Thick Brane Worlds and Their Stability

Shinpei Kobayashi\textsuperscript{1}, Kazuya Koyama\textsuperscript{2} and Jiro Soda\textsuperscript{3}

\textsuperscript{1,2} Graduate School of Human and Environment Studies, Kyoto University, Kyoto 606-8501, Japan
\textsuperscript{3} Department of Fundamental Sciences, FIHS, Kyoto University, Kyoto, 606-8501, Japan

Abstract

Three types of thick branes, i.e., Poincaré, de Sitter and Anti-de Sitter brane are considered. They are realized as the non-singular solutions of the Einstein equations with the non-trivial dilatons and the potentials. The scalar perturbations of these systems are also investigated. We find that the effective potentials of the master equations of the scalar perturbations are positive definite and consequently these systems are stable under the small perturbations.

\textsuperscript{1}E-mail: shinpei@phys.h.kyoto-u.ac.jp
\textsuperscript{2}E-mail: kazuya@phys.h.kyoto-u.ac.jp
\textsuperscript{3}E-mail: jiro@phys.h.kyoto-u.ac.jp
1 Introduction

As the most promising candidate for the unified theory of everything, the superstring theory has been investigated for a long time. The recent discovery of D-branes stimulated a rather old idea, “brane world” [1],[2]. In particular, the Randall-Sundrum model has been advocated as a simple model [3], [4].

In the RS model, the five-dimensional Anti-de Sitter spacetime \((AdS_5)\) and the 3-brane (the four-dimensional Minkowski spacetime) embedded in \(AdS_5\) appear. It is found that there are massless graviton called 0-mode and massive graviton called Kaluza-Klein modes in this system. The massless graviton reproduce the Newtonian gravity on the 3-brane and Kaluza-Klein modes, which are the effect of the existence of the higher-dimension, give correction to the Newtonian gravity [5],[6]. This model succeeds to reproduce the Newtonian gravity in the low energy limit. Stimulated by this success, the idea of our universe is a 3-brane embedded in higher-dimensional spacetime came to be investigated eagerly. Furthermore, the cosmological consequences of the RS model have been investigated and no contradiction with the observation has not been found until now [7]-[22]. The inflationary scenario also seems to be compatible with the brane models [23]-[29].

The RS model is motivated by the unified field theory such as the superstring theory. Such theories are supposed to represent the high energy era of the history of our universe. In the superstring theory, there seems to exist the minimum scale of the length, so we cannot consider the exact 0-width brane. That is, we cannot neglect the thickness of the brane at the string scale. This fact makes the brane world non-singular.

From these reason, “thick brane world” scenario has been investigated [30]-[35]. In this paper, we especially pay attention to the concrete realization of the thick branes. We consider the three types of maximally symmetric branes, that is, Poincaré, de Sitter and Anti-de Sitter branes. As these branes are highly-symmetric, they always exist as the solutions of the Einstein equations if we introduce non-trivial dilatons and suitable scalar potentials. The
existence of non-singular de Sitter and Anti-de Sitter brane is pointed out in this paper for the first time.

And we analyze the stability of these systems through the scalar perturbations of them. We write down the master equations of the scalar perturbations and find the effective potential for the Poincaré, the de Sitter and the Anti-de Sitter brane case, respectively. As a result, due to the positivity of the effective potentials, we find that all of these systems are stable under the scalar perturbations.

At last, we refer to the relations between the thickness and the non-commutativity. As mentioned above, the thickness is needed due to the existence of the minimum length. And the non-commutativity also arises from the existence of the minimum length. So we can naively say that there are some relation between them.

The organization of this paper is as follows: in Sec.2, we see the set-up of the thin brane models at first and we construct three types of maximally symmetric branes. In order to explain the effect of the thickness, we consider the behavior of the graviton in the three types of the thick brane backgrounds. In Sec.3 we analyze the scalar perturbations of the maximally symmetric thick brane systems. We derive the effective potentials and examine the behavior of them. Sec.4 is devoted to the conclusion and the discussion. The relation between the thickness and the non-commutativity is also discussed there.

2 Thick Brane Models

2.1 Thin Brane Models

At first, we see the set-up of the thin brane models. The action of the thin brane model is

$$S = \int d^5x\sqrt{-g_5}\left(\frac{1}{2}R - \Lambda_5\right) - \sigma \int d^4x\sqrt{-g},$$

where we used the unit $8\pi G_5 = 1$ ($G_5$ is the five-dimensional gravitational constant), $g_5$ is the five-dimensional metric and $R$ denotes the five-dimensional Ricci scalar. If we consider
the five-dimensional Anti-de Sitter spacetime, the five-dimensional cosmological constant $\Lambda_5$ is related to the AdS radius $l$ as

$$\Lambda_5 = -\frac{6}{l^2}. \quad (2)$$

The second term of the action is the action of the brane and $\sigma$ denotes the tension of the brane.

