Fractional Kinetics for Relaxation and Superdiffusion in Magnetic Field

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Abstract

We propose fractional Fokker-Planck equation for the kinetic description of relaxation and superdiffusion processes in constant magnetic and random electric fields. We assume that the random electric field acting on a test charged particle is isotropic and possesses non-Gaussian Levy stable statistics. These assumptions provide us with a straightforward possibility to consider formation of anomalous stationary states and superdiffusion processes, both properties are inherent to strongly non-equilibrium plasmas of solar systems and thermonuclear devices. We solve fractional kinetic equations, study the properties of the solution, and compare analytical results with those of numerical simulation based on the solution of the Langevin equations with the noise source having Levy stable probability density. We found, in particular, that the stationary states are essentially non-Maxwellian ones and, at the diffusion stage of relaxation, the characteristic displacement of a particle grows superdiffusively with time and is inversely proportional to the magnetic field.

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1 Introduction

Anomalous random motions and related transport phenomena are ubiquitous in nature. In these phenomena the laws of normal diffusion (ordinary Brownian motion) are altered, e.g., the mean square no longer increases linearly in time, but instead grows slower (sub-diffusion) or faster (super-diffusion) than the linear function. There are a lot of examples from very different fields of applications, see reviews [1], [2] and references therein. The anomalous random motions often exhibit long-time and/or space dependence as well as multi-scaling, or multifractal, behavior [3], [4]. These circumstances require to go beyond the theory of (relatively simple) random Markovian processes as well as beyond the theory of (mono)fractal, or self-affine, processes. The systems, in which anomalous random motions occur are usually essentially non-linear and, in this sense, the random motions are non-linear ones; this circumstance again greatly complicates the problem of an adequate statistical description.

The two basic anomalous fractal random motions are of particular importance, namely, fractional Brownian motion [5], and the Levy motion, whose theory has begun from the works of P. Levy [6]. The former motion is characterized by long-range time correlations, whereas the latter one is characterized by non-Gaussian statistics; in this case the increments of the process may be independent (Levy stable processes or ordinary Levy motion [7]) or have an infinite span of interdependence (fractional Levy motion) [8], [9], [10].

The theory of Levy stable distributions and stable processes naturally serves as the basis for probabilistic description of the Levy motion, since stable distributions obey Generalized Central Limit Theorem, thus generalizing Gaussian distribution [11]. However, the application of the theory of stable processes is limited because of the infiniteness of the mean square and discontinuity of the sample paths. The finite sample size and boundary effects play a decisive role, thus modifying stable probability laws (“truncated Levy distributions”) [12], and violating the property of self-affinity (“spurious multi-affinity”) [13].

The peculiarity of anomalous random motions is that they cannot be described by the standard Fokker-Planck equation because the basic assumptions, namely, the Markov property and the local homogeneity of the space do not hold in these cases. The use of differential equations with partial fractional derivatives is a perspective way for describing such processes. One of the implementations of such an approach is the use of different forms of fractional kinetic Fokker-Planck equation or the corresponding forms of the Master equation with fractional derivatives. Recently, kinetic equations with fractional partial derivatives have attracted attention as a possible tool for the description of diffusion and relaxation phenomena, see review [2] and references therein. However, fractional calculus is far from being a regular tool in physics community, and the solutions to fractional kinetic equations are known in a very few cases. The development of the theory requires, from one hand, the development of microscopic foundations of fractional kinetics and, from the other hand, the development of powerful regular methods for solutions to fractional equations.

The various processes in space and thermonuclear plasmas could serve as important applications of fractional kinetics. Indeed, many of the current challenges in solar system plasmas as well as in plasmas of thermonuclear devices arise from the fundamentally multiscale and nonlinear nature of plasma fluctuation and wave processes. Anomalous diffusion and plasma heating, particle acceleration and macroscopic trans-
fer processes require to go beyond the "traditional" plasma kinetic theory. Fractional kinetics can be useful for describing such processes, just as it occurs in other fields of applications. Our paper is a step in this direction. We consider the motion of a charged Levy particle in a constant external magnetic field and random electric field obeying non-Gaussian Levy statistics. Our problem is a natural generalization of the classical example of the motion of a charged Brownian particle [14]. We solve the fractional Fokker-Planck equation with fractional velocity derivative, study the relaxation processes in phase and real spaces as well, and estimate fractional moments of energy and coordinate. We also perform numerical modelling based on the numerical solution to the Langevin equations and demonstrate qualitative agreement between analytical and numerical results.

