Regularized zero-range model and an application to the triton and the hypertriton

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Abstract

We examine the regularized zero-range model in an application to three-fermion systems – the triton and the hypertriton. We consider bound states and low-energy neutron-deuteron and lambda-deuteron scattering. The model is shown to provide an adequate quantitative description of these systems on a par with finite-range potential models. The well-known correlation between the doublet $nd$ scattering length and the triton binding energy (Phillips line) finds a natural explanation within the model.


Key-words: Zero-range potential; Regularization; Three-body problem; Effective field theory;

1 Introduction

The zero-range potential\(^1\) is a useful concept which employs separation of scales in physical systems [1]. The model allows selection of the relevant degrees of freedom and provides qualitative and often quantitative descriptions of intricate physical phenomena in an intuitive and transparent way. Conceptually it is equivalent to a coordinate space formulation of effective field theories [2].

Particularly, the zero-range model effectively reduces a three-body problem down to a one-body problem and thus provides a useful means of model independent insights into the important properties of various systems [3, 4, 5, 6].

However, a direct application of zero-range potentials to three-body systems presents a problem – a collapse of the system known as the Thomas effect [7]. To have a predictive power the zero-range model needs to be regularized.

Several momentum-space regularization approaches have been recently reported, which basically amount to a cut-off at high momentum [8, 9], or a combination of a cut-off and an additional three-body force [10].

A combination of an energy cut-off and an effective three-body force had also been used in a coordinate-space problem with $\delta$-interactions [11].

Recently we have introduced an alternative coordinate-space regularization approach [12]. Unlike [11] we do not explicitly use $\delta$-functions. The zero-range potential is formulated as a boundary condition for the wave-function at the

\(^1\)also referred to as “pseudo-potential” or “$\delta$-interaction”
origin. Regularization is achieved by a suitable generalization of the boundary condition and can be thought of as a soft cut-off.

In contrast to momentum-space theories [8, 9, 10] our approach does not require an additional three-body parameter to set the scale of the three-body binding energy.

The purpose of this paper is to test the suggested regularized zero-range model in an application to systems of fermions, considering both bound state problems and scattering processes, and comparing, where possible, with finite-range potential models and effective field theories.

We shall give a brief introduction to the regularized zero-range potential model, consider the hyper-spheric formalism of “1+2” scattering, and finally apply the model to the np and Λnp systems.

In the following we shall refer to the regularized zero-range potential as contact interaction.

### 2 Three-body problem with contact interactions

In this chapter we discuss hyper-spheric adiabatic solution of the three-body problem with contact interactions. We use the definition of the two-body Schrödinger equation with contact interaction as a free s-wave equation with the solution \( \psi_k(r) = \sin(kr + \delta)/r \), and the boundary condition at the origin [2, 12]

\[
\left. \frac{1}{r} \frac{d}{dr} (r \psi_k) \right|_{r=0} = k \cot \delta = \frac{1}{a} + \frac{1}{2} R k^2 + P R k^4,
\]

where \( \delta \) is the two-body phase shift, \( a \), \( R \), and \( P \) are the scattering length\(^2\), the effective range and the shape parameter and \( k \) is the wave-number. The \( k^2 \) and \( k^4 \) terms are necessary for the contact interaction model to have a regular three-body ground state solution [12].

#### 2.1 Hyper-spheric adiabatic wave-function

We use the hyper-spheric adiabatic approximation [13] which employs the hyper-spheric coordinates \( \rho \) and \( \Omega \) (see the appendix). The total wave-function \( \Psi \) of a three-body system is written as a product of a hyper-angular function \( \Phi(\rho, \Omega) \) and a hyper-radial function \( f(\rho) \)

\[
\Psi = \rho^{-5/2} f(\rho) \Phi(\rho, \Omega).
\]

