Relativistic Heavy Quarks on the Lattice

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Abstract

Lattice QCD should allow quantitative predictions for the heavy quark physics from first principles. Up to now, however, most approaches have based on the nonrelativistic effective theory, with which the continuum limit can not be taken in principle. In this paper we investigate feasibility of relativistic approaches to the heavy quark physics in lattice QCD. We first examine validity of the idea that the use of the anisotropic lattice could be advantageous to control the $m_Qa$ corrections. Our perturbative calculation, however, reveals that this is not true. We instead propose a new relativistic approach to handle heavy quarks on the isotropic lattice. We explain how power corrections of $m_Qa$ can be avoided and remaining uncertainties are reduced to be of order $(a\Lambda_{\text{QCD}})^2$.

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I. INTRODUCTION

Weak matrix elements associated with $B$ mesons are essential ingredients to determine Cabibbo-Kobayashi-Maskawa matrix elements. In principle lattice QCD provides the opportunity of evaluating these matrix elements from first principles. However it is still difficult to simulate the $b$ quark with high precision on the lattice. The main source of systematic errors originates from the fact that the $b$ quark mass in the lattice unit is large: $m_b a \sim 1 - 2$ in the quenched approximation and $m_b a \sim 2 - 3$ in full QCD with current accessible computational resources. In order to control large $m_Q a$ errors, several ways have been proposed so far: A static approximation with $m_Q \to \infty$ [1], a nonrelativistic QCD [2], a nonrelativistic interpretation applied to results with the Wilson/Sheikholeslami-Wholert (SW) action [3] and an anisotropic lattice with finer temporal lattice spacing $a_t$ while keeping the spatial one $a_s$ modest [4]. Although the $b$ quark can be directly simulated with any of the last three approaches, only the last one has the advantage that we can take the continuum limit, which is a fascinating feature stimulating our interest.

Practical effectiveness of the anisotropic lattice is transparent: with finer temporal lattice spacing time evolutions of all kinds of correlation functions become milder, which benefits us the better signal-to-noise ratio. On the theoretical side, our interest exists in the use of the anisotropic lattice to control $m_Q a$ errors. If they are restricted to only powers of $m_Q a_t$, they can be made small by the anisotropic lattice with smaller $a_t$. Indeed $m_Q a_s$ corrections can be removed at the tree level [5]. Our main concern, however, is whether $m_Q a_s$ corrections could revive perturbatively or nonperturbatively. Up to now no one has successfully eliminated the possibility that $m_Q a_s$ corrections can appear beyond the tree level. Without the proof given the anisotropic lattice is no better than the isotropic one, where one can also eliminate $m_Q a$ corrections at the tree level, in terms of controlling $m_Q a$ corrections.

The first part of this paper is devoted to a one-loop calculation of the quark self energy on the anisotropic lattice. We analyze its $O(g^2 a)$ terms to examine the possibility of appearance of $O(g^2 m_Q a_s)$ contributions. Our results strongly suggest that one-loop radiative
corrections allow the revival of $m_Qa_s$ corrections. Since we find that the anisotropic lattice is not theoretically advantageous any more, in the second part of this paper we propose a new relativistic way to deal with the heavy quarks on the isotropic lattice, analyzing $m_Qa$ corrections carefully. We discuss cutoff effects of the heavy quark system following the on-shell improvement programme [6,7]. An important finding is that leading cutoff effects of order $(m_Qa)^n$ can be absorbed in the definition of renormalization factors for the quark wave function and mass, so that the leading $m_Qa$ correction is reduced to be of order $(m_Qa)^n\Lambda_{\text{QCD}}$. After removing remaining leading cutoff effects of $O((m_Qa)^n\Lambda_{\text{QCD}})$ with parameters in the quark action properly adjusted, we are left with only $O((a\Lambda_{\text{QCD}})^2)$ errors, which are expected to be fairly small. We also provide a nonperturbative method to control the $O((m_Qa)^n)$ corrections involved in renormalization factors.

This paper is organized as follows. In Sec. II we present a one-loop calculation of the quark self energy on the anisotropic lattice and discuss the possibility of the revival of $m_Qa_s$ corrections. In Sec. III we propose a new relativistic approach to handle the heavy quarks on the isotropic lattice avoiding large $m_Qa$ corrections. Our conclusions are summarized in Sec. IV.

II. ANISOTROPIC LATTICE

A. On-shell improvement on the anisotropic lattice

In order to obtain a generic form of the quark action allowed on the anisotropic lattice, let us make the operator analysis according to the Symanzik's improvement programme [6,7]. The lattice theory is described by a local effective theory as

\[
S_{\text{eff}} = S_0 + \sum_{k \geq 1, i} a^k \int d^4x c_{4+k,i}(g)O_{4+k,i}(x),
\]

where $S_0$ denotes the continuum action. $O_{k,i}(x)$ is a local composite operator with $k$ dimensions, which consist of quark mass, quark fields and link variables. These higher dimensional
operators must respect symmetries on the lattice such as the invariance under gauge, parity and charge-conjugation transformations and discrete rotations. The coefficient $c_{k,i}$ is a function of the gauge coupling $g$ to be determined perturbatively or nonperturbatively.

Symanzik’s improvement programme was originally designed to reduce cutoff effects order by order for on-shell and off-shell Green functions. However, we are interested in only on-shell quantities such as hadron masses and matrix elements which require correlation functions at non-zero physical distances. Here it would be better to consider the on-shell improvement procedure that is much simpler but restricted to on-shell quantities [8].

Under the requirement of various symmetries on the lattice, we find the following set of operators with dimension up to five:

\[ \text{dim.3: } O'_3(x) = \bar{q}(x)q(x), \]
\[ \text{dim.4: } O'_{4a}(x) = \bar{q}(x)\gamma_0D_0q(x), \]
\[ O'_{4b}(x) = \sum_i \bar{q}(x)\gamma_iD_iq(x), \]
\[ \text{dim.5: } O'_{5a}(x) = \bar{q}(x)D_0^2q(x), \]
\[ O'_{5b}(x) = \sum_i \bar{q}(x)\gamma_0D_iq(x), \]
\[ O'_{5c}(x) = \sum_i \bar{q}(x)\sigma_{0i}F_{0i}q(x), \]
\[ O'_{5d}(x) = \sum_{i,j} \bar{q}(x)\sigma_{ij}F_{ij}q(x), \]
\[ O'_{5e}(x) = \sum_i \bar{q}(x)[\gamma_0D_0, \gamma_iD_i]q(x), \]

where $D_\mu$ is the covariant derivative on the lattice and $\sigma_{\mu\nu}$ and $F_{\mu\nu}$ are defined as $\sigma_{\mu\nu} \equiv [\gamma_\mu, \gamma_\nu]/2$ and $igF_{\mu\nu} \equiv [D_\mu, D_\nu]$. The subscript 0 denotes the time component, while $i, j = 1, 2, 3$ space components. In terms of these operators, a general form of the quark action on the anisotropic lattice is given by

\[ S_q^\text{imp} = \sum_x \left[ c'_3O'_3(x) + \sum_{i=a,b} c'_{4i}O'_{4i}(x) + \sum_{i=a,...,e} c'_{5i}O'_{5i}(x) \right], \]

where $c'_3, \ldots, c'_{5e}$ are functions of the bare gauge coupling $g$, the bare quark mass $m_0$ and the time and space lattice spacings $a_t, a_s$. Since degrees of freedom of $c'_3$ and $c'_{4a}$ can be
absorbed in the renormalization of the quark mass and the wave function respectively, we choose \( c_3' = m_0 \) and \( c_{4a}' = 1 \) for convenience. We also find \( O_{5a}' \) and \( O_{5e}' \) are related to other operators by using the classical field equation, and hence \( c_{5a}' \) and \( c_{5e}' \) can be set by hand. We eliminate \( O_{5e}' \) by choosing \( c_{5e}' = 0 \) and employ \( O_{5a}' \) to avoid species doubling with \( c_{5a}' \) finite. After all remaining parameters are \( c_{4b}', c_{5b}', c_{5c}' \) and \( c_{5d}' \), which should be tuned (nonperturbatively) in order to remove \( O(a_{t,s}) \) discretization errors, so that we are left with only discretization ambiguities of \( O(a_{t,s}^2) \). This point should be stressed, since previous papers \([5,9]\) claim that only three parameters \( c_{4b}', c_{5c}' \) and \( c_{5d}' \) are enough to be tuned for the \( O(a_{t,s}) \) improvement.