2 Fractional Fokker-Planck equation for charged particle in magnetic field

The history of fractional Fokker-Planck equation (FFPE) for the probability density function (PDF) \( f(\vec{r}, \vec{v}, t) \) in the phase space goes back to the papers by West and Seshadri [15], and by Peseckis [16]. Here we recall briefly the arguments used when deriving FFPE. It is well known that usually the derivation of classical kinetic equations for the Brownian motion is based on the assumption of the finiteness of the second moments of the PDF. This way is not useful here, because, as we shall see, the second moments diverge. Thus, it is expedient to explore the method used by Chandrasekhar [17] for the derivation of the Fokker-Planck equation for the Brownian motion. His method does not require the finiteness of the second moments. In fact, for fractional kinetics, the modification of Chandrasekhar’s method was proposed for the first time in Ref. [16]. Here we proceed mainly along the derivation from Ref. [18]. We consider a test charged particle with the mass \( m \) and the charge \( e \), embedded in constant external magnetic field \( \vec{B} \) and subjected to stochastic electric field \( \vec{E}(t) \). We also assume, as in the classical problem for the charged Brownian particle [14], that the particle is influenced by the linear friction force \(-\nu m \vec{v}\), \( \nu \) is the friction coefficient. For this particle the Langevin equations of motion are

\[
\begin{align*}
\frac{d\vec{r}}{dt} & = \vec{v}, \\
\frac{d\vec{v}}{dt} & = \frac{e}{mc}[\vec{v} \times \vec{B}] - \nu \vec{v} + \frac{e}{m} \vec{E}.
\end{align*}
\]

The statistical properties of the field \( \vec{E}(t) \) are assumed to be as follows.

1. \( \vec{E}(t) \) is homogenous and isotropic.
2. \( \vec{E}(t) \) is a stationary white Levy noise.

The first assumption is the usual one when dealing with the motion of charged particle in a random electric field. In subsequent Sections we consider two possibilities:

(i) \( \vec{E}(t) \) is a 2-dimensional \((d = 2)\) isotropic field in the direction perpendicular to external magnetic field; in this case the motion along the magnetic field is neglected, and

(ii) \( \vec{E}(t) \) is a 3-dimensional \((d = 3)\) field.

The second assumption has a profound meaning.

Indeed, if \( \vec{E}(t) \) is a white Gaussian noise, then we encounter with a classical Brownian problem and, by using Eqs. (2.1), we arrive at the Fokker-Planck equation and get,
as the consequences, Maxwell stationary PDF over the velocity, exponential relaxation to the Maxwell PDF and the normal diffusion law for the particle motion in the real space, as well. Instead, the non-Gaussian Levy statistics of the random force in the Langevin equation (2.1) provides us with a simple and straightforward, at least, from the methodical viewpoint, possibility to consider abnormal diffusion and non-Maxwell stationary states, both properties are inherent to strongly non-equilibrium plasmas of solar system and thermonuclear devices. Returning to Eq. (2.1), it follows from the assumption 2 that the process, which is an integral of $\vec{E}(t)$ over some time lag $\Delta t$,

$$
\bar{L}(\Delta t) = \int_{t}^{t+\Delta t} dt' \vec{E}(t'), \quad 2.2
$$

is an $\alpha$-stable isotropic process with stationary independent increments [7], [9], whose characteristic function is

$$
\hat{\omega}_L(\vec{k},\Delta t) = \exp \left( -D_E \left| \vec{k} \right|^\alpha \Delta t \right), \quad 2.3
$$

where $\alpha$ is called the Levy index, $0 < \alpha \leq 2$, and the positive parameter $D_E$ has the physical meaning of the intensity of the random electric field. If $\alpha = 2$, Eq. (2.3) is a characteristic function of the Wiener process, and, after applying Chandrasekhar’s procedure we arrive at the Fokker-Planck equation. But if $\alpha < 2$ then, by applying the procedure described in detail for the one-dimensional case in Ref. [18], we arrive at the fractional Fokker-Planck equation for the charged particle in the magnetic field and random electric field:

$$
\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{r}} + \Omega [\vec{v} \times \vec{e}_z] \frac{\partial f}{\partial \vec{v}} = \nu \frac{\partial}{\partial \vec{v}} (\vec{v} f) - D (-\Delta_{\vec{v}})^{\alpha/2} f, \quad 2.4
$$

where $\Omega = eB/mc$, $D = e^\alpha D_E/m^\alpha$ and $(-\Delta_{\vec{v}})^{\alpha/2}$ is the fractional Riesz derivative over the velocity. This operator is defined through its Fourier transform as

$$
(-\Delta_{\vec{v}})^{\alpha/2} f (\vec{r},\vec{v},t) \doteq \left| \vec{k} \right|^\alpha \hat{f} (\vec{z},\vec{k},t), \quad 2.5
$$

where $\hat{f}$ is the characteristic function,

$$
\hat{f} (\vec{z},\vec{k},t) = \left\langle \exp \left( i\vec{z}\vec{r} + i\vec{k}\vec{v} \right) \right\rangle, \quad 2.6
$$

the brackets $\left\langle \ldots \right\rangle$ imply statistical averaging.

An explicit representation of the Riesz derivative is realized through hypersingular integral, see the monograph [19] containing detailed presentation of the Riesz differentiation. We also note that at $\alpha = 2$ Eqs. (2.4), (2.5) are reduced to the Fokker-Planck equation of the Brownian motion. In the next Sections we get the solution to Eq. (2.4) and consider physical consequences.