Now we consider the following type of the metric,

$$ds^2 = dy^2 + e^{2\alpha(y)} \gamma_{\mu\nu} dx^\mu dx^\nu, \quad (3)$$

where $y$ denotes the direction of the bulk, $\gamma_{\mu\nu}$ is the metric on the brane, $\mu, \nu$ run the indices of the four-dimension of the brane and $e^{2\alpha(y)}$ is the so-called “warp factor”. The warp factor decides the slicing of the five-dimensional spacetime.

In this situation, we can get three types of branes where the geometry of the branes are maximally symmetric, i.e., Poincaré, de Sitter and Anti-de Sitter branes. Solving the Einstein equations, we find

$$\alpha(y) = \begin{cases} 
  y_0 - |y|, & \text{Poincaré brane}, \\
  \log \left[ \sinh(y_0 - |y|) \right], & \text{de Sitter brane}, \\
  \log \left[ \cosh(y_0 - |y|) \right], & \text{Anti de Sitter brane},
\end{cases} \quad (4)$$

where $y_0$ is constant. In the Poincaré brane case, $y_0$ can be set to 0 without loss of the generality, so $y_0$ does not have the physical meaning. In the de Sitter brane case, $y_0$ decides the range of the bulk. In fact, the warp factor becomes 0 at $y = y_0$, so the range of $y$ is $-y_0 \leq y \leq y_0$ and $y = y_0$ is the horizon of the $AdS_5$. On the contrary, in the Anti-de Sitter brane case $y_0$ denotes only the turning point of the warp factor because the metric does not become 0 at $y = y_0$. So there is no horizon in $AdS_5$ with the Anti-de Sitter slicing. Now, we consider the extension of the thin brane systems to the thick brane ones in the following subsections.
2.2 Construction of the Thick Brane Models

In order to realize the thick brane model, we consider the following action,

\[ S = \int d^5x \sqrt{-g_5} \left[ \frac{1}{2} R - \frac{1}{2} (\partial \varphi)^2 - V(\varphi) \right]. \tag{5} \]

Here \( \varphi \) is the five-dimensional scalar field which depends only on the coordinate of the bulk and \( V(\varphi) \) is its potential. We use the metric,

\[ ds^2 = a^2(z) \left( dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu \right), \tag{6} \]

where conformal-like coordinate \( z \) is defined through the following equation,

\[ z \equiv \int \frac{dy}{a}, \quad a(z) = e^{\alpha(y(z))}. \tag{7} \]

Note that \( \gamma_{\mu\nu} \) denotes the metric of the maximally symmetric four-dimensional spacetimes from now on, so we can write the four-dimensional Ricci tensor and Ricci scalar as follows,

\[ R^{(4)}_{\mu\nu} = 3K \gamma_{\mu\nu}, \tag{8} \]
\[ R^{(4)} = 12K, \tag{9} \]

where \( K \) takes 0, 1 or \(-1\) and these values correspond Poincaré, de Sitter, Anti-de Sitter brane, respectively\(^4\).

Now we can write down the Einstein equations and the equation of motion of the scalar field (matter),

\[ (z,z) : \quad 6\mathcal{H}^2 - 6K = \frac{1}{2} (\varphi')^2 - a^2 V(\varphi), \tag{10} \]
\[ (\mu,\nu) : \quad 3\mathcal{H}' + 3\mathcal{H}^2 - 3K = -\frac{1}{2} (\varphi')^2 - a^2 V(\varphi), \tag{11} \]
\[ \text{matter} : \quad \varphi'' + 3\mathcal{H} \varphi' = a^2 \frac{\partial V}{\partial \varphi}. \tag{12} \]

Here a prime denotes the derivative with respect to \( z \) and we define \( \mathcal{H} \) as follows,

\[ \mathcal{H} \equiv \frac{a'}{a}. \tag{13} \]

\(^4\)From now on, we also set AdS radius \( l = 1 \) and we normalize the curvature radius of the four-dimensional spacetime to the unit.
Figure 1: The warp factor of the thin brane case and that of the thick brane case are shown. The solid line denotes the warp factor of the thin brane case and the dashed line denotes that of the thick brane case.

From eqs.(10) and (11), we get the following equations,

\[ (\varphi')^2 = -3\mathcal{H}' + 3\mathcal{H}^2 - 3K, \]  
\[ V(\varphi) = \frac{1}{2a^2} (3\mathcal{H}' + 9\mathcal{H}^2 - 9K). \]  

From eqs.(14) and (15), one can construct the thick brane model starting from the given warp factor \( a(z) \), as long as \( a(z) \) satisfies the condition,

\[ (\varphi')^2 \geq 0. \]  

Here the equation of motion of the scalar field (12) is satisfied automatically due to the Bianchi identity. Eq.(15) simply determines the functional form of the potential.