**B. Quark and gauge actions**

According to the discussion in the above subsection a general form of the quark action on the anisotropic lattice is given by:

\[
S_q = a_t a_s^3 \sum_x \bar{q}(x) \left[ \gamma_0 D_0 + \nu \sum_i \gamma_i D_i + m_0 - \frac{a_t}{2} r (D_0^2 + \eta \sum_i D_i^2) - a_t \frac{ig}{4} r \left\{ c_E (1 + \eta) \sum_i \sigma_{0i} F_{0i} + c_B \eta \sum_{ij} \sigma_{ij} F_{ij} \right\} \right] q(x),
\]

where parameters in eq. (10) are rewritten as

\[
\begin{align*}
    c_{4b}' &= \nu, \\
    c_{5a}' &= -\frac{r}{2}, \\
    c_{5b}' &= -\frac{r \eta}{2}, \\
    c_{5c}' &= -\frac{g}{4} r c_E (1 + \eta), \\
    c_{5d}' &= -\frac{g}{4} r c_B \eta.
\end{align*}
\]

Here the Wilson parameter \( r \) is taken arbitrary by hand as mentioned in the previous subsection. Note that the quark fields \( q \), the covariant derivatives \( D_0 \) and \( D_i \), the bare quark mass \( m_0 \), the gauge field strengths \( F_{0i} \) and \( F_{ij} \) are dimensionful. They are transformed into dimensionless quantities by
\[ \tilde{q} = a_s^{3/2}q, \quad (17) \]
\[ \tilde{D}_0 = a_t D_0, \quad (18) \]
\[ \tilde{D}_i = a_s D_i, \quad (19) \]
\[ \tilde{m}_0 = a_t m_0, \quad (20) \]
\[ \tilde{F}_{0i} = a_t a_s F_{0i}, \quad (21) \]
\[ \tilde{F}_{ij} = a_s^2 F_{ij}, \quad (22) \]

With this transformation, eq. (11) is rewritten as
\[
S_q(x) = \sum_x \tilde{q}(x) \left[ \gamma_0 \tilde{D}_0 + \frac{\nu}{\xi} \sum_i \gamma_i \tilde{D}_i + \tilde{m}_0 - \frac{r}{2} \left( \tilde{D}_0^2 + \frac{\eta}{\xi^2} \sum_i \tilde{D}_i^2 \right) - \frac{iq}{4r} \left\{ c_E \frac{(1 + \eta)}{\xi} \sum_i \sigma_{0i} \tilde{F}_{0i} + c_E \frac{\eta}{\xi^2} \sum_i \sigma_{ij} \tilde{F}_{ij} \right\} \right] \tilde{q}(x) \quad (23)
\]

with \( \xi = a_s / a_t \) the anisotropy parameter.

From eq. (11) we find the inverse free quark propagator
\[
(a_t S_q^{-1}(p) = i\gamma_0 \sin(p_0 a_t) + \frac{\nu}{\xi} \sum_i \gamma_i \sin(p_i a_s) + m_0 a_t + r(1 - \cos(p_0 a_t)) + \frac{\eta}{\xi^2} \sum_i (1 - \cos(p_i a_s)), \quad (24)
\]

and relevant vertices for the present calculation
\[
V_{10}^A(q, p) = -gT^A \left\{ i\gamma_0 \cos \left( \frac{p_0 a_t + q_0 a_t}{2} \right) + r \sin \left( \frac{p_0 a_t + q_0 a_t}{2} \right) \right\}, \quad (25)
\]
\[
V_{ii}^A(q, p) = -gT^A \left\{ \nu i \gamma_i \cos \left( \frac{p_i a_s + q_i a_s}{2} \right) + \frac{\eta}{\xi} \sin \left( \frac{p_i a_s + q_i a_s}{2} \right) \right\}, \quad (26)
\]
\[
V_{200}^{AB}(q, p) = \frac{a_s}{2\xi} g^2 \frac{1}{2} \left\{ T^A, T^B \right\} \left\{ i\gamma_0 \sin \left( \frac{p_0 a_t + q_0 a_t}{2} \right) - r \cos \left( \frac{p_0 a_t + q_0 a_t}{2} \right) \right\}, \quad (27)
\]
\[
V_{2ii}^{AB}(q, p) = \frac{a_s}{2} g^2 \frac{1}{2} \left\{ T^A, T^B \right\} \left\{ \nu i \gamma_i \sin \left( \frac{p_i a_s + q_i a_s}{2} \right) - \frac{\eta}{\xi} \cos \left( \frac{p_i a_s + q_i a_s}{2} \right) \right\}, \quad (28)
\]
\[
V_{c0}^A(q, p) = -c_E g T^A \frac{r(1 + \eta)}{4\xi} \sum_i \sigma_{0i} \sin(p_i a_s - q_i a_s) \cos \left( \frac{p_i a_s - q_i a_s}{2} \right), \quad (29)
\]
\[
V_{ci}^A(q, p) = -c_E g T^A \frac{r(1 + \eta)}{4} \sigma_{0i} \sin(p_0 a_t - q_0 a_t) \cos \left( \frac{p_i a_s - q_i a_s}{2} \right) - c_B g T^A \frac{r\eta}{2\xi} \sum_j \sigma_{ij} \sin(p_j a_s - q_j a_s) \cos \left( \frac{p_i a_s - q_i a_s}{2} \right), \quad (30)
\]
where \( p \) is for the incoming momentum into the vertex and \( q \) for the outgoing momentum. \( T^A (A = 1, \ldots, N_c^2 - 1) \) is a generator of color \( \text{SU}(N_c) \).

For the gauge part we take the standard Wilson action on the anisotropic lattice:

\[
S_g = \frac{2N_c}{g^2} \sum_n \left[ \xi \sum_i \left( 1 - \frac{1}{2N_c} \text{Tr}(U_0(n) + U_0^\dagger(n)) \right) \\
+ \frac{1}{\xi} \sum_{i<j} \left( 1 - \frac{1}{2N_c} \text{Tr}(U_{ij}(n) + U_{ij}^\dagger(n)) \right) \right]
\]

(31)

with

\[
U_{\mu\nu}(n) = U_{\mu}(n)U_{\nu}(n + \hat{\mu})U_{\nu}^\dagger(n + \hat{\nu})U_{\mu}^\dagger(n).
\]

(32)

The gluon propagator is given by

\[
G^{AB}_{\mu\nu}(k) = a_s^2 \delta_{\mu\nu} \delta_{AB} \frac{\xi^2 4 \sin^2 \left( \frac{k_{a_t}}{2} \right) + 4 \sum_i \sin^2 \left( \frac{k_{a_s}}{2} \right) + \lambda^2 a_s^2}{k_0^2 + \sum_i p_i^2 + m_0^2 + O((p_0 a_t)^2, (p_i a_s)^2)}.
\]

(33)

in the Feynman gauge with the fictitious mass \( \lambda^2 \), which is introduced to work as the infrared cutoff.

\[\text{C. Tree-level analysis on the quark propagator}\]

Expanding the inverse free quark propagator in eq. (24) up to \( O(a_t) \) we obtain

\[
S_q^{-1}(p) = i\gamma_0 p_0 + i\nu \sum_i \gamma_i p_i + m_0 + \frac{r a_t}{2} p_0^2 + \frac{r \eta a_t}{2} \sum_i p_i^2 + O((p_0 a_t)^2, (p_i a_s)^2),
\]

(34)

which yields the following expression for the quark propagator:

\[
S_q(p) = \frac{1}{1 + m_0 r a_t} \frac{-i\gamma_0 p_0 - i\nu \sum_i \gamma_i p_i + m_0 + \frac{r a_t}{2} (p_0^2 + \eta \sum_i p_i^2)}{p_0^2 + \frac{r^2 + m_0^2}{1 + m_0 r a_t} \sum_i p_i^2 + \frac{m_0^2}{1 + m_0 r a_t} m_0^2} + O(a_t^2).
\]

(35)

We consider the tree-level on-shell improvement for this quark propagator requiring that \( S_q(p) \) should reproduce the form

\[
S_q(p) = \frac{1}{Z_q} \frac{-i\gamma_0 p_0 - i \sum_i \gamma_i p_i + m_R}{p_0^2 + \sum_i p_i^2 + m_R^2} + \text{(no pole terms)} + O(a_t^2)
\]

(36)

with the appropriate choice for \( Z_q, Z_m, \nu \) and \( \eta \), where \( Z_q \) and \( Z_m \) denote renormalization factors for the quark wave function and mass defined by
\[ q_R = Z_q^{1/2}q, \quad (37) \]

\[ m_R = Z_mm_0, \quad (38) \]

where \( m_R \) is the pole mass. It should be reminded that \( r \) is a free parameter. We remark that terms without the pole in the quark propagator of eq. (36) yield only contact terms in the configuration space, which do not contribute to on-shell quantities in Green functions. In terms of the inverse quark propagator, the condition eq. (36) is equivalent to

\[ S_q^{-1}(p) = [Z_q - 2Cm_0a_t] (i\psi + m_R) + C\xi (p^2 + m_R^2) + O(a_t(i\psi + m_R)^2) + O(a_t^2) \quad (39) \]

with \( C \) constant. Therefore “(no pole terms)” in eq. (36) are not necessary to be \( O(a_t^2) \).

Comparing the expressions of eqs. (35) and (36), we find at the tree level

\[ Z_q^{-1/2} = 1 - \frac{r}{2}m_0a_t, \quad (40) \]

\[ Z_m = 1 - \frac{r}{2}m_0a_t, \quad (41) \]

\[ \nu = 1, \quad (42) \]

\[ \eta = 1. \quad (43) \]

Up to now three types of quark actions with different choices for \( r \) and \( \eta \) have been proposed: (i) \( r = 1 \) and \( \eta = 1 \) \([5]\), (ii) \( r = \xi \) and \( \eta = 1 \) \([10]\) and (iii) \( r = 1 \) and \( \eta = \xi \) \([9]\). Although the action in the case of (iii) has been most extensively studied numerically, the choice of the parameter \( \eta \) does not meet the condition of eq. (43) that is required from the on-shell improvement at the tree-level. This primitive failure makes us consider that it is not worthwhile to work on the case (iii) in this paper. We focus on only cases (i) and (ii) hereafter.