The function \( \Phi(\rho, \Omega) \) is an adiabatic eigenfunction of the three-body Hamiltonian with a fixed hyper-radius \( \rho \). The corresponding eigenvalue \( \lambda(\rho) \) serves as an effective potential for the hyper-radial function \( f(\rho) \)

\[
\left( -\frac{\partial^2}{\partial \rho^2} + \frac{\lambda(\rho) + 15/4}{\rho^2} - \frac{Q(\rho)}{\hbar^2} E \right) f(\rho) = 0,
\]

where \( E \) is the total energy of the three-body system, \( m \) is the mass unit used in the definition of \( \rho \) (see the appendix), and

\[
Q(\rho) = \int d\Omega \Phi^*(\rho, \Omega) \frac{\partial^2}{\partial \rho^2} \Phi(\rho, \Omega).
\]

\(^2\)Note the sign convention for the scattering length in (1)
In most cases the one-channel adiabatic approximation (2) proves to be quite accurate [14]. However, for higher accuracy several adiabatic eigenfunctions might need to be included. Additional eigenfunctions are also needed, for example, for a description of break-up processes. Although inclusion of additional eigenfunctions is straightforward, in the present context we restrict ourselves to the lowest adiabatic channel only.

2.2 Adiabatic eigenvalue equation.

We assume that contact interactions act only on the s-waves and vanish everywhere except at the origin. The adiabatic eigenfunction $\Phi(\rho, \Omega)$ can be then suitably expressed in terms of the free s-wave Faddeev components $\varphi_\nu(\alpha_i)$

$$\Phi(\rho, \Omega) = \sum_{i=1}^{3} A_i(\rho) \frac{\varphi_\nu(\alpha_i)}{\sin(2\alpha_i)}, \quad (5)$$

$$\varphi_\nu(\alpha_i) = \sin \left[ \nu(\rho) \left( \alpha_i - \frac{\pi}{2} \right) \right], \quad (6)$$

where $\nu = \sqrt{\lambda + 4}$, and $\alpha_i$ is the $i$-th hyper-angle. In the following we shall refer to both $\nu$ and $\lambda = \nu^2 - 4$ as angular eigenvalues.

The quantities $A_i(\rho)$ and $\nu(\rho)$ are to be determined from the boundary conditions (1). The latter can be conveniently reformulated in terms of the hyper-angles as [12]

$$\frac{\rho}{\sqrt{\mu_i}} \left[ \frac{1}{a_i} + \frac{1}{2} R_i \left( \frac{\sqrt{\mu_i \nu}}{\rho} \right)^2 + P_i R_i^3 \left( \frac{\sqrt{\mu_i \nu}}{\rho} \right)^4 \right] = 0, \quad (7)$$

where $a_i$, $R_i$, $P_i$ and $\mu_i$ are the scattering length, effective range, shape parameter and the reduced mass (divided by the mass unit $m$) for the two-body system of particles $j$ and $k$. We assume here and in the following that the indices $\{i, j, k\}$ form a cyclic permutation of $\{1, 2, 3\}$.

For small $\alpha_i$ the wave-function $\Phi$ can be expanded as [13]

$$\sin(2\alpha_i)\Phi = A_i(\rho)\varphi_\nu(\alpha_i)$$

$$+ A_j(\rho)2\alpha_i \frac{\varphi_\nu(\phi_{ji})}{\sin(2\phi_{ji})} + A_k(\rho)2\alpha_i \frac{\varphi_\nu(\phi_{ki})}{\sin(2\phi_{ki})} + O(\alpha_i^2), \quad (8)$$

$$\phi_{ji} = \arctan \left( \frac{m_k(m_1 + m_2 + m_3)}{m_j m_i} \right). \quad (9)$$

