiently high dimension \((k < 4)\) \(\tilde{n}_-\) and \(\tilde{n}_+\) are actually zero, due to the absence of an RR charge for Op-planes in these dimensions. Of course \(n_- + \tilde{n}_- + n_+ + \tilde{n}_+\) is simply the total number of fixed planes \(2^k\), but we prefer to keep the sum explicit, as it clearly reveals the different ways in which the NS charge and RR charge for orientifold planes result in rank reduction.

For a consistent theory one has to cancel the RR tadpole. If \(r\) is positive this can be done by adding \(r\) pairs of Dp branes. If \(r\) is negative, tadpole cancellation requires the addition of \(r\) pairs of anti Dp branes, and this necessarily breaks supersymmetry. Below, we will restrict to \(r\) non-negative, and we will refer to configurations with \(r\) negative as “breaking supersymmetry”. If \(r\) is non-negative, one may loosely refer to \(r\) as the “rank” of the gauge group, although strictly speaking this is not true, as there are also gauge symmetries coming from Kaluza-Klein reduction on the metric and various forms. These however are not manifest in the orientifold set-up.

By letting the coefficients \(b, b_i, b_{ij}, c, c_i, c_{ij}, c_{ijk}\) taking all possible values, one obtains all flat gerbes over \(T^n/\mathbb{Z}_2\). This is too much; many of these gerbes can be related by using coordinate transformations on \(T^n/\mathbb{Z}_2\). We will eliminate these redundancies by reducing the polynomials describing the gerbes to suitably chosen standard forms. Also, we will be less interested in the configurations that break supersymmetry (which in the case of a small number of non-compact dimensions outnumber the supersymmetric configurations), and therefore we will require \(r \geq 0\).

The formula (3.8) is the same for all values of \(k\). If one has a theory on \(T^k/\mathbb{Z}_2\), with a particular configuration of NS charges, one can compactify the theory on (an) additional circle(s). If one does not turn on \(B\)-fields over these extra cycles, one can without difficulty T-dualize. The formula for the configuration on \(T^{k+n}/\mathbb{Z}_2\) remains the same, which implies that the numbers \((n_- + \tilde{n}_-)\) and \((n_+ + \tilde{n}_+)\) are multiplied by powers of 2.

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Upon using a similar procedure on the formula’s for the RR charge, (3.12) should be mapped to (3.13) which in turn gets mapped to (3.14), and so on. We know how the top-form (the coefficient in $C$ with the most indices) transforms; it simply gains the index of the direction in which we dualize. It follows that the same must be true for all the coefficients in the polynomial $C(\{p_i\})$; all gain an index upon T-dualizing, and the map of the polynomials into each other is now obvious. We also see that the number of planes with RR charge stays the same under T-duality. As a consequence of these rules, the formula (3.20) is invariant under T-duality (as it should be).

All this is perfectly consistent with the T-duality rules ($q = p + 1$):

$$Oq^- \leftrightarrow Op^- + Oq^- \quad (3.21)$$
$$Oq^+ \leftrightarrow Op^+ + Op^+ \quad (3.22)$$
$$\tilde{O}q^- \leftrightarrow Op^- + \tilde{Op}^- \quad (3.23)$$
$$\tilde{O}q^+ \leftrightarrow Op^+ + \tilde{Op}^+ \quad (3.24)$$

We conclude that the formalism we have set up is both self-consistent and consistent with known facts about orientifolds. We now turn to study the holonomy formula’s, and derive their implications for the possible configurations of orientifold planes.

4. Models with at most two kinds of planes

Before dealing with the general case, we first study the equations for $B(\{p_i\})$ and $C(\{p_i\})$ separately. This amounts to studying configurations where only one of the $\mathbb{Z}_2$ charges, and hence at most two kinds of orientifold planes are present.

4.1. Models with $Op^-$ and $Op^+$ only

We will start by first classifying all configurations of NS charges that give rise to flat gerbes. As these are reflected in $B(\{p_i\})$, we will set $C(\{p_i\}) = 0$ for the remainder of this subsection. Doing so immediately will give us a list of configurations with $Op^-$ and $Op^+$-planes that give rise to flat gerbes. We will still impose the requirement of supersymmetry. Combining $r \geq 0$ with formula (3.20), one sees that in this context this translates into $n_- \geq n_+$. At the end of the subsection we will add some comments on non-supersymmetric configurations.

We are only interested in configurations modulo the action of coordinate transformations on $T^k/\mathbb{Z}_2$. We therefore fix the polynomial $B(\{p_i\})$ to a particular standard form by using these transformations.

On $T^k$, the symmetry group would be

$$(\mathbb{R}/2\mathbb{Z})^k \ltimes SL(k, \mathbb{Z}),$$

with $(\mathbb{R}/2\mathbb{Z})^k$ the group of translations $\mathbb{R}^k$ up to periodicity $2\mathbb{Z}^k$, and $SL(2, \mathbb{Z})$ the mapping class group of the $k$-torus. The symmetry group of the $\mathbb{Z}_2$ quotient is:

$$\mathbb{Z}_2^k \ltimes SL(k, \mathbb{Z}_2)$$
The $Z_2$ quotient breaks the translation group $(\mathbb{R}/2\mathbb{Z})^k$ to $Z_2^k$ because the periodicity is affected. The group $SL(k, \mathbb{Z})$ is not broken, but elements of these group that have equal entries modulo 2 have the same action on the $Z_2$ quotient (as far as the orientifold planes are concerned), and therefore the group is projected to $SL(k, Z_2)$.

The elements of the $Z_2^k$ subgroup are generated by the affine transformations $x_i \rightarrow x_i + 1$. The elements of $SL(k, Z_2)$ are linear transformations (over the field $Z_2$ one has $GL(k, Z_2) = SL(k, Z_2)$). Acting with these on the polynomial $B(\{p_i\})$, we will now bring it to a standard form.

Pick two coordinates $x_i$ and $x_j$ such that $b_{ij} = 1$. If there are no such coordinates then the quadratic part of the polynomial is in standard form already. If there are, then rename $x_i \leftrightarrow x_1$ and $x_j \leftrightarrow x_2$. In the expression for $B(\{p_i\})$ there now are a number of terms that involve $p_1$. Write these as

$$p_1 \left( p_2 + \sum_{i \neq 2} b_{1i} p_i + b_1 \right)$$

Setting $x_2 \rightarrow \left( x_2 + \sum_{i \neq 2} b_{1i} x_i + b_1 \right)$ eliminates all of these terms except $p_1 p_2$. After this operation the only term in $B(\{p_i\})$ that depends on $p_1$ is $p_1 p_2$. One then isolates all terms containing $p_2$ which can be written as

$$p_2 \left( p_1 + \sum_{i \neq 1} b_{2i} p_i + b_2 \right)$$

and sets $x_1 \rightarrow \left( x_1 + \sum_{i \neq 1} b_{2i} x_i + b_2 \right)$. The only term left that still contains $p_1$ or $p_2$ is the product $p_1 p_2$. One then repeats the procedure with $p_3$ and $p_4$ etc. until done.

We now have transformed to a configuration with only $b_{i,i+1}$ ($i$ odd) possibly non-zero, and if $b_{i,i+1} = 0$, then so are all $b_{j,j+1} = 0$ for $j \geq i$. If $m$ is the smallest odd integer for which $b_{m,m+1} = 0$ then also all $b_i = 0$ for $i < m$. If any of the $b_i$ with $i \geq m$ is non-zero we first interchange (if necessary) $x_i$ with $x_m$, to set $b_m = 1$. Subsequently one transforms

$$x_m \rightarrow \sum_i b_i x_i + b$$

to set all $b_i$ with $i > m$, and $b$ to zero.

With these manipulations we have arrived at what we will use as standard form. The standard form is specified by a number of parameters $b_{i,i+1}$, $i$ odd, $i < m$ for some odd $m$, a parameter $b_m$, and a parameter $b$ (which can only be non-zero if $b_m = 0$). These completely specify the NS charges of a given orientifold configuration on $T^k/Z_2$, up to coordinate transformation (linear or affine).

It is useful to relax for a moment the constraint that $b$ can only be non-zero if $b_m = 0$. Doing this, we note an elegant symmetry: replacing $b$ by $b + 1$, one exchanges $O_p$-planes with NS charge, with planes without NS charge. Therefore this symmetry exchanges $n_-$ and $n_+$. When $b_m = 1$, the transformation $x_m \rightarrow x_m + 1$ precisely has
the effect $b \rightarrow b + 1$. In this case $b \rightarrow b + 1$ is an automorphism of the configuration of orientifold planes, and this implies that if $b_m = 1$, $n_+ = n_-$. This in turn implies that $r = 0$ for these models.