Let us first derive the relation between the bare quark mass \( m_0 \) and the pole mass \( m_p \). Putting \( p_i = 0 \) and \( p_0 = im_p \) into the inverse free quark propagator of eq. (24), the on-shell condition yields

\[ m_p a_t = \log \left| \frac{m_0a_t + r + \sqrt{(m_0a_t)^2 + 2rm_0a_t + 1}}{1 + r} \right|. \quad (44) \]
While in the case (i) with $r = 1$ we can expand $m_p a_t$ in powers of $m_0$ under the condition $m_0 a_t \ll 1$, in the case (ii) with $r = \xi$ the condition $\xi m_0 a_t = m_0 a_s \ll 1$ is necessary. To avoid any confusions we assume $m_0 a_s \ll 1$ from now on. We remark that this assumption does not affect any conclusions in this section.

On the anisotropic lattice we have to be careful about contributions of space doublers. Pole masses of space doublers are written as

$$m_p^d a_t = \log \left| \frac{m_0^d a_t + r + \sqrt{(m_0^d a_t)^2 + 2r m_0^d a_t + 1}}{1 + r} \right|$$

(45)

with

$$m_0^d = m_0 + \frac{2r N_d}{\xi^2 a_t},$$

(46)

where $N_d$ components of spatial momentum $p_i$ are equal to $\pi/a_s$ at the edge of the Brillouin zone. Although doubler pole masses are always heavier than the physical one irrespective of the value of $m_0$, $r$ and $\xi$, their differences in the large limit of $\xi$ are given by

$$(m_p^d - m_p) a_s \rightarrow \frac{2}{\xi} N_d \frac{1}{1 + m_0 a_t} + O \left( \frac{1}{\xi^2} \right) \quad \text{case (i)},$$

(47)

$$(m_p^d - m_p) a_s \rightarrow \sqrt{1 + 2m_0 a_s + 4N_d} - \sqrt{1 + 2m_0 a_s + O \left( \frac{1}{\xi} \right)} \quad \text{case (ii)}.$$ 

(48)

For the case (i) we find the gap $(m_p^d - m_p) a_s$ diminishes as $\xi$ becomes larger. This brings a practical problem in numerical studies of heavy quarks: contributions of doublers could contaminate signals of hadron states. On the other hand, we are free from this problem in the case (ii).

D. One-loop quark self energy

The inverse full quark propagator is written as

$$S_q^{-1}(p) = i\gamma_0 p_0 + \nu i \sum_i \gamma_i p_i + m_0 + \frac{a}{2} p_0^2 + \eta \frac{a}{2} \sum_i p_i^2 - \Sigma(p, m_0),$$

(49)
where we take $a = r a_\text{t}$ ($a = a_\text{t}$ for (i) while $a = a_\text{s}$ for (ii)). One-loop contributions to the quark self-energy $\Sigma(p, m_0)$ consist of two types of diagrams depicted in Figs. 1 (a) and (b), which are expressed by

$$\Sigma_a(p, m) = \int \frac{d^4k}{(2\pi)^4} \sum_A \sum_\mu \left\{ V^A_{1\mu}(p, p + k) S_q(p + k) V^A_{1\mu}(p + k, p) 
+ V^A_{\text{c}}(p, p + k) S_q(p + k) V^A_{\text{c}}(p + k, p) 
+ V^A_{\text{c}}(p, p + k) S_q(p + k) V^A_{\text{c}}(p + k, p) \right\} G^{AA}_{\mu\mu}(k)$$

and

$$\Sigma_b(p) = \int \frac{d^4k}{(2\pi)^4} \sum_A \sum_\mu V^{AA}_{2\mu}(p, p) G^{AA}_{\mu\mu}(k)$$

with

$$\begin{align*}
-\frac{\pi}{a_\text{t}} &\leq k_0 \leq \frac{\pi}{a_\text{t}}, \\
-\frac{\pi}{a_\text{s}} &\leq k_i \leq \frac{\pi}{a_\text{s}}.
\end{align*}$$

Expanding $\Sigma(p, m_0) = \Sigma_a(p, m_0) + \Sigma_b(p)$ in terms of $p$ and $m_0$, we obtain

$$\Sigma(p, m_0) = \frac{g^2}{16\pi^2} C_F \left[ \Sigma_0 \left( \frac{1}{a} + i \gamma_0 p_0 (1 - L + \Sigma_1^t) + i \sum_i \gamma_i p_i (1 - L + \Sigma_1^s) + m_0 (-4L + \Sigma_2) + a p_0^2 ((1 - 3c_{\text{SW}}) L/2 + \sigma_1^t) + a \sum_i p_i^2 ((1 - 3c_{\text{SW}}) L/2 + \sigma_1^s) + am_0 i \gamma_0 p_0 ((5 + 3c_{\text{SW}}) L/2 + \sigma_2^t) + am_0 i \sum_i \gamma_i p_i ((5 + 3c_{\text{SW}}) L/2 + \sigma_2^s) + am_0^2 ((5 - 3c_{\text{SW}}) L + \sigma_3) \right) + O(a^2_{\text{t,s}}) \right]$$

with $L = -\log(\lambda^2 a^2_\text{s})$ the contribution of the infrared divergence. Here we take $c_B = c_E \equiv c_{\text{SW}}$ for the clover coefficient. $\Sigma_0$, $\Sigma_1^t$, $\Sigma_2$ and $\sigma_1^t$, $\sigma_1^s$, $\sigma_2^t$, $\sigma_2^s$, $\sigma_3$ are independent of $a$ but functions of $\xi$, $\eta$ and $r$ parameters. We evaluate these quantities numerically using the Monte Carlo integration routine BASES [11].

From eqs. (49) and (54) we obtain
\[
Z_q^{-1/2} = \left\{ 1 + \frac{g^2}{16\pi^2} C_F \left( -L + \Delta_q^{(0)} \right) \right\} \left\{ 1 + a_q m \left( -\frac{r}{2} + \frac{g^2}{16\pi^2} C_F \Delta_q^{(1)} \right) \right\},
\]
\[
Z_m = \left\{ 1 + \frac{g^2}{16\pi^2} C_F (3L + \Delta_m^{(0)}) \right\} \left\{ 1 + a_t m \left( -\frac{r}{2} + \frac{g^2}{16\pi^2} C_F \Delta_m^{(1)} \right) \right\},
\]
\[
\nu = 1 - \frac{g^2}{16\pi^2} C_F \Delta_{\nu}^{(0)} - \frac{g^2}{16\pi^2} C_F a_t m \Delta_{\nu}^{(1)},
\]
\[
\eta = 1 - 2 \frac{g^2}{16\pi^2} C_F \Delta_{\eta}^{(0)},
\]
where
\[
m = m_0 - \frac{g^2}{16\pi^2} C_F \frac{\Sigma_0}{a},
\]
\[
\Delta_q^{(0)} = \Sigma_t^t,
\]
\[
\Delta_q^{(1)} = r \left( \sigma_1 + \frac{\sigma_1^t}{2} - \Sigma_1^t + \frac{\Sigma_2^t}{2} + \frac{3(1 - c_{SW})}{4} L \right),
\]
\[
\Delta_m^{(0)} = \Sigma_t^t - \Sigma_2^s,
\]
\[
\Delta_m^{(1)} = r \left( \sigma_1^t + \sigma_2^t - \sigma_3 - \Sigma_1^t + \frac{\Sigma_2^t}{2} + 3(c_{SW} - 1) L \right),
\]
\[
\Delta_{\nu}^{(0)} = \Sigma_t^t - \Sigma_1^s,
\]
\[
\Delta_{\nu}^{(1)} = r \left( \sigma_2^t - \sigma_2^s \right),
\]
\[
\Delta_{\eta}^{(0)} = \sigma_1^t - \sigma_1^s.
\]

We find for the anisotropic case that the \( g a \) terms disappear in \( Z_q, Z_m \) for \( c_{SW} = 1 \) as well as \( \nu \) and \( \eta \) for an arbitrary values of \( c_{SW} \), as observed in \( Z_q \) and \( Z_m \) for the isotropic case [12]. Thus \( c_{SW} = 1 \) gives the tree level estimate for \( c_B \) and \( c_E \), and we take this value for the latter numerical calculation in this section. With this choice for \( Z_q, Z_m, \nu \) and \( \eta \) the quark propagator is given by

\[
S_q(p) = \frac{Z_q^{-1}}{p_0^2 + \sum_i p_i^2 + m_R^2} \left\{ -i \gamma_0 p_0 - i \sum_i \gamma_i p_i + m_R \right. \\
+ a(p_0^2 + \sum_i p_i^2 + m_R^2) \left\{ \frac{1}{2} - \frac{g^2}{16\pi^2} C_F \left( -\frac{L}{2} + \sigma_1^t - \frac{\Sigma_1^t}{2} \right) \right\} \right\},
\]
\[
= Z_q^{-1} S_q^R(p) + Z_q^{-1} a \left\{ \frac{1}{2} - \frac{g^2}{16\pi^2} C_F \left( -\frac{L}{2} + \sigma_1^t - \frac{\Sigma_1^t}{2} \right) \right\},
\]
where \( S_q^R(p) \) is the renormalized quark propagator. We again remark that the term without the pole in the quark propagator does not contribute to on-shell quantities in Green
functions.