### 2.3 Thick Poincaré Brane Model

In this paper, we use the following warp factor to make the thick Poincaré brane,\(^5\).

\[ a^2(z) = e^{2\alpha(y(z))} = \left( \frac{1}{e^{2\kappa y(y_0+y)} + e^{-2\kappa y(y_0-y)}} \right)^{1/n}. \]  

When we take the limit \( n \to \infty \), this warp factor approaches to \( e^{2(y_0-|y|)} \). \( n \) can be said to decide the “thickness” of the brane.

\(^5\)There may be many ways to introduce the “thickness”. Here we examine one of the possibilities. The parameter of the thickness which is related to the non-commutative geometry is discussed in Sec.4.
We can see that the spacetime becomes smooth at the location of the brane from Fig.1, so there is no jump of the value of the extrinsic curvature between $y < 0$ and $y > 0$.

In this case, we can find $\varphi(y)$ and $V(\varphi)$ analytically with an arbitrary $n$. In fact, substituting the warp factor (17) into eqs. (14) and (15), we get

$$\varphi(y) = \pm \sqrt{\frac{6}{n}} \arctan(e^{2ny}),$$

(18)

$$V(\varphi) = -6 + 3(n + 2)\sin^2\left(\frac{\sqrt{6n}}{3}\varphi\right).$$

(19)

So we can see explicitly that the condition (16) is satisfied, and that (17) can be realized.

Now, in order to see the effect of the “thickness”, let us consider the gravitational perturbation $h_{\mu\nu}$ in this background. Here the graviton $h_{\mu\nu}$ is the tensor perturbation of the metric, so it can be written as

$$ds^2 = a^2(z) \left[ dz^2 + (\gamma_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \right].$$

(20)

Here $h_{\mu\nu}$ satisfies the transverse-traceless condition,

$$h_{\mu^\nu} = h_{\mu^\nu|\nu} = 0,$$

(21)

where the vertical bar denotes the covariant derivative with respect to $\gamma_{\mu\nu}$. In this system, the equation of $h_{\mu\nu}$ becomes

$$h''_{\mu\nu} + 3H h'_{\mu\nu} + h_{\mu\nu|\lambda}^\lambda - 2K h_{\mu\nu} = 0.$$  

(22)

Here we can define the four-dimensional momentum $p$ as follows,

$$h_{\mu\nu|\lambda}^\lambda - 2K h_{\mu\nu} = p^2 h_{\mu\nu}. $$

(23)

Furthermore, we find that we can decompose $h_{\mu\nu}$ using the polarization tensor $\varepsilon_{\mu\nu}$ which depends only on the four-dimensional coordinates $x^\rho$ as follows,

$$h_{\mu\nu}(z, x^\rho) = \varepsilon_{\mu\nu}(x^\rho) X(z),$$

(24)
Figure 2: The effective potential of the graviton propagating in the thick Poincaré brane system. Here we set $y_0 = 1$. A solid line is $n = 1$ case and a dashed line is $n = 2$ case.

where $\varepsilon_{\mu\nu}$ also satisfies the satisfies the transverse-traceless condition,

$$\varepsilon^\mu_\mu = \varepsilon^{\mu\nu} = 0.$$ (25)

So regardless of the value of $K$, we can rewrite eq.(22) as,

$$X'' + 3\mathcal{H}X' = -p^2 X.$$ (26)

To see how $h_{\mu\nu}$ behaves, we transform this equation into the Schrödinger-type equation and write down the effective potential.

Now we introduce a new function $\chi(z)$ which satisfies the following equation,

$$X(z) = a(z)^{-3/2}\chi(z).$$ (27)

Substituting eq.(27) into eq.(26), we find that the equation of $\chi(z)$ becomes

$$-\chi''(z) + V(z) \cdot \chi(z) = -p^2 \chi(z),$$ (28)

$$V(z) = \frac{3}{2}\mathcal{H}' + \frac{9}{4}\mathcal{H}^2,$$ (29)

where $V(z)$ is the effective potential of this system. Furthermore, substituting the warp factor (17) into eq.(29), we get

$$V(z) = \frac{3e^{2y_0}[5e^{4ny} - (16n + 10) + 5e^{-4ny}]}{4(e^{2ny} + e^{-2ny})^{2+\frac{1}{n}}}.$$ (30)

Figure 3: The effective potential of the graviton in the thick de Sitter brane system. A solid line denotes \( n = 1 \) case and a dashed line denotes \( n = 2 \) case, respectively. Both approach to \( \frac{3}{4} H^2 \) as \( y \to y_0 \). Here we set \( y_0 = 2 \).