### 3 Solution to fractional Fokker-Planck equation

In this Section we solve Eq. (2.4) with initial condition

$$
f (\vec{r},\vec{v},t = 0) = \delta (\vec{r} - \vec{r}_0) \delta (\vec{v} - \vec{v}_0), \quad 3.1
$$

$$
\text{as the consequences, Maxwell stationary PDF over the velocity, exponential relaxation to the Maxwell PDF and the normal diffusion law for the particle motion in the real space, as well. Instead, the non-Gaussian Levy statistics of the random force in the Langevin equation (2.1) provides us with a simple and straightforward, at least, from the methodical viewpoint, possibility to consider abnormal diffusion and non-Maxwell stationary states, both properties are inherent to strongly non-equilibrium plasmas of solar system and thermonuclear devices. Returning to Eq. (2.1), it follows from the assumption 2 that the process, which is an integral of $\vec{E}(t)$ over some time lag $\Delta t$,}
\begin{equation}
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\begin{equation}
\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{r}} + \Omega [\vec{v} \times \vec{e}_z] \frac{\partial f}{\partial \vec{v}} = \nu \frac{\partial}{\partial \vec{v}} (\vec{v} f) - D (-\Delta_{\vec{v}})^{\alpha/2} f, \quad 2.4
\end{equation}
where $\Omega = eB/mc$, $D = e^\alpha D_E/m^\alpha$ and $(-\Delta_{\vec{v}})^{\alpha/2}$ is the fractional Riesz derivative over the velocity. This operator is defined through its Fourier transform as
\begin{equation}
(-\Delta_{\vec{v}})^{\alpha/2} f (\vec{r},\vec{v},t) \doteq \left| \vec{k} \right|^\alpha \hat{f} (\vec{z},\vec{k},t), \quad 2.5
\end{equation}
where $\hat{f}$ is the characteristic function,
\begin{equation}
\hat{f} (\vec{z},\vec{k},t) = \left\langle \exp \left( i\vec{z}\vec{r} + i\vec{k}\vec{v} \right) \right\rangle, \quad 2.6
\end{equation}
the brackets $\left\langle \ldots \right\rangle$ imply statistical averaging.

An explicit representation of the Riesz derivative is realized through hypersingular integral, see the monograph [19] containing detailed presentation of the Riesz differentiation. We also note that at $\alpha = 2$ Eqs. (2.4), (2.5) are reduced to the Fokker-Planck equation of the Brownian motion. In the next Sections we get the solution to Eq. (2.4) and consider physical consequences.

### 3 Solution to fractional Fokker-Planck equation

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\begin{equation}
f (\vec{r},\vec{v},t = 0) = \delta (\vec{r} - \vec{r}_0) \delta (\vec{v} - \vec{v}_0), \quad 3.1
\end{equation}
in the infinite phase space. We pass to the characteristic function (2.6), which obeys the equation
\[ \frac{\partial \hat{f}}{\partial t} + \left( -\vec{z} + \Omega \left( \vec{k} \times \vec{b} \right) + \nu \vec{k} \right) \frac{\partial \hat{f}}{\partial \vec{k}} = -D |\vec{k}|^\alpha \hat{f}. \tag{3.2} \]
with the initial condition
\[ f \left( \vec{z}, \vec{k}, t = 0 \right) = \exp \left( i \vec{z} \vec{r}_0 + i \vec{k} \vec{v}_0 \right). \tag{3.3} \]
Equation (3.2) can be solved by the method of characteristics. The equations of characteristics are
\[ \frac{d\vec{k}}{dt} = \nu \vec{k} + \Omega \left( \vec{k} \times \vec{b} \right) - \vec{z}, \tag{3.4a} \]
\[ \frac{d\hat{f}}{dt} = -D |\vec{k}|^\alpha \hat{f}. \tag{3.4b} \]
Denote \( \vec{k}'(t') \) the value of \( \vec{k} \) at time instant \( t' \). Then, the solution to Eq. (3.4a) can be found after lengthy, but straightforward, calculations:
\[ \vec{K} = e^{\nu(t-t')} \left\{ \left( \vec{K}' \vec{b} \right) \vec{b} + \vec{b} \times \left( \vec{K}' \times \vec{b} \right) \cos \Omega (t-t') \right. \]
\[ + \left( \vec{K}' \times \vec{b} \right) \sin \Omega (t-t') \left\} + \Omega (t-t') \vec{b} \times \left( \vec{K}' \times \vec{b} \right) \right. \]
\[ \vec{K}' = e^{\nu(t-t')} \left\{ \left( \vec{K} \vec{b} \right) \vec{b} + \vec{b} \times \left( \vec{K} \times \vec{b} \right) \cos \Omega (t-t') \right. \]
\[ + \left( \vec{K} \times \vec{b} \right) \sin \Omega (t-t') \vec{b} \times \left( \vec{K} \times \vec{b} \right) \right. \]
\[ \vec{G} \vec{b} + \nu \left[ \vec{b} \times \left( \vec{G} \vec{b} \right) \right] \right. \]
\[ \Omega = \nu \vec{b} + \nu \vec{b} \times \left( \vec{G} \vec{b} \right) \right. \]
\[ \Omega_1^2 = \Omega^2 + \nu^2. \tag{3.6} \]