We show $\xi$ dependences of one-loop coefficients for $Z_q$, $Z_m$, $\nu$ and $\eta$ in Figs. 2, 3, 4 and 5, respectively. Here we consider both $r = 1$ and $r = \xi$ cases with $\eta = 1$, which satisfy the tree-level on-shell condition in eq. (43). Although our main concern is whether $ma_s$ corrections could revive at the one-loop level for the $r = 1$ case, we also present the results of the $r = \xi$ case for comparison. Results for $\Delta_q^{(0)}$, $\Delta_q^{(1)}$, $\Delta_m^{(0)}$ and $\Delta_m^{(1)}$ in the isotropic case are already given in Refs. [13,14]. They show an agreement with our results with the choice of $\xi = 1$. For the $r = 1$ case, Figs. 3(b), 4(b) and 5 show approximate linear dependences on $\xi$ for $\Delta_m^{(1)}$, $\Delta_{\nu}^{(1)}$ and $\Delta_{\eta}^{(0)}$, which tells us that $O(g^2a)$ contributions to $Z_m$, $\nu$ and $\eta$ are effectively of order $g^2ma_s = g^2ma_t\xi$. We observe similar linear dependences for the $r = \xi$ case in Figs. 2(b), 3(b) and 5. From these observations we conclude that $ma_s$ corrections are allowed to revive at the one-loop level.

This is a reasonable conclusion in view of the on-shell improvement. As far as we know there is no symmetry on the anisotropic lattice to prohibit the higher dimensional operators with the form of $(ma_s)^n\mathcal{O}_{4+k} \ (n, k \geq 1)$, where $\mathcal{O}_{4+k}$ denotes $4 + k$ dimensional operators. Unless such symmetry is uncovered, the theoretical advantage of the anisotropic lattice over the isotropic one would be never confirmed.

### III. ISOTROPIC LATTICE

In this section we propose a new relativistic approach to control $m_Qa$ corrections for the heavy quarks on the lattice. The idea is based on the on-shell improvement programme applied to the heavy quarks on the isotropic lattice. This method allows us to obtain the physical quantities in the continuum limit without requiring harsh condition $m_Qa \ll 1$ that is not achievable in near future.

Let us consider the general cutoff effects for the heavy quarks on the lattice. Here we assume that the heavy quark mass $m_Q$ is much heavier than $\Lambda_{\text{QCD}}$, while the light quark mass $m_q$ is lighter than $\Lambda_{\text{QCD}}$. Under this condition we assume that the leading cutoff effects
are

$$f_0(m_Qa) > f_1(m_Qa)a\Lambda_{QCD} > f_2(m_Qa)(a\Lambda_{QCD})^2 > \cdots, \quad (69)$$

where \(f_i(m_Qa) (i \geq 0)\) are smooth and continuous all over the range of \(m_Qa\) and have Taylor expansions at \(m_Qa = 0\) with sufficiently large convergence radii beyond \(m_Qa = 1\). To control the scaling violation effects we want to remove the cutoff effects up to \(f_1(m_Qa)a\Lambda_{QCD}\) by adding the counter terms to the lattice quark action with the on-shell improvement. If \(m_Qa\) is small enough, the remaining \(f_2(m_Qa)(a\Lambda_{QCD})^2\) contributions can be removed by extrapolating the numerical data at several lattice spacings to the continuum limit. Otherwise, in case of sufficiently small lattice spacing, the \(O((a\Lambda_{QCD})^2)\) errors can be neglected. Our aim in this section is to search for the relevant counter terms required in the on-shell improvement and propose a nonperturbative method to determine their coefficients. We also show that the behavior of \(f_i(m_Qa) (i \geq 1)\) in the large \(m_Qa\) region can be discussed by investigating the static limit.

A. On-shell improvement on the isotropic lattice

We first list the allowed operators under the requirement of the gauge, axis permutation and other various discrete symmetries on the lattice, where the chiral symmetry is not imposed. According to the work of Ref. [15], all the operators with dimension up to six are given by

\[
\begin{align*}
\text{dim.3} : & \quad \mathcal{O}_3(x) = \bar{q}(x)q(x), \\
\text{dim.4} : & \quad \mathcal{O}_4(x) = \bar{q}(x)D\bar{q}(x), \\
\text{dim.5} : & \quad \mathcal{O}_{5a}(x) = \bar{q}(x)D_\mu^2 q(x), \\
& \quad \mathcal{O}_{5b}(x) = i\bar{q}(x)\sigma_{\mu\nu}F_{\mu\nu}q(x), \\
\text{dim.6} : & \quad \mathcal{O}_{6a}(x) = \bar{q}(x)\gamma_\mu D_\mu^3 q(x), \\
& \quad \mathcal{O}_{6b}(x) = \bar{q}(x)D_\mu^2 D^\mu q(x),
\end{align*}
\]
\[ O_{6c}(x) = \bar{q}(x) \gamma^5 \gamma^\mu [D_{\mu} F_{\mu\nu}] q(x), \]  
\[ O_{6d}(x) = i \bar{q}(x) \gamma^\mu [D_{\mu} F_{\mu\nu}] q(x), \]  
\[ O_{6e}(x) = \bar{q}(x) \gamma^5 \gamma^\mu [D_{\mu} F_{\mu\nu}] q(x), \]  
\[ O_{6f}(x) = \bar{q}(x) \gamma^\mu [D_{\mu} F_{\mu\nu}] q(x), \]

where \[ \Gamma = 1, \gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma_{\mu\nu}. \] The higher dimensional operators are related to the lower dimensional operators with additional \[ (m_Q a)^n \] corrections with and without classical field equations, or otherwise they have the contributions of order \[ (a \Lambda_{\text{QCD}})^2 \] or less as the cutoff effects.

These operators lead to a following generic form of the quark action on the isotropic lattice:

\[ S^{\text{imp}}_q = \sum_x \left[ c_3 O_3(x) + c_4 O_4(x) + \sum_{i=a,b} c_{5i} O_{5i}(x) + \sum_{i=a,...,f} c_{6i} O_{6i}(x) \right], \]  

where \[ c_3, \ldots, c_{6f} \] are functions of the bare gauge coupling \[ g \] and the power corrections of \[ ma. \] We first remark that the \[ ma \] corrections to the quark mass term and the kinetic term can be absorbed in the renormalizations of the quark mass \[ Z_m \] and the wave function \[ Z_q. \] For the sake of convenience we choose \[ c_3 = m_0 \] and \[ c_4 = 1. \]

In the next step we reduce the number of basis operators with the aid of the classical field equations. It is easily found that \[ O_{5a}, O_{6b}, O_{6c} \] and \[ O_{6e} \] can be related to the quark mass term or the kinetic term. In the on-shell improvement these operators are redundant and can be eliminated from the action of eq. (80). The operator \[ O_{5a}, \] however, is used to avoid the species doubling and the value of its coefficient \[ c_{5a} \] is given by hand.

The remaining operators are \[ O_{5b}, O_{6d}, O_{6e} \] and \[ O_{6f}, \] whose contributions as the cutoff effects are estimated in Table I. We use the classical field equation for \[ \bar{q}(x) \gamma_0 D_0^3 q(x) \] in \[ O_{6a}. \] Here it should be noted that \[ \Lambda_{\text{QCD}} \] means the order \[ \Lambda_{\text{QCD}} \] or less throughout this paper. In some cases the actual contribution may become smaller. For example, the contributions of the bilinear terms whose Dirac matrices consist of off-diagonal components are suppressed by the extra \[ \Lambda_{\text{QCD}}/m_Q \] for the heavy quarks compared to the light quarks. The operator
$\mathcal{O}_{5b}$ is the so-called clover term, for which the nonperturbative method to determine the coefficient $c_{5b}$ in the massless limit is already established [16]. However, the contributions of $(ma)^n\mathcal{O}_{5b}$ ($n \geq 1$) cannot be neglected in the present condition that allows $m_Qa \sim O(1)$. For $O(a\Lambda_{QCD})$ improvement the coefficient $c_{5b}$ has to be adjusted in the mass dependent way. The differences in magnitude between the time and space components in $\mathcal{O}_{6a}$ originate from the violation of rotational symmetry on the lattice with finite lattice spacing. While the contributions of the space components are found to be negligible, those of the time components should be removed. We also find the contributions of the four-quark operators in $\mathcal{O}_{6f}$ are negligible.

The generalization of the above argument to any operators with higher dimensions makes the discussion more transparent. Let us consider an arbitrary operator with $4+k$ dimension, $a^k\mathcal{O}_{4+k}$, where we write the lattice spacing $a$ explicitly. The operator $\mathcal{O}_{4+k}$ contains $l$ pairs of $\bar{q}$ and $q$ and $n$ covariant derivatives $D_\mu$ with $4+k = 3l + n$. Using the classical field equation, some (but not all) of covariant derivatives can be replaced by the quark mass $m$. For $l \geq 2$ the largest possible power of the scaling violation is $(ma)^n(a\Lambda_{QCD})^{3l-4}$. Therefore the operators which contain four or more quarks are irrelevant for the $O(a\Lambda_{QCD})$ improvement. All the relevant contributions come from the quark bilinear operators. With the aid of the classical field equations, they can be reduced to

\begin{align}
(ma)^n a^{-1} \bar{q}(x)q(x) \\
(ma)^{n-1}\bar{q}(x)\gamma_0 D_0 q(x), \quad (ma)^{n-1}\sum_i \bar{q}(x)\gamma_i D_i q(x) \\
(ma)^{n-2} a\bar{q}(x)D_0^2 q(x), \quad (ma)^{n-2} a\sum_i \bar{q}(x)D_i^2 q(x) \\
(ma)^{n-2} ai \sum_i \bar{q}(x)\sigma_{0i} F_{0i} q(x), \quad (ma)^{n-2} ai \sum_{ij} \bar{q}(x)\sigma_{ij} F_{ij} q(x),
\end{align}

for $n \geq 0$. The time and space components of $\mathcal{O}_4$ and $\mathcal{O}_{5a,5b}$ should be treated separately in case of finite $ma$, where the space-time asymmetry reflects the contributions of the higher dimensional operators that break the rotational symmetry. Now we know that the seven operators are needed for the $O(a\Lambda_{QCD})$ improvement. Since three coefficients among these seven operators can be absorbed in $Z_m, Z_q$ and the Wilson parameter $r_t$ for the time derivative.
tive as already explained, the remaining four coefficients have to be actually tuned.