We show the effective potentials in Fig.2. The effective potential shows that the graviton can be localized around \( y = 0 \) (the location of the brane). The width of the brane becomes wider as \( n \) approaches to 0. On the other hand, when \( n \) approaches to \( \infty \), the effective potential approaches the “volcano” potential of the RS model. So we can say that this model with the sufficiently large \( n \) is able to reproduce the four-dimensional gravity (i.e., Einstein gravity) in the low energy limit.

### 2.4 Thick De Sitter Brane Model

Similarly, we can construct the warp factor of the thick de Sitter brane as follows,

\[
a^2(z) = \left( \frac{1}{\sinh^{-2n}(y_0 + y) + \sinh^{-2n}(y_0 - y)} \right)^{1/n}.
\]  

(31)

We cannot calculate \( \varphi(y) \) analytically with an arbitrary \( n \), so we cannot find the explicit form of \( V(\varphi) \), so when we consider the behavior of the graviton or the free test scalar field, we have to calculate it numerically.

Let us consider the behavior of the graviton in the thick de Sitter brane system as well as the previous subsection. Substituting the warp factor (31) into eq.(30), we get the effective potential of the graviton propagating in the thick de Sitter brane system as shown in Fig.3.
The squares of the warp factors of the thin and thick AdS brane. A solid line denotes the warp factor of the thin AdS brane and a dashed line denotes that of the thick AdS brane. Here we set $y_0 = 1$.

The shape of the effective potential of the thick de Sitter brane case is very similar to that of the Poincaré brane case. There is a hole around the location of the brane, so the 0-mode of the graviton can be localized around the brane. Here, we have to note that the asymptotic value of the effective potential of this case is different from that of the Poincaré brane case. The value of the effective potential of the graviton in the thick de Sitter brane system get to $\frac{9}{4}H^2$ (where $H$ is the Hubble constant), but on the other hand, that of the Poincaré brane case approaches to 0 as $y \to \infty$. This phenomenon can be seen in the thin brane analysis as well [23],[26].

2.5 Thick Anti-De Sitter Brane Model

And we can construct the warp factor of the thick Anti-de Sitter brane as follows,

$$a^2(z) = \left[\left(1 + \frac{1}{\sinh^{-2n}(y_0 + y) + \sinh^{-2n}(y_0 - y)}\right)^{1/2n} - \epsilon\right]^2,$$

where $\epsilon$ is a constant which satisfies $0 < \epsilon < 1$. $\epsilon$ is needed in order that the spacetime becomes exact $AdS_5$ only at infinity (i.e., $y = \pm \infty$). Without $\epsilon$, we find that the spacetime also becomes exact $AdS_5$ at $y = \pm y_0$. This will cause technical trouble in the analysis of the scalar perturbation discussed in Sec.3. We have to calculate the fluctuations numerically in
Figure 5: The effective potential of the graviton in the thick Anti-de Sitter brane system. A solid line denotes \( n = 1 \) case and a dashed line denotes \( n = 2 \) case, respectively. Here we set \( \epsilon = 0.4 \) and \( y_0 = 2 \).

this case as well. At the end of this subsection, we show the warp factor of the thick Anti-de Sitter brane in Fig.4 and the effective potential of the graviton propagating in this system in Fig.5. Both the warp factor and the effective potential of the graviton diverge as \( y \to \infty \), which is different from the results of the Poincaré and the de Sitter brane.

3 Stability Analysis

3.1 Perturbation of Scalar-Gravity Coupled Systems

Here we investigate the scalar perturbation of systems having thick Poincaré, de Sitter and Anti-de Sitter brane.

In order to examine them, we set the perturbed metric,

\[
\begin{align*}
    ds^2 &= (g_{MN} + \delta g_{MN})dx^M dx^N \\
    &= a^2(z) [(1 + 2\phi)dz^2 - 2B_{\mu\nu}dx^\mu dx^\nu + ((1 + 2\psi)\gamma_{\mu\nu} - E_{\mu\nu}) dx^\mu dx^\nu].
\end{align*}
\] (33)