In Eq. (3.7) \( \vec{k}'(t') \) is expressed through \( \vec{k} \) with the use of Eqs. (3.5), (3.6). In the next Sections we consider the peculiarities of the relaxation process and of stationary states realized in the framework of this solution.
4 Homogeneous relaxation and stationary states

In this Section we consider homogeneous relaxation, \( \partial / \partial \vec{r} = 0 \) in Eq. (2.4). Obviously, it corresponds to the particular case \( \vec{\kappa} = 0 \) in the equations of Section 3. Setting \( \vec{\kappa} = 0 \) in Eqs. (3.5)-(3.8) we get

\[
\hat{f} \left( \vec{k}, t \right) = \hat{f} \left( \vec{\kappa} = 0, \vec{k}, t \right) = \exp \left\{ i\vec{k}_0 \vec{v}_0 - \frac{D}{\alpha \nu} (1 - e^{-\alpha \nu t}) |\vec{k}|^\alpha \right\}, \tag{4.1}
\]

where

\[
D_{\vec{v}} = \frac{D}{\alpha \nu} (1 - e^{-\alpha \nu t}), \tag{4.2}
\]

\[
\vec{k}_0 = e^{-\nu t} \left\{ (\vec{k} \vec{b}) \vec{b} + \left( \vec{b} \times (\vec{k} \times \vec{b}) \right) \cos \Omega t + (\vec{b} \times \vec{k}) \sin \Omega t \right\}. \tag{4.3}
\]

Let us consider the case \( \vec{v}_0 = 0 \) for the sake of simplicity. The relaxation process, whose characteristic function is given by Eq. (4.1), is not an \( \alpha \)-stable process with independent increments, since \( D_{\vec{v}} \) is not a linear function of time, see Eq. (4.2). The stable process arises asymptotically at small times only,

\[
t << \tau_v = 1/\alpha \nu, \tag{4.4}
\]

when the exponent in Eq. (4.2) can be expanded into power series. On the other hand, after exponential relaxation to the stationary state, that is, at \( t >> \tau_v \) the stochastic process \( \vec{v} (t) \) becomes asymptotically stationary process with the stable PDF independent of \( t, \)

\[
f_{st} \left( \vec{k} \right) = \exp \left( - \frac{D}{\alpha \nu} |\vec{k}|^\alpha \right). \tag{4.5}
\]

We also note that stationary PDF does not depend on the magnetic field.

Another interesting point is related to stationary solutions of fractional kinetic equations. In the theory of Brownian motion equilibrium Maxwell PDF over velocity is reached at \( t >> 1/\nu \). It is characterized by the temperature \( T \) of surrounding medium. The following relation exists between the parameter \( D \) and the friction coefficient \( \nu \) :

\[
D = \frac{\nu k_B T}{m}, \tag{4.6}
\]

where \( k_B \) is the Boltzmann constant. The temperature \( T \) is a measure of a mean kinetic energy of the Brownian particle:

\[
\langle E \rangle = \frac{m \langle v^2 \rangle}{2} = \frac{k_B T}{2}. \tag{4.7}
\]

Equations (4.6) and (4.7) are examples of fluctuation-dissipation relations. For this case the source in the Langevin equation is called the source of internal fluctuations. Relations (4.6) and (4.7) may not be fulfilled, as it takes place, e.g., in auto-oscillation systems [20]. In such a case one says that there is the source of external (relatively to the system considered) fluctuations in Eq. (2.2). However, Maxwell exponential form of stationary solutions retains. As to the Levy motion, the fluctuation - dissipation relations can not be fulfilled, at least, because of the infinity of square velocity: \( \langle v^2 \rangle = \infty \) for \( 0 < \alpha < 2 \). Therefore, we can only say about the Langevin source as about
the source of external fluctuations. Moreover, the stationary solutions do not possess Maxwell form but instead more general form of stable distributions. We also note that at present there is no theory of equilibrium state basing on stable PDFs.

We further study energy distribution in the stationary state,

\[
f_{st}(E) = \int d\vec{v} f_{st}(\vec{v}) \delta \left( E - \frac{m\vec{v}^2}{2} \right) = 4.8
\]

\[
= \int d\vec{v} \left( E - \frac{m\vec{v}^2}{2} \right) \int \frac{d\vec{k}}{(2\pi)^d} \exp \left( -i\vec{k}\vec{v} \right) f_{st}(\vec{k}) .
\]