In conclusion, at all order of $ma$, the generic quark action is written as

$$S_{q}^{\text{imp}} = \sum_{x} \left[ m_{0} \bar{q}(x)q(x) + \bar{q}(x)\gamma_{0}D_{0}q(x) + \nu \sum_{i} \bar{q}(x)\gamma_{i}D_{i}q(x) - \frac{r_{t}a}{2} \bar{q}(x)D_{0}^{2}q(x) \right. $n

$$ \left. - \frac{r_{s}a}{2} \sum_{i} \bar{q}(x)D_{i}^{2}q(x) - \frac{i g a}{2} c_{E} \sum_{i} \bar{q}(x)\sigma_{0i}F_{0i}q(x) - \frac{i g a}{4} c_{B} \sum_{i,j} \bar{q}(x)\sigma_{ij}F_{ij}q(x) \right], \quad (85)$$

where we are allowed to choose $r_{t} = 1$ and the four parameters $\nu$, $r_{s}$, $c_{E}$ and $c_{B}$ are to be adjusted. In general these parameters have the form that $X = \sum_{n} X_{n}(g^{2})(ma)^{n}$ with $X = \nu$, $r_{s}$, $c_{E}$ and $c_{B}$, and $X_{0}$ should agree with the one in the massless $O(a)$ improved theory: $\nu_{0} = 1$, $(r_{s})_{0} = r_{t} = 1$, $(c_{E})_{0} = (c_{B})_{0} = c_{SW} [16]$. Note that $\nu = 1 + O((ma)^{2})$ and $r_{s} = r_{t} + O(ma)$ since the space-time asymmetry arises from Lorentz non-covariant terms such as $O_{6a}$ via the on-shell reduction, accompanied by extra $(ma)^{2}$ factors.

From the above consideration, the leading scaling violation in the massive theory, except for $\sum_{n=1}^{\infty} C_{n}^{\text{impr}}(g^{2}, \log a)(m_{Q}a)^{n}$ in $Z_{q}$ and $Z_{m}$, is $\sum_{n=0}^{\infty} C_{n}^{\text{SW}}(g^{2}, \log a)(m_{Q}a)^{n}a\Lambda_{\text{QCD}}$ for the Wilson quark action, or $\sum_{n=1}^{\infty} C_{n}^{\text{SW}}(g^{2}, \log a)(m_{Q}a)^{n}a\Lambda_{\text{QCD}}$ for the (massless) $O(a)$ improved SW quark action. Here we should notice that the contribution of $(\nu-1) \sum_{i} \bar{q}(x)\gamma_{i}D_{i}q(x)$ is of order $m_{Q}^{2}a^{2}\Lambda_{\text{QCD}}^{2}/m_{Q} \sim m_{Q}a^{2}\Lambda_{\text{QCD}} \sim a\Lambda_{\text{QCD}}$ in the heavy quark region. This implies that once we fix the pole mass from some spectral quantity, the cutoff effects in other spectral quantities are at most of order $a\Lambda_{\text{QCD}}$, not $(m_{Q}a)^{n}$, for the Wilson and the (massless) $O(a)$ improved SW quark actions. It should be noted that the quark wave function does not affect on the spectral quantities. If $\nu$, $r_{s}$, $c_{E}$ and $c_{B}$ are properly adjusted in the mass dependent way, the remaining scaling violations are reduced to $\sum_{n=0}^{\infty} C_{n}^{\text{ours}}(g^{2}, \log a)(m_{Q}a)^{n}(a\Lambda_{\text{QCD}})^{2}$.

This relativistic argument about the on-shell improvement on the massive quarks helps us understand some numerical results for heavy quark physics previously obtained by using the Wilson and SW quark actions under the condition $m_{Q}a \sim O(1)$. We find a good example in Fig. 2 of Ref. [17], which compares the difference between twice of the heavy-light meson mass and the heavy-heavy one, $2m_{HL} - m_{HH}$, obtained by using the pole mass or the kinetic mass defined by $\partial^{2}E_{HH}/\partial p_{t}^{2}$. We observe that $2m_{HL} - m_{HH}$ with the pole mass is consistent with the experimental values within the ambiguities of order $a\Lambda_{\text{QCD}}$ as
expected, while $2m_{HL} - m_{HH}$ with the kinetic mass gets deviated from the experimental values further and further as $m_{QA}$ becomes larger. Recall that the authors of Ref. [3] suggest from a nonrelativistic point of view that the kinetic mass should be used for analyses on the heavy quark quantities. Our relativistic argument, however, tells us that the use of the pole mass makes the remaining cutoff effects $O(a\Lambda_{QCD})$. Since the difference between the kinetic mass and the pole mass is of order $(m_{QA})^2$ at the tree-level, the use of the kinetic mass eventually yields unwanted additional $(m_{QA})^2$ errors. This is the reason why the results of $2m_{HL} - m_{HH}$ with the kinetic mass show considerable deviation from the experimental values.

At the end of this subsection we have to remark one point. The $ma$ corrections in $Z_m$ and $Z_q$, though they are irrelevant for spectral quantities, become important, together with other renormalization factors in case of calculating the quark masses and the matrix elements of various composite operators. In Sec. III F we will show how to calculate these renormalization factors nonperturbatively including the $(m_{QA})^n$ corrections.

### B. Improvement of the axial current

The cutoff effects in the correlation functions of local composite fields are originated from not only the action but also the composite fields themselves. In this subsection we demonstrate the on-shell improvement on the axial current, which is relevant for calculation of the heavy-light pseudoscalar meson decay constants like $f_B$ and $f_D$.

The axial current in the $N_f$ flavor space is given by

$$A^a_\mu(x) = \bar{q}(x)\gamma_\mu\gamma^5\lambda^a/2 q(x),$$

(86)

where $\lambda^a/2$ ($a = 1, \ldots, N_f^2 - 1$) are generators of SU($N_f$). The improvement of this operator is performed in the same way as the improvement of the quark action. After some consideration we find that it is sufficient to consider only the dimension four operators for the $O(a\Lambda_{QCD})$ improvement: the higher dimensional operators can be reduced to $A^a_\mu$ or the
dimension four operators multiplied by \((ma)^n\) using the classical field equations, or otherwise their contributions are of order \((a\Lambda_{\text{QCD}})^2\). The requirement of various symmetries on the lattice allows the following dimension four operators:

\[
\text{dim. 4 : } (\mathbf{A}_a)_{\mu}^a(x) = \bar{q}(x)\gamma_5 D_{\mu} \frac{\lambda^a}{2} q(x) + \bar{q}(x) \overline{D}_{\mu} \gamma_5 \frac{\lambda^a}{2} q(x),
\]

\[
(\mathbf{A}_b)_{\mu}^a(x) = \bar{q}(x)\gamma_5 \sigma_{\mu\nu} D_\nu \frac{\lambda^a}{2} q(x) - \bar{q}(x) \overline{D}_\nu \sigma_{\mu\nu} \gamma_5 \frac{\lambda^a}{2} q(x),
\]  

where we do not take the sum on the index \(\mu\). We find that \((\mathbf{A}_b)_{\mu}^a\) is related to \((\mathbf{A}_4)_{\mu}^a\) and \(\mathbf{A}_a^\mu\) with the aid of the classical field equations, which means the operator \((\mathbf{A}_b)_{\mu}^a\) is redundant for the \(O(a\Lambda_{\text{QCD}})\) improvement.

After all the improved axial current is written as

\[
(\mathbf{A}_\mu^a)^{\text{imp}}(x) = \mathbf{A}_\mu^a(x) + d_\mu (g^2, ma)(\mathbf{A}_4)_{\mu}^a(x).
\]

The parameter \(d_\mu\) has been already calculated in the massless case. Perturbative estimate gives \(d_\mu (g^2, ma = 0) = -0.00756g^2\) [18], which is fairly small in magnitude. From this fact we think that it would be sufficient to evaluate \(d_\mu\) perturbatively in the massive case. The lattice operator \((\mathbf{A}_\mu^a)^{\text{imp}}\) is related to the continuum operator with the renormalization factor \(Z_A\):

\[
(\mathbf{A}_\mu^a)^{\text{con}}(x) = Z_A (g^2, \log a, ma) (\mathbf{A}_\mu^a)^{\text{imp}}(x).
\]

The power corrections of \(ma\) in \(Z_A\) need to be under control to obtain the heavy-light pseudoscalar meson decay constants defined in the continuum regularization scheme. In Sec. III F we will show how to remove the \(ma\) corrections in \(Z_A\) with a nonperturbative method.