On the other hand, if $b_m = 0$, the action of $b \rightarrow b + 1$ does not correspond to any coordinate symmetry. In this case $b \rightarrow b + 1$ has the effect $r \rightarrow -r$. Hence, only one of the configurations can be supersymmetric, and it is not hard to verify that, with the chosen standard form, this is always the one with $b = 0$.

Consequently, the requirement of supersymmetry together with the chosen standard form, leads us to always set $b = 0$.

Putting all information together, one arrives at table 1. The first column of this table lists the maximal dimension of non compact space $d_{\text{max}}$. If a certain model exists in $d_{\text{max}}$, one can further compactify it on an additional $n$–torus, and use T-dualities to deduce the existence of lower dimensional models. The subsequent column lists the non-zero coefficients of the polynomial $B(\{p_i\})$. This specifies the model completely, and determines the number of various kinds of $O^p$ planes, and their location. For easy reference, the numbers $n_-$ and $n_+$ that follow from $B(\{p_i\})$ are given in the next columns. These are functions of the parameter $k$ appearing in $T^k/\mathbb{Z}_2$. Of course one should remember that always $k \geq 9 - d_{\text{max}}$. The last column lists the quantity $r$, which is independent of $k$, as it should be.

<table>
<thead>
<tr>
<th>$d_{\text{max}}$</th>
<th>$b_{ij} = 0$</th>
<th>$b_i = 0$</th>
<th>$2^k$</th>
<th>$0$</th>
<th>$16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>$b_{ij} = 0$</td>
<td>$b_1 = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
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<tr>
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<td>$b_{ij} = 0$</td>
<td>$b_i = 0$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$8$</td>
</tr>
<tr>
<td>7</td>
<td>$b_{ij} = 1$</td>
<td>$b_1 = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>6</td>
<td>$b_{ij} = 1$</td>
<td>$b_3 = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>5</td>
<td>$b_{ij} = 1$</td>
<td>$b_{ij} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>4</td>
<td>$b_{ij} = 1$</td>
<td>$b_5 = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>3</td>
<td>$b_{ij} = 1$</td>
<td>$b_{ij} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>2</td>
<td>$b_{ij} = 1$</td>
<td>$b_7 = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
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<tr>
<td>1</td>
<td>$b_{ij} = 1$</td>
<td>$b_{ij} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>0</td>
<td>$b_{ij} = 1$</td>
<td>$b_{ij} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Table 1: $T^k/\mathbb{Z}_2$ orientifolds with $O^p+$ and $O^p-$ planes only.

The entries in table 1 exhibits a very simple structure, that is actually easy to understand. First of all, in every dimension exactly one new model appears, and therefore there are $k + 1$ possible models for any value of $k$. The models in table 1 can be divided in two classes, depending on the value of $b_m$. For $b_m = 1$, the models have their ranks reduced by powers of 2. The first model in this chain, $b_{ij} = 0$ with rank 16, is just the T-dual of the standard toroidal compactification of type I theory, with its $Spin(32)/\mathbb{Z}_2$ gauge group. The other elements in the chain are T-dual to the various possibilities of compactifying type I theory without vector structure [28, 29, 22, 30].
The values for $b_{ij}$ in the table are also the values for the $B$–field on the $k$–torus in the T-dual type I theory.

The class of models with $b_m = 1$ has equal numbers of $O^+$ and $O^-$ planes. The simplest of these is the model with $d_{max} = 8$. It has one $O^+$ and one $O^-$ plane, and is dual to type IIB on $S^1/\delta \Omega$. This is an orientifold, but here the worldsheet parity $\Omega$ is combined with a translation $\delta$ over half the circumference of the circle $S^1$ [31, 22]. Also the other models with $b_m = 1$ (and therefore $r = 0$) can be understood this way. The existence of these models, which have the same number of various orientifold planes, but inequivalent geometries, was first noted in [2, 3]. These appear at even values for $d_{max}$, and can be understood as duals to compactifications of type IIB on $S^1/\delta \Omega \times T^n$, where there are additional $B$-fields turned on on $T^n$ [16].

Incidentally, we note that, as promised, in low compact dimension ($9 \geq p \geq 6$) our formalism reproduces the known orientifolds [2, 3, 22], even though the original definition of $B$ as the integral over an $\mathbb{R}P^2$ surrounding the $O_p$ plane runs into difficulties here.

Before concluding this section, we briefly return to the class of non-supersymmetric models we discarded, the ones with $b_m = 0, b = 1$. We start with the simplest of these models, described by $b = 1$ and all other coefficients 0. This would be a 10-dimensional theory with $Sp(16)$ group, realized by composing 32 anti-branes with an $O^+$-plane [21], and its T-duals. Also $Sp(16)$ bundles on a $k$-torus allow bundles with twisted boundary conditions, and had we carried out our programme while neglecting supersymmetry, we would have found the T-dual descriptions for all of these. The list would simply be the same list as for the type I duals, but with an extra $b = 1$, and the numbers $n_-$ and $n_+$ interchanged. A last caveat: these models can only be interpreted as representing the flat connections in a classical gauge theory. In absence of supersymmetry, it should be expected that the degeneracy of vacua will be lifted by quantum corrections. The anti $D_p$-branes break supersymmetry and in particular the total configuration is not a BPS-system. As a consequence, it should be expected that a generic configuration is unstable, and the anti $D_p$ branes are dynamically driven to special locations. This looks like a T-dual description of the localization of the wave function that is well known for finite volume non-supersymmetric gauge theories (see e.g. [1] and references therein). Actually, even this final state may be only an approximate one, as it is not unlikely that such a theory may tunnel by non-perturbative effects into a stabler (supersymmetric?) minimum.

4.2. Models with $O^-$ and $\tilde{O}^-p_-$ only

We now turn to classification of flat gerbe coming from RR charges only, meaning that in the remainder of this section we will set $\mathcal{B}(\{p_i\})$ to zero. This automatically gives us the classification of orientifolds $T^k/\mathbb{Z}_2$ with $O^-$ and $\tilde{O}^-$ only. The polynomial $\mathcal{C}(\{p_i\})$ is different for any value of $k$, and we cannot treat all dimensions simultaneously, like in the previous section.
For $k < 4$ there is no formula for $C(\{p_i\})$ because in these models the $O_p$-planes cannot be given an RR charge.

In $k = 4$ one has the simple equation (3.12) which says $C(\{p_i\}) = c$. This equation is invariant under all reparametrizations of $T^k/\mathbb{Z}_2$, and therefore automatically in “standard form”. Setting $c = 0$ reproduces the standard compactification, with 16 $O5^-$ planes. Setting $c = 1$ turns all into $\tilde{O}5^-$. In $k = 5$ one has formula (3.13), which is the simple linear equation $C(\{p_i\}) = \sum_i c_i p_i + c$. We first have to bring this to standard form. If at least one of the $c_i$ is non-zero, one uses, if necessary $x_1 \leftrightarrow x_i$ to set $c_1 = 1$. Subsequently one transforms $x_1 \rightarrow C(\{x_i\})$ (by which we mean $C(\{p_i\})$, with $p_i$ replaced by $x_i$), and ends up with $c_1 = 1$ and all other coefficients zero. Upon using a T-duality in the 1-direction, we see that this is actually T-dual to a configuration that appeared for $k = 4$. The only remaining option is to set all $c_i = 0$, and set $c = 1$. As in $k = 4$, this is invariant under any coordinate transformation. It results in a configuration with 32 $\tilde{O}4^-$ planes.

The one-form in $k = 5$ can be interpreted as a connection on a bundle. In this case we have IIA theory and of course in this case the total space of the bundle is interpreted as the manifold upon which M-theory is compactified. Orientifold planes with RR 1–form charge correspond to twists of this bundle [5, 10], and hence the classification of bundles is the same as the classification of those configurations that can be lifted to M-theory (see the remarks in section 3.2.6 of [3]).

In $k = 6$ we have equation (3.14). This is the same formula as for the $B$-fields, and one can use the tricks of the previous section to rewrite any configuration into a standard form. Again $c = 0$, because of the tadpole requirement. The configurations $c_1 = 1$, and the configuration $c_{12} = 1$ are related by T-dualities to previously considered models. The other ones are new ones.

For $k > 6$ our previous methods fail. Essentially this is because we do not know how to rewrite the tri-linear term in (3.15) or higher order terms appearing for still higher values of $k$ to a standard form. Because of this technical problem our classification of configurations with RR charges will not proceed beyond $k = 6$.