We recall, that the two possibilities can be considered, namely, (i) the random electric field is isotropic in the plane perpendicular to the external magnetic field, and (ii) the field is isotropic in the three-dimensional space. In the former case \( \vec{k} \) and \( \vec{v} \) are two-dimensional vectors in Eq. (4.8), \( d = 2 \), whereas in the latter case \( \vec{k} \) and \( \vec{v} \) are three-dimensional vectors, \( d = 3 \). We consider both cases. After some transforms we get from Eq. (4.8),

\[
f_{st}(E) = \frac{1}{m} \int_0^\infty dk \cdot k J_0 \left( k \sqrt{\frac{2E}{m}} \right) \exp \left( -Dk^\alpha \right) , \quad d = 2, 4.9a
\]

\[
f_{st}(E) = \frac{2}{\pi m} \int_0^\infty dk \cdot k \sin \left( k \sqrt{\frac{2E}{m}} \right) \exp \left( -Dk^\alpha \right) , \quad d = 3, 4.9b
\]

where \( D = D/\alpha v \). The integrals in Eq. (4.9) can be easily calculated in two particular cases:

1. \( \alpha = 2 \)

\[
f_{st}(E) = \frac{1}{2mD} \exp \left( -\frac{E}{2mD} \right) , \quad d = 2, 4.10a
\]

\[
f_{st}(E) = \frac{\sqrt{E}}{\sqrt{2\pi} (mD)^{3/2}} \exp \left( -\frac{E}{2mD} \right) , \quad d = 3, 4.10b
\]

which are the well-known results of the theory of Brownian motion [20].

2. \( \alpha = 1 \).

\[
f_{st}(E) = \frac{1}{2} \left( \frac{mD^2}{2} \right)^{1/2} \frac{1}{(E + mD^2/2)^{3/2}} , \quad d = 2, 4.11a
\]

\[
f_{st}(E) = \frac{\sqrt{2mD^2}}{\pi} \frac{\sqrt{E}}{(E + mD^2/2)^{3/2}} , \quad d = 3.4.11b
\]

Since the \( \alpha \)-stable distribution with \( \alpha = 1 \) is called the Cauchy distribution, Eq. (4.11) corresponds to the case of the Cauchy motion.

From Eq. (4.9) it follows that at large energies \( f_{st}(E) \) has a power law asymptotics for \( 0 < \alpha < 2 \),

\[
f_{st}(k) \propto E^{-(1+\alpha/2)} , \quad 4.12
\]
and, thus only the moments of the order $q$ less than $\alpha/2$ are finite for $\alpha < 2$. For the moments of the energy,

$$\langle E^q \rangle = \int_0^\infty dE E^q f_{st}(E)$$

we get

$$\langle E^q \rangle = (2m)^q D^{2q/\alpha} \Gamma(1+q) \frac{\Gamma(1-q)}{\Gamma(1-2q/\alpha)}, \quad d = 2, 4.13b$$

$$\langle E^q \rangle = \frac{2}{\sqrt{\pi}} (2m)^q D^{2q/\alpha} \frac{\sin \pi q}{\sin 2\pi q/\alpha} \frac{\Gamma(q) \Gamma(3/2 + q)}{\Gamma(2q/\alpha)}, \quad d = 3, 4.13c$$

where $q < \alpha/2 < 1$. The particular cases $\alpha = 2$ and $\alpha = 1$ can be also obtained from Eqs. (4.13) or by direct using $f_{st}(E)$ from Eqs. (4.10), (4.11):

$\alpha = 2$.

$$\langle E^q \rangle = (2mD)^q \Gamma(1+q), \quad d = 2, 4.14a$$

$$\langle E^q \rangle = \frac{(mD)^q}{\sqrt{\pi}} \Gamma(3/2 + q), \quad d = 3, 4.14b$$

for all $q \geq 0$.

$\alpha = 1$.

$$\langle E^q \rangle = \left(\frac{mD^2}{2}\right)^q \frac{1}{\sqrt{\pi}} \Gamma(1+q) \Gamma\left(\frac{1}{2} - q\right), \quad d = 2, 4.15a$$

$$\langle E^q \rangle = \frac{(mD^2)^q}{\pi^{2q-1}} \Gamma\left(\frac{3}{2} + q\right) \Gamma\left(\frac{1}{2} - q\right), \quad d = 3, 4.15b$$

for $q < 1/2$.

We carry out numerical simulation based on the solution of the Langevin equations (2.1) with a two-dimensional isotropic white Levy noise $\mathbf{E}(t)$. The case of a strong magnetic field is simulated. In Fig. 1 typical dependencies $E(t)$ ($t$ is a discrete time, $\Delta t = 10^{-3}$ is a time step) are shown on the left for a) $\alpha = 1.95$ and c) $\alpha = 1.1$. With Levy index decreasing the "jumps" on the trajectories, or "Levy flights", become larger; this effect is due to the power law tails of the PDF of the white Levy noise in the Langevin equation. In this Figure and below, in Figs. 2 - 4, the parameters used in simulation are $\Omega = 2$, $\nu = 0.07$, and $D = 1$. At the right, Figs. b) and d), the "Levy flights" are shown on the $(v_x, v_y)$ plane. Again, large "jumps" are clearly seen in the bottom figure.

In Fig. 2 the stationary PDFs $f_{st}(E)$ are shown for a two-dimensional problem at the left, Figs. a), c) and for a three-dimensional problem at the right, Figs. b), d), respectively. At the top, in Fig. a), b), the linear scale is used, whereas in the bottom, in Figs. c), d), the PDFs are shown in the log-log scale.