The boundary conditions (7) then turn into a set of three linear equations for the coefficients $A_i(\rho)$

$$\sum_{j=1}^{3} M_{ij} A_j(\rho) = 0, \quad (10)$$

$$M_{ii} = \varphi_\nu'(0) - \varphi_\nu(0)L_i,$$

$$M_{ij} = 2 \frac{\varphi_\nu(\phi_{ji})}{\sin(2\phi_{ji})}, \quad i \neq j,$$
where $\varphi'_\nu(0) \equiv \frac{d\varphi_\nu(\alpha)}{d\alpha}|_{\alpha=0}$ and where we have used a short-hand notation

$$L_i \equiv \frac{\rho}{\sqrt{\mu_i}} \left[ \frac{1}{a_i} + \frac{1}{2} R_i \left( \frac{\sqrt{\mu_i}}{\rho} \right) \right]^2 + P_i R_i^3 \left( \frac{\sqrt{\mu_i}}{\rho} \right)^4] . \quad (11)$$

The set of equations (10) has nontrivial solutions only when the determinant of the matrix $M(\rho, \nu)$ is zero, which gives the equation to determine $\nu(\rho)$ and $\lambda(\rho) = \nu^2(\rho) - 4$,

$$\det M(\rho, \nu) = 0 . \quad (12)$$

3 "1+2" scattering

In this chapter we discuss the hyper-spheric formalism for the elastic scattering of a particle against a bound system of the other two. This process is often referred to as "1+2" scattering.

3.1 Adiabatic solution at large distances

A bound two-body subsystem – say, number 3 – with the binding energy $B = \hbar^2 k_3^2 / (2m\mu_3)$ gives rise to an eigenvalue $\nu(\rho)$ which at large $\rho$ asymptotically behaves as [12]

$$\nu(\rho) = \frac{i \rho}{\sqrt{\mu_3}} \kappa_3 , \quad (13)$$

$$\lambda(\rho) = -\frac{2mB}{\hbar^2} \rho^2 - 4, \quad (14)$$

with the corresponding (normalized) adiabatic wave-function (6) being

$$\varphi_\nu(\alpha_3) = \sqrt{2} \frac{\rho \kappa_3}{\sqrt{\mu_3}} \exp \left( -\frac{\rho \kappa_3}{\sqrt{\mu_3}} \alpha_3 \right) . \quad (15)$$

Indeed, $\varphi_\nu(\phi_{3i})$ vanishes exponentially for large $\rho$ and the 3-rd equation in (7) asymptotically decouples from the other two and turns into

$$i\nu = \frac{\rho}{\sqrt{\mu_3}} \left[ \frac{1}{a_3} + \frac{1}{2} R_3 \left( \frac{\sqrt{\mu_3}}{\rho} \right) ^2 + P_3 R_3^3 \left( \frac{\sqrt{\mu_3}}{\rho} \right)^4 \right] , \quad (16)$$

which with the substitution $\nu \rightarrow k\rho / \sqrt{\mu_3}$ takes the form of the usual equation for the two-body bound state

$$ik = k \cot \delta(k) \quad (17)$$

with the solution $ik_3$. Note that since $\varphi_\nu(\phi_{3i})$ vanishes exponentially with $\rho$ the correction to (13) is also exponentially small [12].

The additional term $Q$ in the hyper-radial equation (3) asymptotically approaches $Q \rightarrow -1/(4\rho^2)$, which can be directly verified from (15). Although this term is generally small it is important to ensure the correct asymptotics of the hyper-radial equation in the case of a bound two-body subsystem [12]. For simplicity we shall in the following always use only the leading term $-1/(4\rho^2)$,
often referred to as the Langer correction term. The hyper-radial equation then becomes
\[
\left( -\frac{\partial^2}{\partial \rho^2} + \frac{\lambda(\rho) + 4}{\rho^2} - \frac{2m}{\hbar^2} E \right) f(\rho) = 0,
\] (18)
with the correct "1+2" asymptotics at large \( \rho \)
\[
\left( -\frac{\partial^2}{\partial \rho^2} - \frac{2m}{\hbar^2} (E + B) \right) f(\rho) = 0.
\] (19)