<table>
<thead>
<tr>
<th>$d_{\text{max}}$</th>
<th>$c_{[5..k]} = 0$</th>
<th>$c_{[5..k]} = 1$</th>
<th>$c_{[6..k]} = 1$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$2^k$ 0</td>
<td>$2^k - 16$ 16</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>32</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$c_3 = 0$</td>
<td>$c_{[6..k]} = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2^k - 32$ 32</td>
<td>8</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$c_{12}[7..k] = 1$</td>
<td>$c_3[7..k] = 1$</td>
<td>$2^k - 24$ 24</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$c_{12}[7..k] = c_{34}[7..k] = 1$</td>
<td>$c_4[7..k] = 0$</td>
<td>$2^k - 32$ 32</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$c_{12}[7..k] = c_{34}[7..k] = 1$</td>
<td>$c_5[7..k] = 1$</td>
<td>$2^k - 32$ 32</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$c_{12}[7..k] = c_{34}[7..k] = c_{56}[7..k] = 1$</td>
<td>$c_6[7..k] = 0$</td>
<td>$2^k - 28$ 28</td>
<td>2</td>
</tr>
</tbody>
</table>

**Table 2:** $T^k/\mathbb{Z}_2$ orientifolds with $\tilde{O}p^-$ and $O p^-$ planes only.

Our results of this section are compiled in table 2. The table has the same structure
as table 1 except that now $O^p$ planes, and therefore $n_+$ are absent. Instead $\widetilde{O}^p$ planes appear, with their associated quantity $\tilde{n}_-$. The only other new element in table 2 is the notation we introduced in the coefficients of $C\{p_i\}$. As explained previously, these coefficients gain indices when T-dualizing. We tried to capture this in the notation $[m..k]$. The idea is that this represents a string of subsequent indices ranging from $m$ to $k$. If $k < m$ it represents an empty string. As an example, the notation $c_{12}[7..k]$ stands for $c_{12}$ if $k = 6$, for $c_{127}$ if $k = 7$, for $c_{1278}$ if $k = 8$, etcetera.

The entries in this table have a much more irregular structure than the previous table. The first non-trivial theory in table 2 (the second entry) is T-dual to a type I compactification on $T^4$ with a so-called non-trivial quadruple of holonomies [3, 12, 15]. These belong to a class of only fairly recently discovered compactifications of non-Abelian gauge theories on tori, having flat bundles that are not connected via flat connections to the trivial flat connection, but are also not characterized by their topology.

The entry under $d_{\text{max}} = 4$ is another example of such a compactification, this time T-dual to a type I compactification on $T^5$ with a non-trivial quintuple of holonomies [3, 12]. The interpretation of the theories listed under $d_{\text{max}} = 3$ is less straightforward, although their holonomies are fairly easy to write down. As explained in section 5 of [15], they can be obtained by “superposing” various quadruple configurations, and reducing $Dp$ branes modulo 2.

Besides flat gauge bundles with quadruples and quintuples, there also exist flat gauge bundles with triples. These last bundles were thoroughly studied in a number of papers [4, 12, 13]. Unfortunately, there are severe constraints on the existence of such compactifications in string theory, and as a matter of fact non-trivial triples do not give rise to flat gauge bundles in $Spin(32)/Z_2$ string theory [3]. Although non-trivial flat bundles on higher dimensional tori can be realized, the systematics for these theories is not completely understood yet from the gauge theory side.

We conclude this section with a few remarks on the non-supersymmetric models that were discarded. With the interpretation of an $\widetilde{O}^p$ plane as a bound state of an $O^p$ plane with a $Dp$-brane, and the simple formula for $r$ for the theories in this section

$$r = 16 - \frac{\tilde{n}_-}{2} \quad (4.1)$$

one may interpret these as theories that fail to be supersymmetric because of an insufficient number of $Dp$ branes in the theory. Indeed, were it not for tadpole cancellation, these theories would make perfect sense as dual descriptions to compactifications of $O(n > 32)$ open string theories. With the requirement of tadpole cancellation, one is automatically forced to introduce $-r$ pairs of anti $Dp$ branes. A pair of anti $Dp$ branes will feel an attractive force towards an $\widetilde{O}^p$ plane, and having arrived at this plane should annihilate with the $Dp$ brane there to form a bound state of an $O^p$ plane with an anti $Dp$ brane. Hence these theories will finally have $(\tilde{n}_- + r) \widetilde{O}^p$ planes (remem-
ber that $r$ is negative for non-supersymmetric theories) and $-r$ (non-supersymmetric) bound states of $Op^-$ with a single anti $Dp$ on top.

The positions of these planes are not fixed; by $Dp$ brane-antibrane pair creation in the bulk and letting such pairs annihilate with $Dp$ or anti $Dp$'s at the orientifold planes one can permute $\tilde{O}p^-$ planes with a bound state of $O^-p$ with anti $Dp$. Note however that these processes can never convert an $O^-p$ into an $\tilde{O}p^-$ or the bound state of $O^-p$ and anti $Dp$. Again, in the absence of supersymmetry, the degeneracy in vacuum configurations is not protected; the actual vacuum should be a superposition of all possible semiclassical vacua. Again we are assuming here that this vacuum is “sufficiently” stable, it is conceivable that by non-perturbative effects this theory may tunnel to another vacuum which is not at all resembling this orientifold description. However, under suitable circumstances the time-scale for such a process may be much longer than the time scale that the orientifold would need to find its metastable vacuum.

5. Mixed cases: All kinds of planes

Finally we will tackle the general case, where both $B(\{p_i\})$ and $C(\{p_i\})$ can be non-zero polynomials. We will omit the cases $B(\{p_i\}) = 0$ and $C(\{p_i\}) = 0$ as their descriptions can be found in the previous sections. Because of the problems with $C(\{p_i\})$ for $k > 6$, our classification will terminate at $k = 6$. In $k = 6$ the computations are very lengthy, although relatively straightforward. For the ease of the reader we will first present and discuss our results, and present the justifying computations later.

<table>
<thead>
<tr>
<th>$d_{max}$</th>
<th>$b_{12} = 1$</th>
<th>$n_-$</th>
<th>$n_+$</th>
<th>$\tilde{n}_-$</th>
<th>$\tilde{n}_+$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$b_{12} = 1$</td>
<td>$c_{[5..k]} = 1$</td>
<td>$3 \cdot 2^{k-2} - 12$</td>
<td>$2^{k-2} - 4$</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$b_1 = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1} - 16$</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_{1[6..k]} = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b_{12} = 1$</td>
<td>$c_{1[6..k]} = 1$</td>
<td>$3 \cdot 2^{k-2} - 8$</td>
<td>$2^{k-2} - 8$</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b_{12} = 1$</td>
<td>$c_{1[6..k]} = 1$</td>
<td>$3 \cdot 2^{k-2} - 16$</td>
<td>$2^{k-2}$</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b_{12} = b_{34}$</td>
<td>$c_{1[6..k]} = 1$</td>
<td>$5 \cdot 2^{k-3} - 8$</td>
<td>$3 \cdot 2^{k-3} - 8$</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 3: $T^k/Z_2$ orientifolds with all kinds of planes, $d_{max} > 3$

Our results can be found in the tables 3 and 4. These tables have the same structure as the tables 1 and 2, except that now one has 4 columns to count all the kinds of orientifold planes. We will explain in section 6 why we have attached an index to some of the zero’s in the last column.
<table>
<thead>
<tr>
<th>$d_{\text{max}}$</th>
<th>$b_1 = 1$</th>
<th>$c_{1[7..k]} = 1$</th>
<th>$n_-$</th>
<th>$n_+$</th>
<th>$\tilde{n}_-$</th>
<th>$\tilde{n}_+$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$b_{12} = 1$</td>
<td>$c_{12[7..k]} = c_{34[7..k]} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1} - 32$</td>
<td>0</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = 1$</td>
<td>$c_{1[7..k]} = 1$</td>
<td>$3 \cdot 2^{k-2} - 12$</td>
<td>$2^{k-2} - 12$</td>
<td>12</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = 1$</td>
<td>$c_{12[7..k]} = c_{1[7..k]} = 1$</td>
<td>$3 \cdot 2^{k-2} - 16$</td>
<td>$2^{k-2} - 16$</td>
<td>16</td>
<td>16</td>
<td>0 (_a)</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = 1$</td>
<td>$c_{15[7..k]} = c_{34[7..k]} = 1$</td>
<td>$3 \cdot 2^{k-2} - 16$</td>
<td>$2^{k-2} - 8$</td>
<td>16</td>
<td>8</td>
<td>0 (_c)</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = 1$</td>
<td>$c_{13[7..k]} = c_{2[7..k]} = c_{3[7..k]} = 1$</td>
<td>$3 \cdot 2^{k-2} - 16$</td>
<td>$2^{k-2} - 16$</td>
<td>16</td>
<td>16</td>
<td>0 (_b)</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = 1$</td>
<td>$c_{14[7..k]} = c_{2[7..k]} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1} - 16$</td>
<td>0</td>
<td>16</td>
<td>0 (_b)</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{12[7..k]} = c_{3[7..k]} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1} - 32$</td>
<td>0</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{13[7..k]} = 1$</td>
<td>$5 \cdot 2^{k-3} - 4$</td>
<td>$3 \cdot 2^{k-3} - 12$</td>
<td>4</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{13[7..k]} = 1$</td>
<td>$5 \cdot 2^{k-3} - 8$</td>
<td>$3 \cdot 2^{k-3} - 8$</td>
<td>8</td>
<td>8</td>
<td>0 (_d)</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{13[7..k]} = 1$</td>
<td>$5 \cdot 2^{k-3} - 8$</td>
<td>$3 \cdot 2^{k-3} - 16$</td>
<td>8</td>
<td>16</td>
<td>0 (_d)</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{15[7..k]} = c_{34[7..k]} = 1$</td>
<td>$5 \cdot 2^{k-3} - 8$</td>
<td>$3 \cdot 2^{k-3} - 24$</td>
<td>0</td>
<td>24</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{15[7..k]} = c_{34[7..k]} = 1$</td>
<td>$5 \cdot 2^{k-3} - 8$</td>
<td>$3 \cdot 2^{k-3} - 8$</td>
<td>8</td>
<td>8</td>
<td>0 (_c)</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{15[7..k]} = c_{34[7..k]} = 1$</td>
<td>$2^{k-1}$</td>
<td>$2^{k-1} - 32$</td>
<td>0</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$b_{12} = b_3 = 1$</td>
<td>$c_{15[7..k]} = c_{34[7..k]} = 1$</td>
<td>$9 \cdot 2^{k-4}$</td>
<td>$7 \cdot 2^{k-4} - 28$</td>
<td>0</td>
<td>28</td>
<td>2</td>
</tr>
</tbody>
</table>