In this paper we use the longitudinal gauge \( B = E = 0 \) and we get the following metric,

\[
    ds^2 = a^2(z) \left[(1 + 2\phi)dz^2 + (1 + 2\psi)\gamma_{\mu\nu}dx^\mu dx^\nu\right].
\] (34)
We get equations of the background and those of the perturbation. The equations of the perturbation are

$$(z, z) : 3 \gamma^{\rho \lambda} \psi_{|\rho \lambda} + 12 \mathcal{H} \psi' - 12 \mathcal{H}^2 \phi + 12 K \psi = \varphi_0' \delta \varphi' - \phi(\varphi_0')^2 - a^2 \frac{\partial V}{\partial \varphi_0} \delta \varphi,$$

$$(z, \mu) : -3 \psi'_{|\mu} + 3 \mathcal{H} \phi_{|\mu} = \varphi_0' \delta \varphi_{|\mu},$$

$$(\mu, \nu) : \left(3 \psi'' - 6 \mathcal{H}' \phi - 3 \mathcal{H} \phi' + 9 \mathcal{H} \psi' - 6 \mathcal{H}^2 \phi + \gamma^{\rho \lambda} \phi_{|\rho \lambda} + 2 \gamma^{\rho \lambda} \psi_{|\rho \lambda} + 6 K \psi \right) \delta_{|\nu}$$

$$-\gamma^\mu_{|\mu} \phi_{|\nu} - 2 \gamma^\mu_{|\mu} \psi_{|\nu} = \left(-\varphi_0'' \delta \varphi'' - \phi(\varphi_0')^2 - a^2 \frac{\partial V}{\partial \varphi_0} \delta \varphi \right) \delta_{|\nu},$$

mater : $\delta \varphi'' + 3 \mathcal{H} \delta \varphi' + \left(4 \psi' - \phi' - 6 \mathcal{H} \phi \right) \varphi_0' - 2 \phi \varphi'' + \gamma^{\rho \lambda} \delta \varphi_{|\rho \lambda} = a^2 \frac{\partial^2 V}{\partial \varphi_0^2} \delta \varphi,$

where the vertical bar denotes the covariant derivative with respect to $\gamma_{\mu \nu}$. $\varphi_0$ is the quantity of the background and $\delta \varphi$ is the perturbation of the scalar field. Now we derive the master equation of this system. At first, from eq.(36), we get

$$\delta \varphi = \frac{1}{\varphi_0} (-3 \psi' + 3 \mathcal{H} \phi),$$

and from the off-diagonal part of eq.(37), we get

$$\phi + 2 \psi = 0.$$ (40)

Substituting eqs.(38), (39) and (40) into eq.(35)+(37), we find the master equation of the system,

$$\psi'' + \gamma^{\rho \lambda} \psi_{|\rho \lambda} + \left(3 \mathcal{H} - 2 \frac{\varphi''_0}{\varphi_0} \right) \psi' + \left(4 \mathcal{H}' - 4 \mathcal{H} \frac{\varphi''_0}{\varphi_0} + 6 K \right) \psi = 0.$$ (41)

We transform this equation into the form of Schrödinger equation and get the effective potential to examine the stability of this system. To do so, we define

$$\psi(z, x^\mu) = A(z) F(z, x^\mu),$$

(42)

Here $A(z)$ is

$$A(z) = \frac{\varphi_0'(z)}{a(z)^{3/2}},$$

(43)
We get the Schrödinger-type equation of the system,

\[-F''(z, x) + V_e(z) \cdot F(z, x) = \gamma^\nu F(z, x)_{\nu\lambda},\]  

where \(V_e\) is the effective potential of the system and its concrete form is

\[V_e = -\frac{5}{2} \mathcal{H}' + \frac{9}{4} \mathcal{H}^2 + \mathcal{H} \frac{\varphi''}{\varphi} - \frac{\varphi''}{\varphi} + 2 \left( \frac{\varphi''}{\varphi} \right)^2 - 6K.\]  

We analyze the effective potentials of three types of branes, separately.

### 3.2 Thick Poincaré Brane Case

In the thick Poincaré brane case, we can expand \(F(z, x)\) as follows,

\[F(z, x) = \int \frac{d^4p}{\sqrt{2\pi}^4} f(z)e^{ip\cdot x},\]  

so we find that the equation for \(f_p(z)\) becomes

\[-f_p''(z) + V_e(z) \cdot f_p(z) = m^2 f_p(z),\]  

where \(m\) is the four-dimensional mass which satisfies \(m^2 = -p^2\). In four-dimensional flat spacetime, \(p\) can be expressed as \(p^2 = -\omega^2 + k^2\). \(\omega\) is the eigenvalue of the time-direction and \(k\) is the norm of the eigenvalue of the three-dimensional space. If there is only time-dependence in this system (i.e., \(k = 0\)), \(\omega = m\). So if there is a solution with an imaginary \(m\), we can say that this system is unstable.