3.2 Scattering wave-function and phase shift

Within the contact interaction model the two-body bound state wave-function simplifies to \( \exp(-\kappa_3 r_{12})/r_{12} \), where \( r_{12} \) is the relative coordinate. The wave-function describing an elastic scattering of the third particle against the bound system of the first two must then have the following asymptotic form
\[
\Psi \to \frac{1}{r_{12}} \exp(-\kappa_3 r_{12}) \frac{1}{r_{12}} \sin(k r_{(12)3} + \delta) ,
\] (20)
where \( r_{(12)3} \) is the relative distance between the third particle and the center of mass of the first two, \( \delta \) is the sought "1+2" phase shift, and the scattering energy is \( \hbar^2 k^2/(2m \mu_3) \).

We shall show now that the three-body wave-function (2) indeed takes this asymptotic form at large \( \rho \). The asymptotic form (19) of the hyper-radial equation has a solution
\[
f(\rho) = \sin(q \rho + \Delta) ,
\] (21)
where \( \Delta(q) \) is the hyper-radial phase and \( q^2 = 2m(E + B)/\hbar^2 > 0 \). Multiplying the hyper-radial function (21) and the adiabatic function (15) gives the total wave-function
\[
\Psi = \rho^{-5/2} \sin(q \rho + \Delta(q)) \frac{1}{\sin(2 \alpha_3)} \sqrt{\frac{2 \rho \kappa_3}{\sqrt{\mu_3}}} \exp\left( -\frac{\rho \kappa_3}{\sqrt{\mu_3}} \alpha_3 \right) .
\] (22)
For \( \rho \kappa_3/\sqrt{\mu_3} \gg 1 \) the exponent in (22) is non-vanishing only when \( \alpha_3 \ll 1 \). Therefore in this region \( \rho \alpha_3 = x \) and \( \rho = y \) (neglecting terms of the order of \( \alpha_3^2 \)) and the wave-function (22) becomes
\[
\Psi \simeq \frac{1}{x} \exp\left( -\frac{\kappa_3}{\sqrt{\mu_3}} x \right) \frac{1}{y} \sin(q y + \Delta(q)) .
\] (23)
Recalling that \( x = \sqrt{\mu_3} r_{12} \) and \( y = \sqrt{\mu_{(12)3}} r_{(12)3} \) (see the appendix) we obtain
\[
\Psi \simeq \frac{1}{r_{12}} \exp\left( -\kappa_3 r_{12} \right) \frac{1}{r_{(12)3}} \sin(k r_{(12)3} + \Delta(q)) ,
\] (24)
where \( k = q \sqrt{\mu_{(12)3}} \). This is exactly the sought asymptotic form (20). The hyper-radial phase shift \( \Delta(q) \) is then equal to the "1+2" phase shift \( \delta(k) \).

Thus the "1+2" phase shift \( \delta(k) \) can be easily calculated within the hyperspherical approach by solving the hyper-radial equation (3) with the wave-vector \( q = k/\sqrt{\mu_{(12)3}} \) and extracting the hyper-radial phase \( \Delta(q) \) from the asymptotic form (21). The "1+2" phase shift \( \delta(k) \) is equal to the hyper-radial phase \( \Delta(q) \). The "1+2" scattering length can then be found as the limit \( a^{-1} = \lim_{k \to 0} k \cot \delta \).
Table 1: Parameters of the contact interaction (1) in the $^3S_1$ (triplet) $np$ and $^1S_0$ (singlet) $nn$ channels. Here $a$ is the scattering length, $R$ – the effective range, $P$ – the shape parameter, $E_d$ is the energy of the bound $np$ state (deuteron), and $E_{virt}$ is the energy of the virtual (anti-bound) $nn$ state.