**Table 4:** $T^k/\mathbb{Z}_2$ orientifolds with all kinds of planes, $d_{\text{max}} = 3$

The theories described in the tables 3 and 4 are “mixed” in multiple ways. First of all, “mixed” refers to the presence of both NS and RR charges in the characterization of the orientifold planes. But they are also mixed in the sense that their dual description
involves a mixture of the elements of the dual descriptions in the previous sections.

First consider the models where less than half of the Op planes carries an NS charge. As explained in section 4.1 such models are T-dual to type I theories without vector structure. In section 4.2 we explained that for topologically trivial, flat Spin(32)/Z₂ bundles there are components in the gauge theory moduli space that are not connected to the trivial flat bundle via flat connections, and that these give rise to new orientifold models. It should come as no surprise that also for topologically non-trivial bundles there are many different components in the moduli space of flat bundles [4], and that these give rise to different orientifold descriptions [3, 14].

The models where half of the Op-planes carry NS charges can still be identified with duals to the type IIB-theory on S¹/δΩ. Because in this case n− + ñ− = n+ + ñ+, a glance at formula (3.20) will teach us supersymmetry requires that ñ− = 0. This is also intuitively clear; as the tadpole is already cancelled because of the balance between planes with and without NS charge, there can be no excess Dp-branes forming bound states with Op- planes.

This argument does not rule out the presence of Õp+ planes in these theories, and indeed various solutions with these planes exist. It is however not easy to describe accurately the distinction between these theories, and the ones where one replaces the Õp+ planes by Op+ -planes. Both are duals of the type IIB-theory on S¹/δΩ, and their difference is due to rather subtle RR phases on the torus upon which the theory is compactified, which are rather poorly understood at present.

Also for the (large number of) non-supersymmetric theories that are discarded there, it is hard to make general statements. In principle the starting point is again clear: when one computes r to be negative, one adds anti Dp branes to cancel the tadpole. Anti Dp branes can annihilate with Dp branes at Õp- planes, and the endpoint of the whole process is depending on r and n−. If −r > n− there will be free anti Dp branes left. These will then probably be driven to some (meta)stable non-supersymmetric configuration of minimal vacuum energy. If −r < n− all Dp branes at Õp- planes will be annihilated, and one has a configuration where all Dp and anti Dp branes form bound states with Op- planes, and the true vacuum will be a superposition of all possible configurations. Finally if −r = n−, all anti Dp branes will annihilate, and the final state is, at least semiclassically, unique⁵.

We will now turn to the actual computations that lead to the tables 3 and 4. We start with the easiest computation, for k = 4. If at any place the reader gets weary of the computations, he is encouraged to skip the remainder of this section and turn to section 6.

5.1. 5+1 dimensional theories

Our general line of attack in this subsection and the following ones will be as follows.

---

⁵An example of such an orientifold is T⁶/Z₂, with B = p₁p₂ + 1, and C = p₁p₂. This demonstrates that this set of models is not empty.
First we fix a configuration of NS charges. This amounts to picking an entry from table 1, and inserting the coefficients in the polynomial $B(\{p_i\})$. After having fixed this we consider in all generality all possibilities for $C(\{p_i\})$. We have to do so, because the fact that we have already used part of the symmetries to fix $B(\{p_i\})$ means that we have less freedom to manipulate $C(\{p_i\})$.

It will be clear that there are two factors determining the difficulty of this process: 1) When we will go to lower dimensions (higher $k$) the polynomial $C(\{p_i\})$ will become a more complicated expression; and 2) the more coefficients of $B(\{p_i\})$ are fixed, the fewer symmetries we have at our disposal. This is why the first computations we will present are almost trivial, while by the end of the third subsection they will have grown very tedious. In these subsections, when we have found a supersymmetric model we will compute the row of 5 numbers $(n_-, n_+, \tilde{n}_-, \tilde{n}_+, r)$ and list it.

Let us apply the procedure to the case of $k = 4$. We fix $B$, by picking one of the first 4 non-trivial entries in table 1 (as we want something that is not in our tables yet, we require $B(\{p_i\}) \neq 0$). For $p = 5$ the expression for $C(\{p_i\})$ is simple: equation (3.12) has only one coefficient. To look for new configurations we have to set $C(\{p_i\}) = 1$.

Then we have the following cases:

- $b_1 = 1$: Setting $c = 1$ breaks supersymmetry.
- $b_{12} = 1$: Setting $c = 1$ gives the model $(0,0,12,4,2)$.
- $b_{12} = b_3 = 1$: $c = 1$ breaks supersymmetry.
- $b_{12} = b_{34} = 1$: $c = 1$ breaks supersymmetry.

Hence in 6 dimensions, after a few straightforward computations we found one new model. This model made a brief appearance in [3], and was more thoroughly described in [16]. In the next subsection we will find models that have not appeared in the literature thus far.

5.2. 4+1 dimensional theories

Again we start by fixing $B(\{p_i\})$, by picking an entry from table 1. The expression for $C(\{p_i\})$ is still relatively simple (see eq. 3.13).

We will use a number of symmetries and properties of the expressions for $B(\{p_i\})$ to fix the polynomial $C(\{p_i\})$ to a standard form without affecting $B(\{p_i\})$. These coordinate transformations play an even more prominent role in fixing a standard form for the 4-dimensional theories.

A very useful and important property is, that the set of indices on the non-zero $b_{ij}$, $b_i$ has an upper bound. For $k = 5$ this leads immediately to the following simplification. Let $m$ be the smallest integer that does not occur in the indices appearing on the non-zero entries of $b_{ij}, b_i$. If any of the $c_i$ with $i \geq m$ is non-zero, one first relabels coordinates (only coordinates with $i \geq m$ should be relabelled !) to set $c_m = 1$. Subsequently one
can transform away all $c_i$ with $i \neq m$, as well as $c$, by setting $x_m = \mathcal{C}(\{x_i\})$, while leaving all other coordinates invariant. Because $c_m$ was non-zero, this transformation is non-singular, and because the coordinates $x_i$ with $i \neq m$ are not affected, this does not change the expression for $\mathcal{B}(\{p_i\})$. We hence end up with a configuration given by $\mathcal{B}(\{p_i\})$ and $\mathcal{C}(\{p_i\}) = p_m$. Now applying a T-duality in the $m$-direction, we see that this configuration is related to one that already appeared one dimension higher. Summarizing: If any of the $c_i$ with $i \geq m$ is non-zero, the configuration is related to one considered before and therefore it is either ruled out, or already in our table. Therefore we can restrict to $c_i = 0$ for $i \geq m$.

We now turn to the $c_i$, with $i < m$. If $b_{i,i+1} = 1$ for some (odd) $i$, then one can assume that $c_{i+1} = 0$, because if it is not, we can set it to zero in the following way: If $c_i = 0$, $x_i \leftrightarrow x_{i+1}$ (other coordinates invariant) leaves $\mathcal{B}(\{p_i\})$ invariant, but sets $c_{i+1}$ to zero; if $c_i = 1$, setting $x_i \rightarrow x_i + x_{i+1} + 1$ leaves $\mathcal{B}(\{p_i\})$ invariant but removes $p_{i+1}$ dependence of $\mathcal{C}(\{p_i\})$.