C. Benefit of chiral symmetry

Although the above discussions are free from the chiral symmetry, it is also interesting to look into what can be changed by the presence of the chiral symmetry. For the convenience we treat the quark mass matrix in the \(N_f\) flavor space
as a spurious field $\mathcal{M}$ which transforms like

$$\mathcal{M} \rightarrow V_R \mathcal{M} V_L^\dagger, \quad (92)$$

$$\mathcal{M}^\dagger \rightarrow V_L \mathcal{M}^\dagger V_R^\dagger, \quad (93)$$

under $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$, where $V_L$ and $V_R$ are elements of the fundamental representation of $\text{SU}(N_f)_L$ and $\text{SU}(N_f)_R$ respectively. In terms of quark fields $q$ and $\bar{q}$, $\text{SU}(N_f)_L$ and $\text{SU}(N_f)_R$ act on the left and right handed components,

$$q_L = \frac{1 - \gamma_5}{2} q, \quad (94)$$

$$\bar{q}_L = \frac{1 + \gamma_5}{2} \bar{q}, \quad (95)$$

$$q_R = \frac{1 + \gamma_5}{2} q, \quad (96)$$

$$\bar{q}_R = \frac{1 - \gamma_5}{2} \bar{q}, \quad (97)$$

whose transformation properties are given by

$$q_L \rightarrow V_L q_L, \quad (98)$$

$$\bar{q}_L \rightarrow \bar{q}_L V_L^\dagger, \quad (99)$$

$$q_R \rightarrow V_R q_R, \quad (100)$$

$$\bar{q}_R \rightarrow \bar{q}_R V_R^\dagger. \quad (101)$$

We work all the calculations assuming this symmetry and at the end of calculations we can choose $\mathcal{M} = \mathcal{M}^\dagger = M$.

With this artifice let us consider which operators among $O_3, \ldots, O_{6f}$ in the quark action are allowed under $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$ symmetry. We easily find that the dimension three and five operators are not allowed. As for the dimension six operators, some four-fermi
operators are excluded. We also observe that the power corrections of $m_Q a$ emerge as the form

$$M^{2n} \cdot (SU(N_f)_L \times SU(N_f)_R \text{ invariant operators})$$

with $n \geq 0$, which means

$$\bar{q}(x) M^{2n+1} q(x),$$

$$\bar{q}(x) M^{2n} \gamma_\mu D_\mu q(x),$$

$$i\bar{q}(x) M^{2n+1} \sigma_{\mu\nu} F_{\mu\nu} q(x)$$

are allowed, while

$$\bar{q}(x) M^{2n} q(x),$$

$$\bar{q}(x) M^{2n+1} \gamma_\mu D_\mu q(x),$$

$$i\bar{q}(x) M^{2n} \sigma_{\mu\nu} F_{\mu\nu} q(x)$$

are forbidden. This may be advantageous in controlling the cutoff effects as $m_Q a$ becomes smaller away from one. Even for the chiral non-invariant quark action such as the SW quark action, however, the leading cutoff effects except for $Z_q$ and $Z_m$ are $(m_Q a)^n a \Lambda_{QCD}$ with $n \neq 0$, which are of the same order as those in the chirally symmetric actions, once the coefficient of $O_{5b}$ in the quark action is nonperturbatively tuned in the massless limit.

As for the improvement of the axial current, the similar argument can be applied. The dimension four operators $(A_{4a})_\mu$ and $(A_{4b})_\mu$ are not allowed by the chiral symmetry and the power corrections of $m_Q a$ are restricted to the form of eq. (102). As an example,

$$\bar{q}(x) M^{2n+1} \gamma_5 D_\mu \frac{\lambda^a}{2} q(x) + \bar{q}(x) M^{2n+1} D_\mu \gamma_5 \frac{\lambda^a}{2} q(x),$$

with $n \geq 0$ are allowed.

**D. $m_Q a$ corrections at tree-level**

In order to control the $m_Q a$ corrections it should be essential to nonperturbatively determine the renormalization factors $Z_m$ and $Z_q$ and the four parameters $\nu$, $r_s$, $c_E$ and $c_B$. 

20
However, we think it is instructive to first investigate the \(m_Qa\) corrections at the tree-level.

\(Z_q, Z_m, \nu\) and \(r_s\) can be determined by demanding that the tree-level quark propagator
\(S_q(p)\) derived from eq. (85) should reproduce the relativistic form
\[
S_q(p_0, p_i) = \frac{1}{Z_q} \frac{-i\gamma_0 p_0 - i \sum_i \gamma_i p_i + m_p}{p_0^2 + \sum_i p_i^2 + m_p^2} + \text{(no pole terms)} + O((p_i a)^2)
\] (110)
around the pole. Imposing \(p_i = 0\) we first obtain
\[
m_p = \log \left| \frac{m_0 + r_t + \sqrt{m_0^2 + 2r_t m_0 + 1}}{1 + r_t} \right|,
\] (111)
\[
Z_m = \frac{m_p}{m_0},
\] (112)
\[
Z_q = \cosh(m_p) + r_t \sinh(m_p).
\] (113)
We then find with finite spatial momenta
\[
\nu = \frac{\sinh(m_p)}{m_p},
\] (114)
\[
r_s = \frac{\cosh(m_p) + r_t \sinh(m_p)}{m_p} - \frac{\sinh(m_p)}{m_p^2}.
\] (115)
We should notice that the \(m_Qa\) corrections start at \(O((m_Qa)^2)\), not \(O(m_Qa)\), in the \(\nu\)
parameter as expected. Figure 6 illustrates the \(m_p a\) dependences of \(Z_q, Z_m, \nu\) and \(r_s\) in case
of \(r_t = 1\). We observe that the \(m_p a\) dependences of \(\nu\) is relatively mild compared to those
of \(Z_q, Z_m\) and \(r_s\).

To fix the \(c_E\) and \(c_B\) parameters we consider the quark-quark scattering amplitude depicted in Fig. 7. The improvement condition is that \(c_E\) and \(c_B\) should be chosen to reproduce the following form of scattering amplitude at the on-shell point removing the \(m_Qa\) corrections,
\[
T = -g^2 \bar{u}(p') \gamma_{\mu} u(p) D_{\mu\nu}(p - p') \bar{u}(q') \gamma_{\nu} u(q)
- g^2 \bar{u}(q') \gamma_{\mu} u(p) D_{\mu\nu}(p - q') \bar{u}(p') \gamma_{\nu} u(q) + O((p_i a)^2, (q_i a)^2, (p_i' a)^2, (q_i' a)^2),
\] (116)
where \(D_{\mu\nu}\) is the gluon propagator on the lattice. Notice that with the use of the Gordon
identity (; on-shell condition for external spinors \(u, \bar{u}\)) the quark-gluon interaction induced
by the clover term can be transformed into the ordinary quark-gluon vertex.
\[ \bar{u}(p') \sum_l i \sigma_0 (\sin(p'_l) - \sin(p_l)) u(p) \]
\[ = \bar{u}(p') \frac{1}{\nu} [\nu i \sin(p'_0) + i \sin(p_0) + \gamma_0 \{ 2(m_0 + r_t) - r_t (\cos(p'_0) + \cos(p_0)) \} \]
\[ + r_s \sum_l (2 - \cos(p'_l) - \cos(p_l)) ] u(p), \quad (117) \]
\[ \bar{u}(p') i \sigma_0 (\sin(p'_0) - \sin(p_0)) u(p) + \bar{u}(p') \sum_l \nu i \sigma_0 (\sin(p'_l) - \sin(p_l)) u(p) \]
\[ = \bar{u}(p') [\nu (i \sin(p'_0) + i \sin(p_0)) + \gamma_i \{ 2(m_0 + r_t) - r_t (\cos(p'_0) + \cos(p_0)) \} \]
\[ + r_s \sum_l (2 - \cos(p'_l) - \cos(p_l)) ] u(p). \quad (118) \]

This improvement procedure with the finite quark mass is an extension of the previous work [19] that determined the \( c_{SW} = c_E = c_B \) parameter up to one-loop level in the massless case. After some algebra with the aid of eqs. (117) and (118), we obtain
\[ c_E = r_t \nu, \quad (119) \]
\[ c_B = r_s, \quad (120) \]
where \( \nu \) and \( r_s \) are already determined from the on-shell improvement on the quark propagator.

You may have already noticed that our values for \( \nu, r_s \) and \( c_E \) are different from those derived in Ref. [3]. This difference originates from whether the on-shell improvement is implemented in the relativistic way or in the nonrelativistic way. (more precisely, in case that the Lagrangian does not retain the rotational invariance on the Euclidean space-time, we need both the momentum operator and the Hamiltonian to discuss the rotational symmetry.) For example, both methods give the same relation for the \( \nu \) and \( r_s \) parameters from the dispersion relation:
\[ \nu^2 + r_s \sinh(m_p) = \frac{\sinh(m_p)}{m_p} \{ \cosh(m_p) + r_s \sinh(m_p) \}. \quad (121) \]

In the nonrelativistic approach, however, \( \nu \) and \( r_s \) are not distinguishable due to the lack of relativistic informations. Generally speaking, we do not have sufficient number of nonrelativistic conditions to fix the coefficients of the relativistic operators with higher dimensions because the degrees of freedom of quarks are smaller in the nonrelativistic approximation.
compared to the relativistic case. Although the nonrelativistic approach with the Wilson type quark action [3] has been considered to work better than the NRQCD in the charm quark region where the sizable relativistic effects are expected, this is not necessarily assured because this approach does not meet the relativistic on-shell improvement.