Next we see the effective potential \(V_e\). In this system, \(V_e\) becomes

\[V_e = \frac{e^{2y_0} [(16n^2 + 16n + 3)e^{4ny} + 32n^2 + 80n - 6 + (16n^2 + 16n + 3)e^{-4ny}]}{4(e^{2ny} + e^{-2ny})^2 + 4}.\]  

Clearly \(V_e\) is positive definite and it approaches to 0 as \(y \to \pm \infty.\) The solution which has an imaginary \(m\) cannot exist because it will necessarily diverge either at \(y = \infty\) or at \(y = -\infty\). So we can conclude that the thick Poincaré brane is stable under the scalar

\(^6\)Note that we can discuss using \(y\) instead of \(z\) because \(z\) is monotonic function of \(y\) from eq.(7).
perturbation. It is interesting that there is no bound state on the brane, which is different from the tensor perturbation (See Fig.2.). The thickness parameter $n$ decides the height of the effective potential. The effective potential becomes higher as $n$ becomes larger. In the thin brane limit (i.e., $n \to \infty$), the height of the effective potential get to infinity, and the thin brane becomes a singular object which has no width.

3.3 Thick De Sitter Brane Case

In the thick de Sitter brane case, we expand $F(z, x^\mu)$ as follows,

$$F(z, x^\mu) = \int d^3 k dm f_m(z) g_{km}(t) e^{ikx}. \quad (49)$$

Here we introduce the four-dimensional mass $m$ through the equation of $g_{km}(t)$ as follows,

$$\ddot{g}_{km}(t) + 3H \dot{g}_{km}(t) + \left(k^2 e^{-2t} + m^2\right) g_{km}(t) = 0, \quad (50)$$

where a dot denotes the derivative with respect to $t$ and $H$ is the Hubble constant. Then solving eq.(50), we find that $g_{km}$ becomes

$$g_{km}(\eta) = \frac{\sqrt{\pi}}{2} H(-\eta)^{3/2} e^{-\pi \beta/2} H^{(1)}_{i\beta}(-k\eta), \quad (51)$$
where \( H^{(1)} \) is the first Hankel function\(^7\). Here, \( \eta \) is conformal time defined as

\[
\eta \equiv -e^{-t},
\]

and \( \beta \) is defined as follows,

\[
\beta \equiv \sqrt{m^2 - \frac{9}{4}}.
\]

To see the time evolution of the scalar perturbation, we examine \( g_{km}(\eta) \) with various \( m \). At first, we expand \( g_{km}(\eta) \) near \( \eta \sim 0 \) (i.e., \( t \to \infty \)). \( g_{km}(\eta) \) becomes

\[
g_{km}(\eta) = \frac{\sqrt{\pi}}{2} e^{-\pi \beta/2} \left[ \frac{k^{i \beta}}{2^{i \beta} \Gamma(1 + i \beta)} \left( 1 + i \frac{\cos(i \beta \pi)}{\sin(i \beta \pi)} \right) (-\eta)^{\frac{1}{2} + i \beta} - \frac{i k^{-i \beta}}{2^{-i \beta} \sin(i \beta \pi) \Gamma(1 - i \beta)} (-\eta)^{\frac{1}{2} - i \beta} \right].
\]

In the region of \( m^2 \geq \frac{9}{4} \), \( \beta \) is a real number, so \( g_{km} \) oscillate near \( \eta \sim 0 \) and we can say that \( g_{km} \) is stable. Next we consider the region of \( 0 < m^2 \leq \frac{9}{4} \). In this region, \( \beta \) becomes an imaginary number. Now we define a new variable \( \zeta \) as

\[
\beta = \sqrt{m^2 - \frac{9}{4}} \equiv i \zeta.
\]

The first term in the bracket of eq.(54) behaves as \((-\eta)^{\frac{1}{2} + \zeta}\) and the second term behaves as \((-\eta)^{\frac{1}{2} - \zeta}\). So both terms converge to 0 as \( \eta \) approaches to 0 because \( \zeta \) is \( 0 \leq \zeta < \frac{3}{2} \). From above discussion, we can conclude that the solution whose \( m^2 \) is \( 0 < m^2 \leq \frac{9}{4} \) is also stable.

At last, we treat the region of \( m^2 \leq 0 \). In this region, \( \zeta \) is larger than \( \frac{3}{2} \), so the first term of eq.(54) does not converge when \( \eta \) gets to 0. It means that if there is the solution with \( m^2 \leq 0 \), this system must be unstable.

Then we examine the effective potential of the thick de Sitter brane. We calculate it numerically and the result is shown in Fig.7. In this case, \( y = \pm y_0 \) is the horizon of \( AdS_5 \), so the spacetimes is defined in the region \(-y_0 \leq y \leq y_0\). Fig.7 shows that the effective potential of the scalar perturbation is positive at \(-y_0 \leq y \leq y_0\) and we can conclude that there is no

\(^7\)We set \( H = 1 \) from now on.
Figure 7: The effective potential of the scalar perturbation with the thick de Sitter brane. Here we set $n = 1, y_0 = 1$.

solution with $m^2 \leq 0$ because it cannot be normalizable. So unstable solutions do not exist, as a result, this system is stable under the scalar perturbation the same as the thick Poincaré brane case.