The previous trick will already eliminate many $c_i$’s, but we can still do more if we have $b_{12} = b_{34} = 1$. In the way just described one sets $c_2$ and $c_4$ to zero, but one can also eliminate a non-zero $c_3$. Either $c_1 = 0$, and one uses $(x_1, x_2) \leftrightarrow (x_3, x_4)$; or $c_1 = 1$ in case one uses $x_1 \rightarrow x_1 + x_3$, $x_4 \rightarrow x_2 + x_4$. In either case $\mathcal{B}(\{p_i\})$ remains invariant, but $c_3$ is eliminated. Hence in this case we only have to consider non-zero $c_1$ and possibly $c_5$.

The following argument also presents a simplification in 5-dimension, but will in particular be a crucial shortcut to some computations in 4-dimensions. As remarked before, in models where the number of planes with NS charge is equal to the number of planes without NS charge, it is impossible to have $\tilde{O}p^-$ planes while preserving supersymmetry. So, for these theories, instead of computing $r$ case by case, we can also simply demand absence of $\tilde{O}p^-$ planes in these models. This amounts to requiring that, whenever $\mathcal{B}(\{p_i\})$ gives the value 0 at a certain plane, we should also require $\mathcal{C}(\{p_i\})$ to give 0 at that particular plane. Solving for this constraint fixes most, sometimes even all coefficients in the polynomial $\mathcal{C}(\{p_i\})$.

After these preliminary considerations we simply check the remaining options case by case:

- $b_1 = 1$: As explained, we set $c_i = 0$ for $i > 1$. Supersymmetry requires absence of $\tilde{O}4^-$ planes. Then one quickly deduces that $c = 0$ (because that would turn planes at $p_i = 0$ into $\tilde{O}4^-$). Hence the only non-trivial possibility is $c_1 = 1$, which gives $(16, 0, 0, 16, 0)$.

- $b_{12} = 1$: We can set $c_i = 0$ for $i > 1$, then the only options left are:
  1. $c = 1$: Breaks supersymmetry.
  2. $c_1 = 1$: Leads to the model $(16, 0, 8, 8, 4)$.
  3. $c_1 = c = 1$: Leads to $(8, 8, 16, 0, 0)$.
\[ b_{12} = b_3 = 1 : \text{Set } c_i = 0 \text{ for } i > 3 \text{ and } c_2 = 0. \text{ Supersymmetry requires absence of } \widetilde{O4}^- \text{ planes. Then one quickly deduces that } c = 0 \text{ (because that would turn planes at } p_1 = p_3 = 0 \text{ into } O4^-) \text{ that } c_1 = 0 \text{ (with planes at } p_1 = 1, p_3 = 0), \text{ and } c_3 = 0 \text{ (with planes at } p_1 = p_3 = 1). \text{ Hence supersymmetry requires } \mathcal{C}(\{p_i\}) = 0, \text{ and we find no new models.}
\]

\[ b_{12} = b_{34} = 1 : \text{Set } c_i = 0 \text{ for } i > 1.
\]

1. \( c = 1 \): Breaks supersymmetry.
2. \( c_1 = 1 \): Leads to \( (12, 4, 8, 8, 0) \).
3. \( c_1 = c = 1 \): Breaks supersymmetry.

\[ b_{12} = b_{34} = b_5 = 1 : \text{Set } c_2 = c_3 = c_4 = 0. \text{ Supersymmetry requires absence of } O4^- \text{ planes. Then one quickly deduces that } c = 0 \text{ (because that would turn the plane at } p_1 = 0 \text{ into } \widetilde{O4}^-) \text{ and that } c_1 = 0 \text{ (with planes at } p_1 = 1), \text{ and } c_5 = 0 \text{ (with planes at } p_1 = p_5 = 1). \text{ Hence supersymmetry requires } \mathcal{C}(\{p_i\}) = 0 \text{ and we find no new models.}
\]

With a little more effort than in the previous subsection, we have identified in total 4 new models.

### 5.3. 3+1-dimensional theories

In this dimension computations get very tedious. This is to some extent clear from the expression for \( B(\{p_i\}) \) (3.8) and \( \mathcal{C}(\{p_i\}) \) (3.14). In a very literal sense, they are both equally complicated, but after having fixed a standard form of \( B(\{p_i\}) \) there are only relatively few possibilities left to manipulate \( \mathcal{C}(\{p_i\}) \).

One category of theories is still (relatively) simple to tackle: the ones with equal number of planes with NS charge and planes without NS charge. Supersymmetry imposes the absence of \( \widetilde{O4p}^- \) planes, and having realized that, the computation to be done is straightforward:

\[ b_1 = 1.
\]

The absence of \( \widetilde{O3}^- \) planes restricts the polynomial \( \mathcal{C}(\{p_i\}) \) to

\[ \mathcal{C}(\{p_i\}) = p_1 \left( \sum_{j \neq 1} c_{1j} p_j + c_1 \right) \quad (5.1)
\]

If at least one of the \( c_{1j} \) is non-zero, we use (if necessary) \( x_2 \leftrightarrow x_j \) to set \( c_{12} = 1 \). Subsequently we transform \( x_2 \rightarrow \left( \sum_{j \neq 1} c_{1j} x_j + c_1 \right) \). Hence we are left with the following inequivalent possibilities:

1. \( c_{12} = 1 \): One can use T-duality in 2-direction to relate this to \( b_1 = c_1 = 1 \) for \( p = 4 \).
2. $c_1 = 1$: This leads to $(32,0,0,32,0)$.

- $b_{12} = b_3 = 1$.
  Absence of $\tilde{O}3^-$ restricts the polynomial $C(\{p_i\})$ to
  \[ C(\{p_i\}) = c_{12}p_1p_2 + c_{13}p_1p_3 + c_{23}p_2p_3 + c_3p_3 \]  
  \[ \text{(5.2)} \]
  with
  \[ c_{12} + c_{13} + c_{23} + c_3 = 0 \]
  This gives us 8 remaining possibilities to be checked. However, we still have some remaining symmetries that are generated by the transformations: $x_1 \rightarrow x_1 + x_2 + 1$, other coordinates invariant; $x_1 \leftrightarrow x_2$, other coordinates invariant; and, $x_1 \rightarrow x_1 + x_2$, $x_3 \rightarrow x_2 + x_3$. These leave $B(\{p_i\})$ invariant but have a non-trivial effect on $C(\{p_i\})$. Using these symmetries, one can reduce to 2 inequivalent options:
  1. $c_{12} = c_{13} = 1$: Leads to $(32,16,0,16,0)$.
  2. $c_{12} = c_3 = 1$: Leads to $(32,0,0,32,0)$

- $b_{12} = b_{34} = b_5 = 1$.
  Again we require absence of $\tilde{O}3^-$ planes. A somewhat lengthy computation shows that this requirement restricts the polynomial $C(\{p_i\})$ to
  \[ C(\{p_i\}) = c_{12}p_1p_2 + c_{34}p_3p_4 + c_5p_5 \]  
  \[ \text{(5.3)} \]
  with
  \[ c_{12} = c_{34} = c_5 \]
  Hence the only new model comes from setting $c_{12} = c_{34} = c_5 = 1$, which gives $(32,0,0,32,0)$.

The most tedious computations are the ones that will follow now. We found it convenient to invoke a simple computer program for parts of the analysis. Nevertheless, a number of computations was still done by hand. Equivalence or inequivalence of certain models can often be fairly easily checked, and allows us to avoid wasting computer time on configurations that have already been analyzed.

- $b_{12} = 1$. We use coordinate transformations in $x_i$ with $i = 3, 4, 5, 6$ to bring the part of $C(\{p_i\})$ with terms involving (only) these coordinates to standard form.
  We will start with the easiest case:
  1. $c_{34} = c_{56} = 1$.
     Redefining $x_i$ ($i = 3, 4, 5, 6$) one can set $c_{1i}, c_{2i}$ and $c_i$ to zero. Using $x_1 \leftrightarrow x_2$, $x_1 \rightarrow x_1 + x_2 + 1$ also $c_2$ can be set to zero. The only possibly non-zero
coefficients of $C$ (apart from $c_{34}$ and $c_{56}$) are then $c_{12}$, $c_1$, $c$. This gives in total 8 models, but a computation reveals that all have $r < 0$, and therefore they are not supersymmetric.

2. $c_{34} = 1$, $c_{56} = 0$.

We may set coefficients $c_{3i}$, $c_{4i}$, $c_3$ and $c_4$ to zero by redefining $x_3$ and $x_4$. If $c_{15} = c_{16} = c_{25} = c_{26} = 0$, one sets $c_2 = 0$ by the transformations $x_1 \leftrightarrow x_2$ and $x_1 \rightarrow x_1 + x_2 + 1$. One can now also use $x_5 \leftrightarrow x_6$, and $x_5 \rightarrow x_5 + x_6$, to set $c_6$ to zero. If $c_5$ non-zero, one can transform to $c_1 = c = 0$. Summarizing, with $c_{34} = 1$, all $c_{ij} = 0$ except possibly $c_{12}$, we only need to consider non-zero values for $c_{12}$, $c_1$, $c_5$ and $c$. Computing $r$ for these various possibilities one finds that $r \geq 0$ for:

- $c = 0$: This is related by T-duality in the 3 and 4 directions to a model considered previously.
- $c_{12} = 1$: Leads to the model $(36,4,12,12,2)$.