E. Large $m_Qa$ and static limit

Although we have restricted ourselves to the case of finite $m_Qa$ so far, it is worthwhile to show that we can derive the static quark action from eq. (85) by taking $m_Qa \to \infty$. In terms of the heavy quark field $h(x)$ defined by

$$q(x) = \frac{e^{iEt}}{\sqrt{m_0}} h(x),$$

(122)

where $E = \pm im_p$, the lattice action in eq. (85) becomes

$$S_q^{\text{imp}} = \sum_x \left[ \left( 1 + \frac{r_t}{m_0} + \frac{3r_s}{m_0} \right) \bar{h}(x)h(x) - \frac{e^{iE}}{2m_0} \bar{h}(x)(r_t - \gamma_0)U_0(x)h(x + \hat{0}) - \frac{e^{-iE}}{2m_0} \bar{h}(x + \hat{0})(r_t + \gamma_0)U_0(x)\bar{h}(x) \right] + O \left( \frac{\nu, c_E, r_s, c_B}{m_0} \right).$$

(123)

Taking $m_0 \to \infty$ and setting $r_t = 1$ for simplicity, we obtain

$$S_q^{\text{imp}} \simeq \sum_x \left[ \bar{h}(x)h(x) - \bar{h}(x + \hat{0}) \frac{1 + \gamma_0}{2} U_0(x)\bar{h}(x) \right]$$

(124)

for the heavy quark ($E = im_p$), or

$$S_q^{\text{imp}} \simeq \sum_x \left[ \bar{h}(x)h(x) - \bar{h}(x) \frac{1 - \gamma_0}{2} U_0(x)h(x + \hat{0}) \right]$$

(125)

for the heavy anti-quark ($E = -im_p$), since

$$\frac{\nu}{m_0} = \frac{c_E}{m_0} \sim O \left( \frac{1}{m_p} \right),$$

(126)

$$\frac{r_s}{m_0} = \frac{c_B}{m_0} \sim O \left( \frac{1}{m_p} \right),$$

(127)

$$\frac{e^{mp}}{m_0} \simeq 1,$$

(128)
where \( m_p = \log|1 + m_0| \) is used. We replace \( \frac{1 + \omega}{2} h(x) = h(x) \) from the property that \( \gamma_0 h(x) = h(x) \) for the quark and \( \gamma_0 h(x) = -h(x) \) for the anti-quark. Thus we exactly obtain the static quark action, where the quark moves forward, or the static anti-quark action, where the anti-quark moves backward in time.

The existence of the static limit may give some constraints on the mass dependent scaling violation for \( m_Qa \gg 1 \). As an explicit example, we take the heavy-light pseudoscalar meson decay constant \( f_{HL} \). One can extract \( f_{HL} \) from the correlation function of the heavy-light current \( A_0^{HL}(x) = \bar{q}_H(x)\gamma_0\gamma_5q_L(x) \) with zero spatial momentum following

\[
Z_q^{(0)} \langle A_0^{HL}(t)A_0^{HL}(0) \rangle \simeq f_{HL}^2 m_{HL} e^{-m_{HL}t},
\]

where \( Z_q^{(0)} \) denotes the renormalization factor of the quark wave function at the tree-level. On the other hand, in the static limit, the current is given by \( \bar{h}(x)\gamma_0\gamma_5q_L(x) = \sqrt{m_0}e^{m_Qt}A_0^{HL}(x) \), and the correlation function

\[
m_0 \langle A_0^{HL}(t)A_0^{HL}(0) \rangle \simeq C^2 e^{-\Delta E t}
\]

has a well-defined limit. From the relation \( Z_q^{(0)} = e^{m_p} \simeq m_0 \) with \( r_t = 1 \) and \( m_{HL} \simeq m_Q + \Delta E \), we obtain

\[
f_{HL} = \frac{C}{\sqrt{m_{HL}}},
\]

Since the continuum limit can be taken in the static theory, \( C \) should behave as

\[
\frac{C}{\Lambda_{QCD}^{3/2}} = w_0 + w_k (a\Lambda_{QCD})^k + O((a\Lambda_{QCD})^{k+1})
\]

where \( k = 1 \) for the Wilson light quark action or \( k = 2 \) for the (nonperturbatively tuned) SW light quark action. Therefore the following relation holds for \( m_Qa \to \infty \):

\[
\frac{f_{HL}}{\Lambda_{QCD}} = \sqrt{\frac{\Lambda_{QCD}}{m_{HL}}} \left[ w_0 + w_k (a\Lambda_{QCD})^k + O((a\Lambda_{QCD})^{k+1}) \right].
\]

On the other hand, the decay constant should behave

\[
\frac{f_{HL}}{\Lambda_{QCD}} = v_0 (\Lambda_{QCD}/m_Q) \\
\times \left[ 1 + v_k^L (a\Lambda_{QCD})^k + v_n^H (m_Qa)(a\Lambda_{QCD})^n + O((a\Lambda_{QCD})^{k+1}, (a\Lambda_{QCD})^{n+1}) \right]
\]

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from our consideration in the previous sections, where $v_k^L$ is the scaling violation caused by the light quark action, and $v_n^H(m_Qa)$ comes mainly from the Wilson/SW heavy quark action for $n = 1$ or from the action of eq. (85) with the nonperturbatively tuned $\nu, r_s, c_E$ and $c_B$ for $n = 2$. Comparing eqs. (133) and (134), we find

$$v_0(\Lambda_{QCD}/m_Q) \rightarrow \sqrt{\frac{\Lambda_{QCD}}{m_{HL}}} w_0$$

(135)

and

$$v_k^L \rightarrow w_k, \quad v_n^H(m_Qa) \rightarrow 0 \quad \text{for } n < k$$

(136)

$$v_k^L + v_n^H(m_Qa) \rightarrow w_k, \quad \text{for } n = k$$

(137)

$$v_k^L \rightarrow w_k, \quad v_n^H(m_Qa) \rightarrow w_n \quad \text{for } n > k$$

(138)

in the limit $m_Qa \rightarrow \infty$, where $w_k$ and $w_n$ are universal constants independent of the choice of the heavy quark action. Note that we expect $v_0$ can be expanded in terms of $m_q/\Lambda_{QCD}$ near the chiral limit.

The important point is that the function $v_n^H(m_Qa)$ becomes constant (or even vanishes) in the limit $m_Qa \rightarrow \infty$. On the other hand, we also know that the behavior of $v_n^H(m_Qa)$ is benign under the condition $m_Qa \gtrsim 1$. From these facts it would be reasonable to assume that the function $v_n^H(m_Qa)$ behaves modestly all over the range of $m_Qa$ from the massless limit to the static limit. This assumption is supported by previous numerical results for the heavy-light decay constants $f_{Ds}$ and $f_{Bs}$ which are obtained by using the pole mass under the condition $m_Qa \gtrsim 1$ with the Wilson quark action. Figures 4 and 5 in Ref. [17] show that the naive continuum extrapolation with a linear form for $f_{Ds}$ and $f_{Bs}$ gives “reasonable” values in the continuum limit. This outcome is understood as follows: Since $v_n^H(m_Qa)$ is a modest function in terms of $m_Qa$, the leading scaling violation effects $v_1^H(m_Qa) a \Lambda_{QCD}$ for $f_{Ds}$ and $f_{Bs}$ are effectively removed by the naive linear extrapolation. However, the $O(a \Lambda_{QCD})$ cutoff effects cannot be completely removed. The remnant is estimated to be $|v_1^H(m_Qa_{\max}) - v_1^H(m_Qa_{\min})|a_{av} \Lambda_{QCD}$, where $a_{\max}$ and $a_{\min}$ are the maximum and minimum of the lattice spacing used for the continuum extrapolation, and $a_{av} = a_{\max}a_{\min}/(a_{\max} - a_{\min})$. 25
Although we have focused on the case of heavy-light pseudoscalar meson decay constants, the above arguments on scaling violation effects can be easily generalized to any observables which can be defined in the static limit.

**F. Nonperturbative renormalization**

Let us turn to a nonperturbative determination of $Z_q$, $Z_m$, $\nu$, $r_s$, $c_B$ and $c_E$. We first consider the $\nu$ and $r_s$ parameters. Since the rotational symmetry breaking due to the $m_Q a$ corrections deviates the $\nu$ and $r_s$ parameters from one, it would be a reasonable way to adjust them such that the correct dispersion relations are reproduced for some hadronic states. We think a set of dispersion relations for the heavy-heavy and heavy-light mesons is a good choice. A previous study demonstrated a clear distinction between the kinetic masses defined by $\partial^2 E_{HH}/\partial p_i^2$ for the heavy-heavy meson and $\partial^2 E_{HL}/\partial p_i^2$ for the heavy-light meson (see Fig. 1 in Ref. [17]), which tells us that the dispersion relations for the heavy-heavy and heavy-light mesons give two independent conditions to fix both $\nu$ and $r_s$ parameters. We point out that to avoid the ambiguities coming from the clover term it would be better to consider the dispersion relations of the spin averaged meson states over the pseudoscalar and vector channels. This is motivated by an observation that the clover term causes the hyperfine splitting.