The PDFs estimated according to Eqs. (4.9) for the Levy index $\alpha = 1.1$ are depicted by solid lines, whereas the PDFs estimated according to Eqs. (4.10) for $\alpha = 2.0$ (Brownian motion) are depicted by dotted lines. The bottom figures clearly show the power asymptotics of the PDFs. The black points on the left figures indicate the PDF obtained in numerical simulation of a two-dimensional problem, the parameters used in numerical simulation are the same as in Fig. 1. The quantitative agreement between analytical and numerical results is obvious.
In Fig. 3, as the result of numerical solution of the Langevin equations, the moments \( \langle E^q \rangle \) of the energy are shown for the Levy index \( \alpha = 1.6 \) and for different orders \( q \), see from the bottom to the top: \( q = 0.12 \), which is less than \( \alpha/2 \), \( q = 0.8 \), which is equal \( \alpha/2 \), and \( q = 2.0 \), which is greater than \( \alpha/2 \). The stationary level of the \( q \)-th moment, estimated according to Eq. (4.13b) is indicated by dotted line in the bottom figure. It is seen that, with \( q \) increasing, the moments of the energy strongly fluctuate; this is the numerical manifestation of the fact that these moments diverge at \( q \geq \alpha/2 \).

5 Non-Homogeneous Relaxation and Superdiffusion

We turn to the relaxation in non-homogeneous case. Since general analysis of Eqs. (3.5)-(3.8) is rather complicated and taking in mind that we already have information about velocity relaxation, we study evolution of a simpler PDF instead of \( f(\vec{r}, \vec{v}, t) \), namely,

\[
f(\vec{r}, t | \vec{r}_0, \vec{v}_0) = \int d\vec{v} f(\vec{r}, \vec{v}, t | \vec{r}_0, \vec{v}_0), \tag{38}
\]

whose characteristic function is

\[
\hat{f}(\vec{\kappa}, t | \vec{r}_0, \vec{v}_0) = \exp \left\{ i \vec{\kappa} \vec{r}_0 + i \vec{\kappa} \vec{v}_0 (1 - e^{-\nu t}) - \frac{D |\vec{\kappa}|^\alpha}{\nu^\alpha} \int_0^t dt' |\vec{\kappa}'(t')|^\alpha \right\}, \tag{41}
\]

For one-dimensional case this result was obtained in [18].

For the case of a strong magnetic field, \( \Omega >> \nu \), we get, again by using Eqs. (5.3), (5.2) and (3.7),

\[
\hat{f}(\vec{\kappa}, t | \vec{r}_0, \vec{v}_0) = \exp \left\{ i \vec{\kappa} \vec{r}_0 + i \vec{\kappa} \vec{v}_0 - D \int_0^t dt' |\vec{\kappa}'(t')|^\alpha \right\}, \tag{42}
\]
\[ \vec{k}'(t') = \left( \frac{\vec{z} \vec{b}}{\nu} \right) (1 - e^{-\nu \tau}) + \frac{e^{-\nu \tau}}{\Omega} \left( \vec{b} \times \left( \vec{z} \times \vec{b} \right) \right) \sin \Omega \tau + 5.6 \quad (43) \]
and \( \vec{k}_0 \) is given by Eq. (5.6) at \( t' = 0 \) (that is, \( \tau = t \)).

For simplicity we put \( \vec{r}_0 = 0 \), and \( \vec{v}_0 = 0 \), as in previous Section. Further, we are interested in diffusion across the magnetic field and, therefore we set \( \left( \vec{z} \vec{b} \right) = 0 \) in Eq. (5.6), thus not considering the motion of a particle along the magnetic field. For the characteristic function we get

\[ \hat{f}(\vec{z}, t) \equiv \hat{f}(\vec{z}, t | 0, 0) = \exp \left\{ -\frac{D}{\Omega^\alpha} \int_0^t d\tau \left( 1 - 2e^{-\nu \tau} \cos \Omega \tau + e^{-2\nu \tau} \right)^{\alpha/2} \right\} , \quad (5.7) \]

where \( \vec{z} \) is two-dimensional vector in the plane perpendicular to \( \vec{B} \), \( \vec{z} \equiv |\vec{z}| \).