Notice that the shape of the effective potential is very different from that of the graviton or that of the free test scalar field case. The effective potentials of the graviton or the free scalar field have the hole in their potentials at the location of the branes. On the contrary, the effective potential of the scalar perturbation has no hole at the location of the brane.

Furthermore, the mass gap has disappeared in the effective potential of the scalar perturbation. There exists the mass gap in the effective potentials of the graviton or the free test scalar fields, that is, the effective potentials approach to $\frac{9}{4}H^2$ not to 0 as $y \to \pm y_0$ (Here we write the Hubble constant $H$ explicitly) as shown in Fig.3. The disappearance of the mass gap is peculiar to the scalar perturbation of the thick de Sitter brane.
3.4 Thick Anti-De Sitter Brane Case

Let us consider the thick AdS brane case as well as the thick de Sitter brane case. In the AdS brane case, we decompose $F(z, x^\mu)$ in eq. (44) as

$$F(z, x^\mu) = \int dm f_m(z)g_m(x^\mu),$$

(56)

and we get the following equation for $g_m(x^\mu)$,

$$\gamma^{\rho\lambda}g_m(x^\mu)|_{\rho\lambda} = m^2 g_m(x^\mu),$$

(57)

where $m$ is the four-dimensional mass. It is known that (57) can be solved with suitable harmonic functions and there is Breitenlohner-Freedman bound which allows the tachyonic mass to some extent from the condition of the normalization [37]-[40]. From Breitenlohner-Freedman bound, the mass $m$ is bounded as

$$m^2 \geq -\frac{9}{4}.$$  

(58)

It means that if there is the solution having $m^2$ larger than $-\frac{9}{4}$, that solution is stable in spite of having the tachyonic mass. On the contrary, if there is the solution having $m^2$ below $-\frac{9}{4}$, this system comes to have an unstable solution.
Then we examine the effective potential of the thick AdS brane case. Substituting the warp factor (32) into eq.(45) and setting $K = -1$, we can get $V_e$. We show the numerical result of $V_e$ in Fig.8.

We see that the effective potential is positive definite also in this case. As the effective potential is positive, there is no solution having $m^2$ below $-\frac{2}{3}$, so we can conclude that the thick AdS brane is stable for the scalar perturbation as well as the thick Poincaré and the thick de Sitter brane cases.

Next we pay attention to the shape of the effective potential. The upheaval around $y = 0$ (around the thick brane) is the same as that can be seen in the Poincaré and the de Sitter brane case. This represents the repulsive behavior of the thick brane. But at $y = \pm \infty$, the behavior of the effective potential of the thick AdS brane case is very different from the Poincaré brane case or the de Sitter brane case. The effective potential of the AdS case diverges as $y \to \pm \infty$. This is because the warp factor of the thick AdS brane diverges as $y \to \pm \infty$. On the other hand, the warp factor of the thick Poincaré brane converges to 0 as $y \to \pm \infty$ and that of the thick de Sitter brane becomes 0 at $y = \pm y_0$.

Note that we treat the thick brane metrics which coincide with exact $AdS_5$ asymptotically. Though we may be able to consider the metric which coincides exact $AdS_5$ at finite $y$, it makes $\varphi' = 0$ and the effective potential appears to diverge at that point. In fact, if we take $\epsilon = 0$ in (32), we find that $\varphi' = 0$ at $y = \pm y_0$ and the effective potential diverges there. But note that the warp factor itself does not diverge at $y = \pm y_0$. It means that this divergence has a different origin from the divergence at infinity. And this divergence can be thought not to be physical. If we take a suitable variable and rewrite the effective potential with it, we will see no divergence at $y = \pm y_0$. 

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4 Conclusion and Discussion

In this paper we proposed the three types of the thick brane models and analyzed the stability of them. We get three types of the thick branes as the solutions of the Einstein equations with non-trivial dilatons and potentials. These solutions are non-singular in the whole spacetime even at the location of the brane.

At first, we considered the graviton in these thick brane systems. The effective potential of the graviton shows that there is a bound state around the brane, so the 0-mode of the graviton is bounded around the brane. Such gravitons make Newtonian gravity on the thick brane and if we take the thin brane limit, we get the same result as the RS model.

Next we analyzed the scalar perturbations of the thick brane systems. We write the master equations of the scalar perturbations explicitly, and we get the effective potentials of the the scalar perturbations in thick Poincaré, de Sitter and Anti-de Sitter brane case, respectively. As a result, we see all of the thick branes are stable under the scalar perturbations. This result comes from the positivity of the effective potentials of the scalar perturbations. But the shape of the effective potentials are very different from that of the graviton. In all thick brane cases, there is no bound state because there is no hole around the location of the brane ($y = 0$) in the effective potentials. On the contrary, there is an upheaval around the location of the brane. So we conclude that the three types of the thick branes behaves repulsively against the scalar perturbations.