<table>
<thead>
<tr>
<th></th>
<th>$V_{np}(^3S_1)$</th>
<th>$a$, fm</th>
<th>$R$, fm</th>
<th>$P$</th>
<th>$E_d$, MeV</th>
</tr>
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<tr>
<td>I</td>
<td>-5.26</td>
<td>1.54</td>
<td>0.084</td>
<td>-2.2</td>
<td></td>
</tr>
<tr>
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<td>0.143</td>
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<td></td>
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<tr>
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<td>0.256</td>
<td>-2.2</td>
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<table>
<thead>
<tr>
<th></th>
<th>$V_{nn}(^1S_0)$</th>
<th>$a$, fm</th>
<th>$R$, fm</th>
<th>$P$</th>
<th>$E_{virt}$, MeV</th>
</tr>
</thead>
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<td>0.086</td>
<td>-0.1</td>
<td></td>
</tr>
<tr>
<td>II</td>
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<td>1.91</td>
<td>0.132</td>
<td>-0.1</td>
<td></td>
</tr>
<tr>
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<td>1.64</td>
<td>0.238</td>
<td>-0.1</td>
<td></td>
</tr>
</tbody>
</table>

4 Contact interactions for $np$ and $nn$ systems

The basic quantities that characterize the $np$ interaction at low energies are the phase shifts and the deuteron binding energy. We therefore choose the contact interaction parameters $a$, $R$, and $P$ in (1) by fitting the phase shifts while keeping the deuteron binding energy fixed at -2.2 MeV. For the $nn$ system we instead keep the energy of the virtual $s$-state at -0.1 MeV. Our parametrization does not allow change of sign of the phase shifts at large energies. Therefore we can only aim at describing the phases below 100 MeV for $nn$ and below 150 MeV for $np$ scattering.

For each pair of particles we consider three sets of parameters labelled I, II, and III (see Table 1), where II is more or less the best possible fit, while I and III represent small variations towards better description of correspondingly lower and higher energies.

The $np$ and $nn$ phase shifts for these models are shown, correspondingly, on Fig. 1 and Fig. 2. We have used $\hbar c=197.3$ MeV fm and $m_n=m_p =939$ MeV. Potentials I provide a better description at lower energies while overestimate the phases at larger energies. Potentials III are better in the higher energy region at the expense of the lower energies, while potentials II represent a reasonable compromise.

The larger $P$-parameters give smaller phase shifts at higher energy or, correspondingly, weaker interaction. One can say that the $P$-parameter smoothly cuts off the interaction at higher energies, thus regularizing the three-body problem [12].

5 Application to the $nnp$ system

The spins of three nucleons can couple to a total spin $S$ of 3/2 and 1/2, which for relative $s$-waves is also the total angular momentum. In the $S=1/2$ (doublet) case there is a bound state of a proton and two neutrons – a triton. No three-body bound state is found in the $S=3/2$ (quartet) case. We assume that the neutron and the proton have the same mass $m=939$ MeV, which is also used as the mass scale in the definition of $\rho$ (see appendix).
Figure 1: The $^3S_1$ np phase shift $\delta$ as function of the center-of-mass energy $E_{cm}$ for the contact interactions from Table 1. The circles represent the results from the Argonne $v_{18}$ potential [15].

Figure 2: The $^1S_0$ nn phase shift $\delta$ as function of the center-of-mass energy $E_{cm}$ for the contact interactions from Table 1. The circles represent the results from the Argonne $v_{18}$ potential [15].
5.1 Quartet \(nd\) scattering

5.1.1 Adiabatic eigenfunction

In the case of particles with spins we have to include also the spin-functions \(\zeta\) in the adiabatic wave-function

\[
\Phi(\rho, \Omega) = \sum_{i=1}^{3} \zeta_i A_i \frac{\varphi_\nu(\alpha_i)}{\sin(2\alpha_i)} .
\]

(25)

For \(S=3/2\) the spin functions \(\zeta_i\) are represented by

\[
\zeta_i = \uparrow_1 \uparrow_2 \uparrow_3 ,
\]

(26)

where \(\uparrow_i\) designates the spin-function of the \(i\)-th particle where the \(z\)-projection of the spin operator is equal to \(+1/2\).