At \( \tau >> 1/\nu \) (the diffusion stage of relaxation) the expression (5.7) gets a simple form,

\[ \hat{f}(\vec{z}, t) = \exp \left\{ -\frac{D}{\Omega^\alpha} \vec{z}^\alpha t \right\} , \quad (5.8) \]

which is the characteristic function of an \( \alpha \)-stable isotropic process, compare with Eq. (2.3). Now we consider the PDF and diffusion process in more detail. By taking inverse Fourier transform from Eq. (5.8) we get the PDF,

\[ f(\vec{r}, t) = \int \int \frac{d\vec{z}}{4\pi^2} e^{-i\vec{z} \vec{r}} \hat{f}(\vec{z}, t) = \int_0^t \frac{d\vec{z}}{2\pi} \cdot \vec{z} J_0 (\vec{z} \vec{r}) \exp \left\{ -\frac{D}{\Omega^\alpha} \vec{z}^\alpha \right\} , \quad \vec{r} \equiv |\vec{r}| . \quad (5.9) \]

The particular cases of Eq. (5.9) are

1. \( \alpha = 2 \),

\[ f(\vec{r}, t) = \frac{\Omega^2}{4\pi Dt} \exp \left( -\frac{\Omega^2}{4Dt} \vec{r}^2 \right) , \quad (5.10) \]

and

2. \( \alpha = 1 \),

\[ f(\vec{r}, t) = \frac{Dt/\Omega}{2\pi \left( \vec{r}^2 + (Dt/\Omega)^2 \right)^{3/2}} . \quad (5.11) \]

In general case \( 0 < \alpha < 2 \) the asymptotics of the PDF at large \( r \) behaves as

\[ f(\vec{r}, t) \propto \frac{Dt}{\Omega^\alpha \vec{r}^{2+\alpha}} \quad (5.12) \]
It implies that the $q$-th moments of $r$ diverge at $q \geq \alpha$. The expression for the moments follows from Eq. (5.9):

$$\langle r^q \rangle \equiv M_r (t; q, \alpha) = \left[ \frac{(Dt)^{1/\alpha}}{\Omega} \right]^q C_2 (q; \alpha), \quad 5.13$$

where

$$C_2 (q; \alpha) = \int_0^\infty d\varkappa_1 \varkappa_1^{1+q} \int_0^\infty d\varkappa_2 J_0 (\varkappa_1 \varkappa_2) e^{-\varkappa_2^2}, \quad 5.14$$

The integral over $\varkappa_2$ behaves as $\varkappa_1^{2-\alpha}$ at large $\varkappa_1$, thus $C_2 (q; \alpha)$ diverges at upper limit of $\varkappa_1$ at $q \geq \alpha$. The particular cases following from Eqs. (5.13), (5.14) are as follows:

1. $\alpha = 2$,

$$\langle r^q \rangle = \left( \frac{4D}{\Omega^2} \right)^{q/2} t^{q/2} \Gamma(1 + \frac{q}{2}), \quad q > 0, \quad 5.15$$

and

2. $\alpha = 1$,

$$\langle r^q \rangle = \frac{1}{2} \left( \frac{Dt}{\Omega} \right)^q \mathbb{B} \left( 1 + \frac{q}{2}; 1 - \frac{q}{2} \right), \quad 0 < q < 15, \quad 5.16$$

where $\mathbb{B}$ is the beta-function.

From Eq. (5.15) we get a classical diffusion law for the square displacement of charged particle across the magnetic field:

$$\langle r^2 \rangle \propto \frac{t}{B^2}, \quad 5.17$$

One can introduce ”the typical displacement” of a particle defined as

$$\Delta r = \langle r^q \rangle^{1/q}, \quad 5.18$$

From Eq. (5.15) it follows that at any $q$ (not only at $q = 2$) we have for the Brownian particle,

$$\Delta r \propto \frac{t^{1/2}}{B}, \quad 5.19$$

with the prefactor which, of course, depends on $q$. We recall that usually just $t$ and (especially in plasma physics) $B$-dependences serve as indicator of normal or anomalous diffusion. Therefore, we can use the typical displacement (5.18) as a measure of anomalous diffusion rate at $0 < \alpha < 2$ and any $q < \alpha$. Indeed, it follows from Eq. (5.13) that for the anomalous diffusion

$$\Delta r \equiv \langle r^q \rangle^{1/q} \propto t^{1/\alpha} \quad 5.20$$

Expression (5.20) teaches us that in our model diffusion is anomalously fast over $t$, since $\alpha < 2$; this diffusion is also called superdiffusion. At the same time it retains classical scaling over $B$ (that is, given by Eqs. (5.17) or (5.19)). We also note that
the obtained $t$ - dependence is the typical superdiffusion law within the framework of fractional kinetics, see Refs. [21], [22], [18], [23].

Basing on the numerical solution to the Langevin equations (2.1) we estimated numerically the moments $M_r (t; q, \alpha)$ by averaging over 100 realizations, each consisting from 50000 time steps. In Fig. 4 the $q$-th root of the $q$-th moment (characteristic displacement $\Delta r$) is shown versus $t$ in a log-log scale for the three values of $q$, and for the Levy index equal 1.2, see from bottom to top: $q = 0.12$, which is less than $\alpha$, $q = 1.1$ which is nearly $\alpha$, and $q = 2.0$ (variance), which is greater than $\alpha$. The dashed lines have the slope $1/\alpha$, which is, in fact, the theoretical value of the diffusion exponent at $q < \alpha$, see Eq. (5.20). At $q < \alpha$ the numerical curve is well fitted by the dotted line. At $q \geq \alpha$ theoretical value of the moment is infinite, and in numerical simulation the moment strongly fluctuates.