If $c_{15} = 1$, one may set $c_{12},c_{16}$ and $c_1$ to zero by redefining $x_5$. With $c_{16} = 1$ but $c_{15} = 0$ one uses $x_5 \leftrightarrow x_6$. Thus in case of $c_{15} = c_{34} = 1$ we only need to consider non-zero $c_2, c_5, c_6$ and $c$. For these coefficients there is only one supersymmetric model:

- $c_{15} = 1$: Leads to $(32,8,16,8,0)$.

With $c_{25} = 1$, but $c_{16} = c_{15} = 0$, one uses $x_1 \leftrightarrow x_2$. With $c_{15} = c_{25} = 1$, but $c_{16} = 0$ one uses $x_1 \rightarrow x_1 + x_2 + 1$. When $c_{25} = c_{16} = 1$ one can absorb any $c_{1i}$ ($i \neq 6$) and $c_1$ upon redefining $x_6$, and $c_{2i}$ ($i \neq 5$) and $c_2$ upon redefining $x_5$. Hence for this subclass of theories one only has to consider non-zero-values for $c_5, c_6$ and $c$. All of these turn out to result in $r < 0$.

With $c_{26} = 1$ but $c_{15} = c_{16} = c_{25} = 0$ one uses $x_1 \leftrightarrow x_2$ to reduce to a previously considered case. If $c_{26} = c_{15} = 1$, one uses $x_1 \leftrightarrow x_2$. If $c_{26} = c_{16} = 1$ use $x_5 \leftrightarrow x_6$. If $c_{26} = c_{15} = c_{16} = 1$ use $x_1 \leftrightarrow x_2$.

In case $c_{26} = c_{25} = 1$, redefine $x_5$ to absorb $c_{26}$. Any remaining model with $b_{12} = c_{34} = 1$ can be transformed to one of the previously considered models, and we can pass on to the next class.

3. $c_{34} = c_{56} = 0$.

As long as $c_{ij} = 0$, $C(\{p_i\})$ takes the simpler form (3.13), we can use the same set of tricks as in the $p = 4$ case, and hence we only consider non-zero values for $c_1$ and $c$. This results in one supersymmetric model:

- $c_1 = 1$: Leads to the model $(32,0,16,16,0)$.

If $c_{12} = 1$, we can set $c_2 = 0$ by using $x_1 \leftrightarrow x_2$ and $x_1 \rightarrow x_1 + x_2 + 1$. If $c_{12} = c_3 = 1$, one can absorb other $c_i$ and $c$ by redefining $x_3$. Considering the remaining possibilities gives the supersymmetric models:
- $c_{12} = 1$: Leads to (48,0,0,16,8).
- $c_{12} = c_1 = 1$: Leads to (32,16,16,0,0).

If at least one of the $c_{1j}$ ($j \neq 2$) is non-zero, one can use coordinate transformations in $x_i$ ($i = 3, \ldots, 6$) to set all $c_{1j}$ except $c_{13}$, and $c_1$ to zero. If $c_2$ non-zero can set $c = 0$ by redefining $x_2 \to x_1 + x_2 + 1$. Possible values for $c_i$ with $i > 3$ can all be absorbed in $c_4$ by using coordinate transformations in $x_i$ with $i = 4, 5, 6$. Hence we capture all of these theories by setting $c_{13} = 1$, and considering all values for $c_2$, $c_3$, $c_4$ and $c$. This leads to the following set of supersymmetric models.

- $c_{13} = 1$: Related to a higher dimensional model by using T-duality in the 3-direction.
- $c_{13} = c_3 = 1$: Also related by T-duality in the 3-direction to a higher dimensional model.
- $c_{13} = c_3 = c_2 = 1$: Leads to (32,0,16,16,0).

We next consider $c_{23} = 1$, $c_{13} = 0$; in this case one uses $x_1 \leftrightarrow x_2$. If $c_{23} = c_{13} = 1$, use $x_1 \to x_1 + x_2 + 1$.

If $c_{23} = c_{14} = 1$, one can redefine $x_3$ and $x_4$ to absorb all other $c_{1j}$, $c_{2j}$, $c_1$ and $c_2$. If either one of $c_3$ or $c_4$ is zero, they can both be set to zero in the following way: If $c_3 = 0$ and $c_4 = 1$, use $x_1 \to x_1 + x_2 + 1, x_3 \to x_3 + x_4$; If $c_3 = 1$ and $c_4 = 0$, use $x_2 \to x_1 + x_2 + 1, x_4 \to x_3 + x_4$. Hence in this case we only need to check for values of $c_3 = c_4$ and $c$. This results in a single supersymmetric model

- $c_{23} = c_{14} = 1$: Leads to (32,8,16,8,0)

If $c_{23} = 1$ and $c_{15} = 1$ and/or $c_{16} = 1$ one uses coordinate transformations in $x_4, x_5, x_6$ to set $c_{14} = 1$ and proceeds as previously.

If $c_{24} = 1$ and $c_{23} = 0$ one uses $x_3 \leftrightarrow x_4$. If $c_{24} = c_{23} = 1$ then redefine $x_3$ to absorb all other $c_{2j}$ and $c_2$, and hence set $c_{24} = 0$. Finally, if $c_{25}$ or $c_{26}$ is non-zero, one again permutes $x_4, x_5, x_6$, such that $c_{24}$ is set to one, and proceeds as before.

This finishes the computation of configurations with $b_{12} = 1$

- $b_{12} = b_{34} = 1$.

We would like to bring the part with quadratic terms of the expression for $C(\{p_i\})$ to standard form, but our freedom is limited to the coordinates $x_5$ and $x_6$. Therefore the “standard form” simply amounts to stating the value of $c_{56}$, all other symmetries have to come from the transformations that leave $B(\{p_i\})$ invariant.

1. $c_{56} = 1$. 

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In principle one could go through the whole tedious procedure of the fixing of symmetries. However, running a computer program computing all possible configurations with $b_{12} = b_{34} = c_{56} = 1$ (without actually fixing any other symmetries) shows that none of these theories is supersymmetric. Therefore our efforts would be in vain anyway, and we simply pass on to $c_{56} = 0$.

2. $c_{56} = 0$.

At first we set $c_{i5}, c_{j6} = 0$.

Symmetries of the B-field configuration are respected by a group of symmetries (see $p = 4$), generated by

$$ (x_1, x_2, x_3, x_4) \rightarrow (x_2, x_1, x_3, x_4) \quad (5.4) $$
$$ (x_1, x_2, x_3, x_4) \rightarrow (x_1 + x_2 + 1, x_2, x_3, x_4) \quad (5.5) $$
$$ (x_1, x_2, x_3, x_4) \rightarrow (x_3, x_4, x_1, x_2) \quad (5.6) $$
$$ (x_1, x_2, x_3, x_4) \rightarrow (x_1 + x_3, x_2, x_3, x_2 + x_4) \quad (5.7) $$

Of course there are also symmetries involving $x_5$ and $x_6$, but because $B(\{p_i\})$ does not involve these coordinates, we do not mention them at this point.

The set of coefficients $c_{12}, c_{13}, c_{14}, c_{23}, c_{24}$ and $c_{23}$ can a priori take 64 values, but under the above symmetries many of these are related. We decompose the orbits of the symmetries on the space of possible values of the set of above $c_{ij}$’s (there turn out to be 6 of these) and pick a representative from each orbit. Then we run a computer calculation on each representative, checking on supersymmetric configurations if combinations of $c_i$’s and $c$ are turned on. This immediately leads to the following results:

- $c_{ij} = 0$: If not $c_i = c = 0$ then supersymmetry is broken. The remaining configuration has $C(\{p_i\}) = 0$ and can be found in section 4.1.
- $c_{12} = 1$: Only $c_i = c = 0$ is a supersymmetric configuration; this leads to $(36,12,4,12,2)$.
- $c_{13} = 1$: Supersymmetric configurations are $c_i = 0$, $c_1 = 1$ and $c_3 = 1$. These are equivalent by symmetries: $x_1 \rightarrow x_1 + x_3, x_4 \rightarrow x_2 + x_4$, takes $c_{13} = 1, c_i = 0$ to $c_{13} = c_3 = 1$, while $x_3 \rightarrow x_1 + x_3, x_2 \rightarrow x_2 + x_4$, takes $c_{13} = 1, c_i = 0$ to $c_{13} = c_1 = 1$. Therefore they are really only one model, with characteristic $(32,16,8,8,0)$.
- $c_{13} = c_{24} = 1$: Supersymmetric configurations are $c_i = 0$, $c_2 = c_3 = 1$ and $c_1 = c_4 = 1$. These are also related to each other by the symmetries $x_1 \rightarrow x_1 + x_3, x_4 \rightarrow x_2 + x_4$, and $x_3 \rightarrow x_1 + x_3, x_2 \rightarrow x_2 + x_4$, and hence all lead to the same model, $(32,8,8,16,0)$.
- $c_{12} = c_{13} = c_{24} = 1$: None of the theories in this orbit is supersymmetric.
- $c_{12} = c_{34} = 1$: Only $c_1 = c = 0$ is supersymmetric. This leads to the model $(40,0,0,24,4)$. 