Let the proton be particle number 1 and the neutrons number 2 and 3. Two neutrons with total spin 1 can not be in an s-state. Therefore the Faddeev component number 1 must vanish identically. Under exchange of the neutrons the angular functions \(\varphi_\nu(\alpha_2)\) and \(\varphi_\nu(\alpha_3)\) in (25) transform into each other while the spin functions \(\zeta_i\) do not change. Therefore the antisymmetric wave-function for \(S=3/2\) is

\[
\Phi(\rho, \Omega) = A_2 \zeta_2 \left( \frac{\varphi_\nu(\alpha_2)}{\sin(2\alpha_2)} - \frac{\varphi_\nu(\alpha_3)}{\sin(2\alpha_3)} \right) .
\]

(27)

5.1.2 Eigenvalue equation

The system of three equations (10) reduces in this case to only one equation which includes triplet neutron-proton parameters

\[
\varphi_\nu'(0) - \frac{4}{\sqrt{3}} \varphi_\nu\left(\frac{\pi}{3}\right) = L_{np} \varphi_\nu(0) ,
\]

(28)

where we have used that \(\phi_{ji} = \pi/3\) for three particles with the same mass and where \(L_{np}\), see eq. (11), contains triplet neutron-proton parameters \(a_{np}\), \(R_{np}\), and \(P_{np}\)

\[
L_{np} = \frac{\rho}{\sqrt{\mu}} \left[ \frac{1}{a_{np}} + \frac{1}{2} R_{np} \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^2 + P_{np} R_{np}^3 \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^4 \right] .
\]

(29)

Using the explicit expressions (6) for the functions \(\varphi_\nu\) the eigenvalue equation (28) can be written as

\[
\frac{\nu \cos(\nu \frac{\pi}{2}) + \frac{4}{\sqrt{3}} \sin(\nu \frac{\pi}{2})}{-\sin(\nu \frac{\pi}{2})} = \frac{\rho}{\sqrt{\mu}} \left[ \frac{1}{a_{np}} + \frac{1}{2} R_{np} \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^2 + P_{np} R_{np}^3 \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^4 \right] ,
\]

(30)

where we have dropped the index at the reduced mass \(\mu\) as we assumed that all three nucleons have the same mass.
Quartet adiabatic eigenvalue

\[ \lambda = 12 - \frac{3}{8} \left( \frac{\rho}{\sqrt{R}} \right)^{3} \frac{\rho^{3}}{\pi P_{np} R_{np}^{3}}. \]  

For large distances, due to the bound two-body subsystem (deuteron) the eigenvalue is asymptotically approaching

\[ \nu \to \frac{i}{\sqrt{R}} \kappa_{d}, \]  
\[ \lambda \to -\frac{2mB_{d}\rho^{2}}{\hbar^{2}} - 4, \]  

where \( B_{d} = 2.2 \text{ MeV} \) is the deuteron binding energy and \( \kappa_{d} = \sqrt{2m\mu_{d}/\hbar^{2}} \) \( (\kappa_{d} \approx 0.23 \text{ fm}^{-1}) \).

5.1.3 Adiabatic eigenvalues and results

The solutions \( \lambda(\rho) \) of the equation (30) for the three \( np \) potentials from Table 1 are shown on Fig. 3. The difference between the curves is not big and, moreover,
Table 2: Quartet $nd$ scattering length, $^4a_{nd}$. Here $V_{np}$ refers to the $np$ contact interaction from Table 1.

<table>
<thead>
<tr>
<th>$^4a_{nd}$, fm</th>
<th>$V^{(I)}_{np}$</th>
<th>$V^{(II)}_{np}$</th>
<th>$V^{(III)}_{np}$</th>
<th>EFT [16]</th>
<th>Exp. [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7.03</td>
<td>-6.86</td>
<td>-6.67</td>
<td>-6.33</td>
<td>-6.35</td>
<td>±0.02</td>
</tr>
</tbody>
</table>

the region, where the curves differ appreciably, is largely limited to $\rho \leq 5$ fm. However, in this region $\lambda + 4 > 0$ and, consequently, the effective potential $(\lambda + 4)/\rho^2$ in the hyper-radial equation (18) is repulsive. Therefore the hyper-radial wave-function in this region is small and thus insensitive to the small variations in the eigenvalues $\lambda$.