In our numerical simulation the scaling given by Eq. (5.20) was also checked in more detail. The results are presented in Fig. 5. At the left, in Figure a), we show the $1/B$ - dependence of the characteristic displacement $\Delta r$. The Levy index $\alpha$ is equal 1.2, and $q$ is equal 0.12. The values of $\Delta r$ obtained in numerical simulation are shown by black points, which are well fitted by straight dotted line. This fact confirms that $\Delta r$ is inversely proportional to the magnetic field. At the right in Figure b) we show the exponent $\mu$ in the relation $\Delta r \sim t^\mu$ versus $1/\alpha$. The black points indicate the result of numerical simulation, whereas the dotted line shows the straight line $\mu = 1/\alpha$. We conclude that the right figure confirms the obtained theoretical dependence $\Delta r \sim t^{1/\alpha}$.

6 Conclusion

In this paper we propose fractional Fokker-Planck equation (FFPE) for the description of the motion of a charged particle in constant magnetic field and stochastic electric fields. The latter is assumed to be a white Levy noise. We also assume that the particle is influenced by a linear friction force. Such formulation is a natural generalization of the classical problem for the Brownian particle [14]. It allows us to consider in a simplest way the peculiarities of the motion stipulated by non-Gaussian Levy statistics of a random electric field.

The main results are as follows:

1. The general solution to FFPE for a charged particle in constant magnetic field is obtained. In case of the absence of magnetic field this solution lead to the results obtained previously. However, the general solution also allows us to study the opposite case of a strong magnetic field in detail.

2. The properties of stationary states are studied for two- and three-dimensional motions. The velocity relaxation is studied in detail, and non-Maxwellian stationary states are found, for which the velocity PDF is a Levy stable distribution.

The energy PDFs are obtained, which have power law tails. This circumstance leads to divergence of the energy mean. In the real experiments as well as in numerical simulation the divergence manifests itself in large fluctuations of the mean value with time.

3. The superdiffusion of a charged particle across magnetic field is studied within the framework of FFPE. The fractional moments of space displacement are estimated, and anomalous dependence of characteristic displacement $\Delta r$ versus time, $\Delta r \sim t^{1/\alpha}$, $\alpha < 2$, is found. The typical displacement is inversely proportional to the magnetic
field. Therefore the diffusion described by our FFPE demonstrate anomalous behavior with time and remains classical with respect to the magnetic field dependence.

4. We carry out numerical simulation based on the solution to the Langevin equations for a charged particle in constant magnetic and random electric fields. We study numerically the process of relaxation and stationary energy states for different Levy indexes as well as superdiffusion process. The results of numerical simulation are in qualitative agreement with analytical estimates.

In summary, we believe that fractional kinetics will be a useful complementary tool for understanding and description of variety of non-equilibrium phenomena in space and thermonuclear plasmas. More elaborated fractional kinetic equations based on more sophisticated Langevin equations can be constructed, which, in particular, lead to a finite variance of the displacement of a particle and to anomalous $B$-dependence. On the other hand, a consistent development of the theory of essentially non-Gaussian plasma fluctuations is also of interest.

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References


Figure Captions

Fig. 1. Numerical solution to the Langevin equations (2.1). At the left: typical dependences of the energy $E$ versus discrete time $t$ for a) $\alpha = 2.0$ (Gaussian noise term in the Langevin equations), and c) $\alpha = 1.1$. At the right: $(v_x - v_y)$ plane for b) $\alpha = 2.0$, and d) $\alpha = 1.1$.

Fig. 2. Stationary energy PDFs are shown in linear scale (at the top) and in log-log scale (in the bottom), for two-dimensional motion (in the left) and three-dimensional motion (in the right), respectively. The PDF with $\alpha = 1.1$ is shown by solid curves, the PDF with $\alpha = 2$ is shown by dotted lines. The points indicate the PDF obtained in numerical simulation.

Fig. 3. The results of numerical simulation based on the solution to the Langevin equations. The $q$-th moments of the energy vs $t$ for different orders of $q$ and for $\alpha = 1.6$. From bottom to top: $q = 0.12 < \alpha/2$; $q = 0.8 = \alpha/2$; $q = 2.0 > \alpha/2$. In the bottom figure the dotted vertical line indicates the velocity relaxation time $\tau_v$, whereas the dotted horizontal line shows the analytical value of the moment.

Fig. 4. The results of numerical simulation based on the solution to the Langevin equations. The $q$-th root of the $q$-th moment as a function of $t$ in a log-log scale for different orders of $q$ and for $\alpha = 1.2$. From bottom to top: $q = 0.12 < \alpha$; $q = 1.2 = \alpha$; $q = 2.0 > \alpha$. The tangent of a slope of dashed lines equals to $1/\alpha$.

Fig. 5. At the left: the $q$-th root of the $q$-th moment versus $1/B$. The black points, which are fitted by straight dotted line, show the result of numerical simulation. At the right: the power $\mu$ in the relation $\langle r^q \rangle^{1/q} \propto t^\mu$ versus $\alpha$. The black points, which are fitted by straight dotted line result from numerical simulation.