Furthermore, the asymptotic behavior of the effective potential of the thick de Sitter brane is also different from that of the graviton or the free test scalar field propagating in the fixed background. The effective potential of the graviton approaches to $\frac{2}{3}H^2$, not to 0, as $y \to \pm y_0$. This mass gap can be seen in various analyses related to de Sitter brane models. But in the scalar perturbation of the thick de Sitter brane system, the mass gap disappeared. That is, the effective potential approaches to 0 as $y \to \pm y_0$. This phenomenon is peculiar to the scalar perturbation including the back reaction.
And the shape of the effective potential of the thick AdS brane system are distinctive. It diverges as \( y \to \pm \infty \), which is caused by the divergence of the warp factor at \( y = \pm \infty \). In the thick Poincaré or de Sitter brane case, the warp factors do not diverge in the whole spacetimes.

Consequently, we have concluded that all of the three branes are stable under the scalar perturbations, but we should note that these analysis have been done classically. From the analogy of the two analyses on the Schwarzschild black holes (i.e., the analysis by Regge-Wheeler and the analysis by Hawking), the thick de Sitter brane might be unstable quantum mechanically. In fact, as the de Sitter spacetime has a temperature, the spacetime may radiate and get to the Poincaré spacetime\[41\]. So we have to treat this system by quantum mechanically in order to get the answer. We leave this issue for the future work.

At last we refer the relation between the thick branes and the non-commutativity. We can make smooth warp factors in various ways. For example, to construct the thick Poincaré brane, we can introduce another “thickness” parameter \( \lambda \theta^2 \),

\[
a^2(z) = e^{2\alpha(y(z))} = \frac{1}{e^{2y} + \lambda \theta^2 e^{-2y}}. \tag{59}
\]

If we set \( \lambda \theta^2 \) to 0, AdS\(_5\) spacetime with the Poincaré slicing is recovered. Here we introduce the new variable \( \xi \), which is defined as

\[
e^{-2\xi} \equiv \sqrt{\lambda \theta^2} e^{-2y}. \tag{60}
\]

Using \( \xi \), we can rewrite the warp factor (59) into the following form,

\[
a^2(\xi) = \frac{1}{\sqrt{\lambda \theta^2}} \cdot \frac{1}{e^{2 \xi} + e^{-2 \xi}}. \tag{61}
\]

So (59) coincides with (17) with \( n = 1 \) and \( e^{2y_0} = 1/\sqrt{\lambda \theta^2} \). For this reason, we can say that (59) is one of the warp factors of the thick Poincaré brane models and that \( \lambda \theta^2 \) is one of the parameters of the thickness.

On the other hand, \( \lambda \theta^2 \) seems to be related to the non-commutative geometry. From the discussion of the AdS/CFT correspondence, it is known that there is the classical so-
olution of the supergravity which corresponds to $\mathcal{N} = 4$ super Yang-Mills theory in the non-commutative spacetime [42],[43]. This classical solution is given by

$$ds^2 = dy^2 + \left( \frac{1}{e^{2y} + \lambda \theta^2 e^{-2y}} \right) \eta_{\mu\nu} dx^\mu dx^\nu + d\Omega_5^2,$$

where $d\Omega_5^2$ is the metric of $S^5$ and clearly (59) coincides with the warp factor of (62).

$\mathcal{N} = 4$ Super Yang-Mills theory in the non-commutative spacetime is realized on the D3-branes with the non-zero expectation value of the B-field. In this context, $\lambda$ is t’Hooft coupling of $\mathcal{N} = 4$ Super Yang-Mills theory in the non-commutative spacetime and $\theta$ is the expectation value of the B-field. So we can call $\lambda \theta^2$ the non-commutative parameter.

From above discussion, we can interpret $\lambda \theta^2$ in two ways, that is, as the thickness parameter and as the non-commutative parameter. So we can expect (59) contains some effects due to the non-commutativity and the thickness of the brane can be considered as the very indication of it.

There is another thing we should mention here. In this paper, we construct the thick de Sitter and the thick Anti-de Sitter brane system. When we consider the analogy to the thick Poincaré brane case, we cannot deny the possibility that the thick de Sitter and the thick Anti-de Sitter brane system also have the corresponding some field theories. If so, the corresponding theories may be the quantum filed theories in the curved spacetime with the non-commutative coordinate. As we have not known the theories in the non-commutative curved spacetime, this topic is very interesting.

Anyway, we want to derive thick brane systems from the ten-dimensional supergravity or from the eleven-dimensional M-theory. We expect that we might be able to interpret the non-commutativity in the context of these theories through this derivation. We leave these themes for future works.
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References