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Next consider \( c_{15} = 1 \). We can now absorb other \( c_{1i} \) and \( c_1 \) by redefining \( x_5 \). Combining this with the orbit structure one is left with:

- \( c_{15} = 1 \): The only supersymmetric model has \( c_i = c = 0 \), but is related by T-duality to a higher dimensional model.
- \( c_{15} = c_{24} = 1 \): All values for \( c_i \) and \( c \) give tadpoles.
- \( c_{15} = c_{34} = 1 \): The only supersymmetric model is \( c_i = c = 0 \), leading to \((32, 8, 8, 16, 0)\).

If \( c_{16} = 1 \) we use \( x_5 \leftrightarrow x_6 \). If \( c_{15} = c_{16} = 1 \) one absorbs \( c_{16} \) by redefining \( x_5 \).

Next we turn to \( c_{25} = 1 \). Now we absorb all other \( c_{2i} \) and \( c_2 \) in a redefinition of \( x_5 \)

- \( c_{25} = 1 \): use \( x_1 \leftrightarrow x_2 \).
- \( c_{13} = c_{25} = 1 \): use \( x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4 \).
- \( c_{34} = c_{25} = 1 \): use \( x_1 \leftrightarrow x_2 \).

If \( c_{15} = c_{25} = 1 \), one first absorbs the other \( c_{1i} \) and \( c_{2i} \) in a redefinition of \( x_5 \). Then use \( x_1 \rightarrow x_1 + x_2 + 1 \) to eliminate \( c_{25} \).

If \( c_{25} = c_{16} = 1 \), one first absorbs the other \( c_{1i} \) and \( c_{2j} \) in redefinitions of \( x_5 \) and \( x_6 \). The only two-index coefficient in \( C(\{p_i\}) \) that is not fixed yet is \( c_{34} \), but a check reveals that both \( c_{34} = 0 \) and \( c_{34} = 1 \) always result in supersymmetry breaking configurations. When \( c_{25} = c_{16} = c_{15} = 1 \) one again absorbs \( c_{1i} \) and \( c_{2i} \) (with \( i \neq 5 \)) in a redefinition of \( x_5 \), and proceeds as previously. With \( c_{26} = 1, c_{25} = 0, c_{26} \leftrightarrow x_6 \). If \( c_{26} = c_{25} = 1 \): Redefine \( x_5 \) to absorb \( c_{26} \).

Next consider \( c_{35} = 1 \). Absorb all other \( c_{3i} \) in a redefinition of \( x_5 \). The following options remain:

- \( c_{35} = 1 \): Use \( x_1 \leftrightarrow x_3, x_2 \rightarrow x_4 \).
- \( c_{12} = c_{35} = 1 \): Use \( x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4 \).
- \( c_{24} = c_{35} = 1 \): Use \( x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4 \).
- \( c_{12} = c_{24} = c_{35} = 1 \): These all break supersymmetry.

If \( c_{35} = c_{15} = 1 \), one absorbs all other \( c_{3i} \) and \( c_{1i} \) in a redefinition of \( x_5 \). The only coefficient left to consider is \( c_{24} = 0, 1 \), but now one can always use \( x_1 \rightarrow x_1 + x_3, x_4 \rightarrow x_2 + x_4 \) to relate this to a previously considered model.

With \( c_{35} = c_{16} = 1 \). Again one absorbs all other \( c_{1i} \) and \( c_{3i} \) in redefinitions of \( x_5 \) and \( x_6 \), and one is left with only \( c_{24} = 0, 1 \). All possible models with these coefficients break supersymmetry.

With \( c_{35} = c_{25} = 1 \) absorb all other \( c_{2i} \) and \( c_{3i} \) in a redefinition of \( x_5 \). Then use \( x_2 \leftrightarrow x_2 + x_3, x_4 \rightarrow x_1 + x_4 \). The cases \( c_{35} = c_{25} = 1 \) with \( c_{15}, c_{16} = 0, 1 \) are treated similarly. With \( c_{35} = c_{26} = 1 \), first transform away \( c_{2i} \) and \( c_{3i} \), then use \( x_1 \leftrightarrow x_2 \). With \( c_{35} = c_{26} = c_{25} = 1 \), transform away \( c_{2i}, c_{3i} \) and \( c_{26} \).
With \( c_{36} = 1 \) use \( x_5 \leftrightarrow x_6 \). With \( c_{35} = c_{36} = 1 \), absorb \( c_{36} \) in a redefinition of \( x_5 \).

With \( c_{45} = 1 \) absorb all other \( c_{4i} \) in a redefinition of \( x_5 \). After this one is left with possible non-zero values for \( c_{12} \) and \( c_{13} \). By using \( x_1 \leftrightarrow x_4, x_2 \leftrightarrow x_3 \) this can always be reduced to a previously considered model. With \( c_{45} = c_{15} = 1 \) one redefines \( x_5 \) to absorb other \( c_{1i} \) and \( c_{4i} \). Then use \( x_1 \rightarrow x_1 + x_4, x_3 \rightarrow x_2 + x_3 \). With \( c_{45} = c_{16} = 1 \), absorb other \( c_{1i} \) and \( c_{4i} \) in redefinitions of \( x_5 \) and \( x_6 \). Then use \( x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4 \). With \( c_{45} = c_{25} = 1 \), first absorb \( c_{2i} \) and \( c_{4i} \) in a redefinition of \( x_5 \). Then use \( x_2 \rightarrow x_2 + x_4, x_3 \rightarrow x_1 + x_3 \). With \( c_{45} = c_{25} = 1 \) and \( c_{15} \) and/or \( c_{16} = 1 \), absorb all other \( c_{1i}, c_{2i} \) and \( c_{4i} \) with coordinate definitions of \( x_5 \) and \( x_6 \), and use again \( x_2 \rightarrow x_2 + x_4, x_3 \rightarrow x_1 + x_3 \).

With \( c_{45} = c_{26} = 1 \), use \( x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4 \).

With \( c_{45} = c_{35} = 1 \), absorb all other \( c_{3i} \) and \( c_{4i} \) in a redefinition of \( x_5 \), and then use \( x_3 \rightarrow x_3 + x_4 + 1 \). With \( c_{45} = c_{36} = 1 \), absorb all other \( c_{3i} \) and \( c_{4i} \) and use \( x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4 \). The same for \( c_{45} = c_{36} = c_{35} = 1 \).

Finally, for \( c_{46} = 1 \) and \( c_{45} = 0 \), one uses \( x_5 \leftrightarrow x_6 \), and for \( c_{46} = c_{45} = 1 \) one can use \( x_5 \rightarrow x_5 + x_6 \).

We have explicitly or implicitly considered all models with \( b_{12} = b_{34} = 1 \), and we pass on to the last category of theories.

- \( b_{12} = b_{34} = b_{56} = 1 \).

There are only relatively few, to be precise 4 D3 branes to be possibly distributed over the O3 planes. One possibility is requiring absence of O3\(^-\) planes, which automatically implies \( c_{12} = c_{34} = c_{56} \), all others zero. Therefore, the only non-trivial solution to this constraint is

\[- c_{12} = c_{34} = c_{56} = 1 \text{ leading to } (36,0,0,28,2)\]

Running a check with the computer on other configurations with \( b_{12} = b_{34} = b_{56} = 1 \) immediately reveals that there are no other supersymmetric solutions, than the above one and the one with \( C(\{p_i\}) = 0 \).

In this subsection we have identified in total 17 new models, that were collected in table 4.