Nonperturbative determination of $c_B$ and $c_E$ is a little bit troublesome. Although in the massless case the clover coefficient can be determined with the aid of the PCAC relation, this method does not work with the massive case. The reason is that the chiral symmetry allows the clover terms with the odd power of $ma$ corrections $(ma)^{2n+1+i\bar{q}(x)\sigma_{\mu\nu}F_{\mu\nu}q(x)} \ (n \geq 0)$ as discussed in Sec. III C. This implies that even the chirally symmetric quark actions, e.g., the domain wall and the overlap quark actions, suffer from this difficulty. However, we can at least evaluate $c_B$ and $c_E$ perturbatively up to one-loop level by extending the calculation in Sec. III D. In this case the remaining cutoff effects are of order $\alpha^2 a\Lambda_{QCD}$, which might be small enough for numerical studies.
As for a nonperturbative determination of renormalization factors, we consider the use of the Schrödinger functional (SF) method [20]. The renormalization of the quark mass is made through the renormalizations of the axial current and the pseudoscalar density using the PCAC relation:

$$\bar{m}(\mu = 1/L) = \frac{Z_A(\bar{g}, \bar{m}L, g, ma)\partial_\mu \bar{q}(x)\gamma_\mu \gamma_5 q(x)}{Z_P(\bar{g}, L, \bar{m}L, g, ma)\bar{q}(x)\gamma_5 q(x)}.$$  \hspace{1cm} (139)

where $L$ is the physical box size and $\bar{g}$ and $\bar{m}$ are the renormalized coupling and quark mass in the SF scheme. $Z_A$ and $Z_P$ with the finite quark mass are defined by

$$Z_A(\bar{g}, \bar{m}L, g, ma) = \frac{\sqrt{c_m f_1}}{f_A(x_0 = L/2)},$$  \hspace{1cm} (140)

$$Z_P(\bar{g}, L, \bar{m}L, g, ma) = \frac{\sqrt{c_m f_1}}{f_P(x_0 = L/2)},$$  \hspace{1cm} (141)

where $c_m$ is a mass-dependent constant, and $f_A$, $f_P$ and $f_1$ are the correlation functions given by

$$f_A(x_0) = -\frac{1}{3} \int d^3\bar{y}d^3\bar{z}\bar{q}(x)\gamma_\mu \gamma_5 q(x)\bar{\zeta}(\bar{y})\gamma_5 \zeta(\bar{z}),$$  \hspace{1cm} (142)

$$f_P(x_0) = -\frac{1}{3} \int d^3\bar{y}d^3\bar{z}\bar{q}(x)\gamma_5 q(x)\bar{\zeta}(\bar{y})\gamma_5 \zeta(\bar{z}),$$  \hspace{1cm} (143)

$$f_1 = -\frac{1}{3L^6} \int d^3\bar{u}d^3\bar{v}d^3\bar{y}d^3\bar{z}\bar{\zeta}'(\bar{u})\gamma_5 \zeta'(\bar{v})\bar{\zeta}(\bar{y})\gamma_5 \zeta($$  \hspace{1cm} (144)

with $\zeta$, $\zeta'$, $\bar{\zeta}$ and $\bar{\zeta}'$ the boundary quark fields. We illustrate the function $f_{A,P}$ and $f_1$ in Fig. 8. As for the boundary conditions we refer to the description in Ref. [21].

Although each of $Z_A$, $Z_P$, $\langle \bar{q}(x)\gamma_\mu \gamma_5 q(x) \rangle$ and $\langle \bar{q}(x)\gamma_5 q(x) \rangle$ has the power corrections of $ma$, they are canceled out in the combinations $Z_A\langle \bar{q}(x)\gamma_\mu \gamma_5 q(x) \rangle$ and $Z_P\langle \bar{q}(x)\gamma_5 q(x) \rangle$ and the $O((a\Lambda_{\text{QCD}})^2)$ uncertainties are left. This assures us to take the continuum limit for the renormalized matrix elements and also for the renormalized quark mass of eq. (139). These quantities, however, are defined in the SF scheme with the finite quark mass and hence different from those renormalized in the SF scheme at the massless point, which are exactly what we want. Therefore, we still need the finite renormalization factor to make the conversion between the two renormalization descriptions. This can be obtained by taking the continuum limit of the ratio

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for several $g'$ chosen to satisfy $ma' \ll 1$. We remark that the physical size $L$ for the SF scheme can be taken to be much smaller than that for the measurement of the spectral quantities and the various matrix elements, allowing us to access the finer lattice spacing with $g'$.

IV. CONCLUSIONS

In this paper we have first examined the validity of the idea of anisotropic lattice by making a perturbative calculation of the quark self energy up to $O(g^2a)$. Our results show that the $m_Qa_s$ corrections revive through one-loop diagram. We also find that on the anisotropic lattice the four parameters must be adjusted to remove all the terms of order $a$ even in the massless case. From a theoretical point of view the anisotropic lattice is not necessarily advantageous over the isotropic one.

In the second part of this paper we have presented a new relativistic approach to the heavy quarks on the lattice. The idea is based on the relativistic on-shell improvement with the finite $m_Qa$ corrections. We have shown that the cutoff effects can be reduced to $O((a\Lambda_{QCD})^2)$ putting the $(m_Qa)^n$ corrections on the renormalization factors of the quark mass $Z_m$ and wave function $Z_q$. As far as the spectral quantities such as hadron masses are concerned, the $(m_Qa)^n$ corrections in $Z_m$ and $Z_q$ do not matter: $Z_q$ does not affects on the spectral quantities and the $(m_Qa)^n$ corrections in $Z_m$ can be handled by employing the pole mass fixed from some spectral quantity. On the other hand, in case of calculating the quark mass or the various hadron matrix elements we can control them by determining the renormalization factors nonperturbatively.

The leading scaling violation for various types of actions are summarized as follows. $f_1(m_Qa)a\Lambda_{QCD}$ for the Wilson quark ($f_1(0) \neq 0$) and the $O(a)$ improved SW quark
\( f_1(0) = 0 \), while \( f_2(m_Q a)(a \Lambda_{QCD})^2 \) for our proposed action with mass-dependently tuned \( \nu, r_s, c_E \) and \( c_B \). Therefore, if the magnitude of \( f_1(m_Q a) \) is \( O(1) \), the scaling violation for heavy hadron masses with the \( O(a) \) improved SW quark might be as bad as that for light hadron masses with the ordinary Wilson quark. For sufficiently small \( m_Q a \), we can remove the leading scaling violations by extrapolating the data at several lattice spacings to the continuum limit. Even if \( m_Q a \sim O(1) \), \( f_2(m_Q a)(a \Lambda_{QCD})^2 \) in our proposed action is expected to be negligibly small in case of \( (a \Lambda_{QCD})^2 \sim 0.01 \).

Our relativistic approach has the strong point over the nonrelativistic ones: the finer the lattice spacing becomes, the better the approach works. This is a desirable feature because we can take the full advantage of configurations with finer lattice spacing generated to control the cutoff effects on the light hadron physics. We are going to perform a numerical simulation to test the ideas presented in this paper.

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FIG. 1. One-loop diagrams for the quark self energy.
FIG. 2. $\xi$ dependences of (a) $\Delta_q^{(0)}$ and (b) $\Delta_q^{(1)}$ in the renormalization constant of the quark wave function. $c_F$ and $c_B$ are chosen to be 1. Errors are within symbols.
FIG. 3. $\xi$ dependences of (a) $\Delta_m^{(0)}$ and (b) $\Delta_m^{(1)}$ in the renormalization constant of the quark mass. $c_E$ and $c_B$ are chosen to be 1. Errors are within symbols.
FIG. 4. $\xi$ dependences of (a) $\Delta^{(0)}_\nu$ and (b) $\Delta^{(1)}_\nu$ in the $\nu$ parameter. $c_E$ and $c_B$ are chosen to be 1. Errors are within symbols.
FIG. 5. $\xi$ dependences of $\Delta^{(0)}_\eta$ in the $\eta$ parameter. $c_E$ and $c_B$ are chosen to be 1. Errors are within symbols.
FIG. 6. Tree-level values for $Z_q, Z_m, \nu$ and $r_s$ as functions of $m_p a$. We choose $r_t = 1$. 
FIG. 7. Tree-level diagrams for the quark-quark scattering.

FIG. 8. Quark diagrams contributing to (a) $f_{A,P}(x)$ and (b) $f_1$. $O_{A,P}$ are inserted at the point $x$. $C$ and $C'$ denote the boundary conditions for the gauge fields at $t = 0$ and $T$. 
TABLE I. Expected magnitude of the cutoff effects due to the higher dimensional operators in
the quark action. $m_Q$ denote the heavy quark masses.

<table>
<thead>
<tr>
<th>operator</th>
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<tr>
<td>$\mathcal{O}<em>{5b} : \ i \sum_i \bar{q}(x)\sigma_0 i F</em>{ib} q(x)\ $</td>
<td>$a\Lambda_{\text{QCD}}$</td>
<td>$a\Lambda_{\text{QCD}}^2 / m_Q$</td>
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<tr>
<td>$\ i \sum_{ij} \bar{q}(x)\sigma_{ij} F_{ij} q(x)\ $</td>
<td>$a\Lambda_{\text{QCD}}$</td>
<td>$a\Lambda_{\text{QCD}}$</td>
</tr>
<tr>
<td>$\mathcal{O}_{6a} : \bar{q}(x)\gamma_0 D^3_0 q(x)\ $</td>
<td>$(a\Lambda_{\text{QCD}})^2 (m_Q a)^3 \bar{q}(x)q(x), \ (m_Q a)^2 \bar{q}(x)\gamma_i D_i q(x), \ (m_Q a)\bar{q}(x) D^2 q(x)\ $</td>
<td>$(a\Lambda_{\text{QCD}})^2 (a\Lambda_{\text{QCD}})^2$</td>
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<tr>
<td>$\bar{q}(x)\gamma_i D^2_i q(x)\ $</td>
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<tr>
<td>$\mathcal{O}<em>{6d} : \ i \sum_i \bar{q}(x)\gamma_i [D_0, F</em>{i0}] q(x)\ $</td>
<td>$(a\Lambda_{\text{QCD}})^2 a^2 \Lambda_{\text{QCD}}^3 / m_Q$</td>
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<tr>
<td>$\ i \sum_i \bar{q}(x)\gamma_0 [D_i, F_{i0}] q(x)\ $</td>
<td>$(a\Lambda_{\text{QCD}})^2 (a\Lambda_{\text{QCD}})^2$</td>
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<tr>
<td>$\ i \sum_i \bar{q}(x)\gamma_i [D_j, F_{ij}] q(x)\ $</td>
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