6. S-duality in 4 dimensions

All our orientifold theories on \( T^6/\mathbb{Z}_2 \) result in low energy effective 4 dimensional \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theories. As is well known, these theories exhibit an \( SL(2,\mathbb{Z}) \) S-duality symmetry. This provides an interesting check on our results and methods.
First of all we note that the S-duality symmetry was already manifest in our formalism. In 4 dimensions the RR charges are described by (3.14), which is isomorphic to the formula for the NS charges (3.8). The group $SL(2, \mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (6.1)$$

Because the formula’s (3.8) and (3.14) are over the field $\mathbb{Z}_2$, the manifest S-duality group is not $SL(2, \mathbb{Z})$, but its reduction modulo 2, $SL(2, \mathbb{Z}_2)$. This group is generated by matrices $S$ and $T$, reduced modulo 2. The effect of the matrix $S$ is the interchange of the formula’s (3.8) and (3.14). The transformation $T$ leaves (3.14) invariant, but replaces (3.8) by

$$(B + C)(\{p_i\}) \equiv \int_{M_p} (B + C) = \sum_{i<j} (b_{ij} + c_{ij})p_ip_j + \sum_i (b_i + c_i)p_i + (b + c) \quad (6.2)$$

Recall that $B(\{p_i\})$ gave 1 on $O_3^+$ and $\tilde{O}_3^+$, and 0 on $O_3^-$ and $\tilde{O}_3^-$, whereas $C(\{p_i\})$ resulted in 1 on $\tilde{O}_3^+$ and $\tilde{O}_3^-$, and 0 on $O_3^+$ and $O_3^-$. Hence formula (6.2) gives the value 1 on $O_3^+$ and $\tilde{O}_3^-$, and 0 on $O_3^-$ and $O_3^+$.

From the action of $S$ and $T$ on the formula’s describing the charges we see that $S$ interchanges $O_3^+$ with $\tilde{O}_3^-$, and therefore $n_+$ with $\tilde{n}_-$, while $T$ interchanges $O_3^+$ with $\tilde{O}_3^+$, and hence $n_+ \leftrightarrow \tilde{n}_+$. These two transformations generate the whole permutation group $S_3 \cong SL(2, \mathbb{Z}_2)$, acting on the 3 numbers in the triple $(\tilde{n}_-, n_+, \tilde{n}_+)$. 

One can now distinguish essentially 3 possibilities. First, if $\tilde{n}_- = n_+ = \tilde{n}_+$, then the configuration forms a singlet under $SL(2, \mathbb{Z}_2)$. There are actually 2 models in our tables with this property: the trivial compactification, with $(64, 0, 0, 0, 64)$, and a much more interesting model, with $(40, 8, 8, 8, 4)$. 

Instead of the 3 numbers $\tilde{n}_-, n_+, \tilde{n}_+$ being all equal, they could also be all different. This possibility is however not realized at all, as can be seen by inspection of our tables. We don’t know whether this is just a numerical coincidence, or there is a deeper level of understanding of this fact possible.

All configurations in our tables that are not singlets under $SL(2, \mathbb{Z}_2)$ have 2 out of the 3 numbers $\tilde{n}_-, n_+, \tilde{n}_+$ equal. In this case, some transformations in $SL(2, \mathbb{Z}_2)$ have a trivial effect on the triple $(\tilde{n}_-, n_+, \tilde{n}_+)$. Only a $\mathbb{Z}_3$ subgroup is manifest, and the orbits of the group consist of 3 elements.

It is a nice consistency check on our results to divide the models in our tables in orbits under $SL(2, \mathbb{Z}_2)$. Closure of all orbits gives us added confidence that our classification is indeed complete. In most cases this is a fairly simple exercise, as often the set of numbers $(\tilde{n}_-, n_+, \tilde{n}_+)$ specifies the model completely (in 4 dimensions $n_- = 64 - \tilde{n}_- - n_+ - \tilde{n}_+$, and $r$ follows from (3.20)).

There exist however models with equal $(n_-, n_+, \tilde{n}_-, \tilde{n}_+, r)$ that are nevertheless not equivalent. Earlier examples of models with this property involved $O^-p$ and $O^+p$ planes only [2, 3, 16] (see our table 1), but we now see that this phenomenon also occurs in the
mixed cases. We now also need the information provided by the coefficients of $B\{p_i\}$ and $C\{p_i\}$ as these specify the geometry of all planes.

In this case we always have $r = 0$ (we do not know whether there is a deeper reason that such models should have $r = 0$, it follows from inspection of the tables). For some configurations whose $SL(2,\mathbb{Z}_2)$ orbit cannot be determined by the set of numbers $(n_-, n_+, \tilde{n}_-, \tilde{n}_+, r)$, we have attached an index to the number $r = 0$ in tables 3 and 4. Theories with the same index belong to the same $SL(2,\mathbb{Z}_2)$ orbit. The theories in table 4 with $r = 0$, but not labelled by an index are S-dual to theories in tables 1 and 2. The reader should have no difficulty identifying these models and their duals.

Rather interestingly, our whole formalism is $SL(2,\mathbb{Z})$ dual, regardless whether in the end one finds a supersymmetric theory or not, and therefore also assigns dual theories to non-supersymmetric theories. It would be interesting to see to what extent physics in these non-supersymmetric “dual” theories is related to each other, the more because after the anti-brane annihilation process that turns $\tilde{\Sigma}p^-$ into a bound state of $Op^-$ with an anti $Dp$-brane, the manifest duality between various descriptions will have dissipated.

7. Conclusions

By computing the holonomies of gerbes defined by the discrete charges of the orientifold planes, the problem of classification of orientifold planes on $T^k/\mathbb{Z}_2$ can be reduced to rather simple polynomial equations over the field $\mathbb{Z}_2$. Moreover, these polynomials represent a very compact way of storing information about the orientifold model, as many relevant quantities can be deduced from them. The techniques we used have their roots in [3], where they were applied to K3’s of the form $T^4/G$ ($G$ a finite group), but are useful in the present problem as well. In principle they may be relevant to any situation involving orientifold and/or orbifold fixed point sets with more than one connected component, in the presence of discrete torsion and similar degrees of freedom.

Preservation of supersymmetry reduces to a simple inequality. With these powerful techniques the classification of supersymmetric orientifolds was carried out for all $k \leq 6$. The results can be found in the tables 1, 2, 3 and 4.

All our theories are dual to toroidal compactifications of either type I theory, or the type IIB theory on $S^1/\delta\Omega$, with suitable holonomies on the torus. This has not played a large role in our analysis in this paper. Instead we refer the reader to [3, 14, 16, 18, 22] for various ideas and results.

For the benefit of the reader, we collect some information from our 4 tables in another table summarizing our results.

Table 5 lists the number of maximally supersymmetric orientifold models in various dimensions, where $d$ is the number of spatial dimensions. It should be stressed that this is not a classification of irreducible components in the string moduli space. On the one hand, there are many components in the string moduli space that do not have
an orientifold description. On the other hand, it is known that a single irreducible component in the string moduli space can give rise to multiple orientifold descriptions [3].

One of the things that immediately draws attention is the explosive growth of number of theories for $d < 6$. This can be viewed as the analogue of the existence of triples, and other non-standard compactifications in gauge theory, where for sufficiently high compact dimension there are many irreducible components in the moduli space of flat connections [4, 12, 13]. Another typical fact is that theories with $r = 4$ and $r = 2$ appear simultaneously, at $d = 5$, but that then subsequently there are no new values for $r$ encountered in the rest of the table. Furthermore, except for $r = 0$, the numbers $r$ are even powers of 2. Comparing with for example [4, 14], one sees that this does not follow from the possible bundles on $T^k$ (as there exist flat $Spin(32)/\mathbb{Z}_2$ bundles with rank 12 and rank 5, for example), and not even from supersymmetry (the previously mentioned bundles give perfectly valid vacua for supersymmetric gauge theories on tori), but from constraints that are intrinsic to string theory [3]. It would be nice to understand the numerology better\(^6\).

We also discussed the implications of S-duality of 4-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories. Among the orientifold configurations we found, there are two singlets under the $SL(2,\mathbb{Z})$ duality group, one with $r = 16 \ (64,0,0,0,16)$ and one with $r = 4 \ (40,8,8,8,4)$. Especially the latter one may be an interesting theory to study, the behavior under S-duality suggests a highly symmetric spectrum, and with $r = 4$ gauge groups of relatively large rank are possible. Note that it is even possible to occupy at least one of each kind of $Op$-plane with a $Dp$-brane pair. It may also be interesting to compute the lattice for the heterotic version of this theory, along the lines of [3]. The remaining theories organize in orbits of each 3 elements, which is reflected in table 5, because if one omits the singlets, all the numbers in the last line of the table are divisible by 3.

\(^6\)In [2] the rank 5 and rank 12 models are interpreted as string theories without supersymmetry. This involves the discrete cosmological constant of ref. [11]. In spite of the presence of such a cosmological constant the authors of [2] seem to assume a flat background away from the orientifold planes. It is not clear to this author that this is self-consistent.
Although discussed in less detail, it should also be clear that the present techniques provide a rich source of examples of non-supersymmetric theories, similar to the ones analyzed in [19, 21].

To take this classification to still lower dimension, we need new insights concerning the rewriting of (3.15) to some kind of standard form. An estimate on the needed time to perform a computer analysis without any new information, does not give much hope that continuing this classification to $p = 2$ by brute force can be done in the near future.

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