The quartet $nd$ scattering lengths calculated using these three adiabatic eigenvalues in the hyper-radial equation (18) are given in Table 2. All three contact interaction models give similar results which are very close to the experiment and to the result of Bedaque and van Kolck [16] obtained within the effective field theory approach.

5.2 Doublet $nd$ scattering and the triton

5.2.1 Adiabatic eigenfunction

For the doublet ($S=1/2$) state we have to include in the adiabatic function (25) three different spin-functions, $\sigma_{nn}$,$\tau_{2p}$, $\tau_{p3}$ where the two neutrons, $n_1$ and $n_2$, are in the singlet state while the neutron-proton pairs, $n_2p$ and $pn_1$ are in the triplet state.

\[
\sigma_{nn} = ((n_1 \otimes n_2) \otimes p)_{\frac{1}{2}} = \frac{1}{\sqrt{2}} (|1213 - 121\bar{3}|)_{\uparrow 1},
\]

\[
\tau_{2p} = ((n_2 \otimes p)_{\frac{1}{2}} \otimes n_1)_{\frac{1}{2}} = \sqrt{\frac{2}{3}} (|1\bar{3}1\bar{1}|)_{\downarrow 2} - \sqrt{\frac{1}{6}} (|131\bar{1} + 1\bar{3}1\bar{1}|)_{\uparrow 2},
\]

\[
\tau_{p3} = ((p \otimes n_1)_{\frac{1}{2}} \otimes n_2)_{\frac{1}{2}} = \frac{1}{2} (|1\bar{1}\bar{2}|)_{\downarrow 3} - \frac{1}{\sqrt{6}} (|1\bar{1}\bar{2} + 1\bar{1}\bar{2}|)_{\uparrow 3},
\]

where $(a \otimes b)_s$ denotes coupling of the spins of the particles (or systems of particles) $a$ and $b$ to the total spin $s$.

The wave-function must be antisymmetric under exchange of the two identical neutrons, which in our case are particles number 2 and 3. Under transformation $2\longleftrightarrow 3$ the wave-functions undergo the following transformations

\[
\varphi_\nu(\alpha_1) \rightarrow \varphi_\nu(\alpha_1), \quad \varphi_\nu(\alpha_2) \leftrightarrow \varphi_\nu(\alpha_3)
\]

\[
\sigma_{nn} \rightarrow -\sigma_{nn}, \quad \tau_{2p} \leftrightarrow \tau_{p3},
\]

where $\varphi_\nu(\alpha_i)$ are the usual solutions (6) to the free Faddeev equations. The antisymmetric hyper-angular function can therefore be written as

\[
\Phi(\rho, \Omega) = \sigma_{nn} A_1 \frac{\varphi_\nu(\alpha_1)}{\sin 2\alpha_1} + \tau_{2p} A_2 \frac{\varphi_\nu(\alpha_2)}{\sin 2\alpha_2} - \tau_{p3} A_2 \frac{\varphi_\nu(\alpha_3)}{\sin 2\alpha_3}.
\]
5.2.2 Eigenvalue equation

The hyper-angular function must satisfy the spin-projected boundary conditions

\[
\frac{\partial}{\partial \alpha_1} \left[ \alpha_1 \left( \sigma_{nn}^\dagger \Phi \right) \right]_{\alpha_1=0} = L_{nn} \left[ \alpha_1 \left( \sigma_{nn}^\dagger \Phi \right) \right]_{\alpha_1=0},
\]  
\[
\frac{\partial}{\partial \alpha_2} \left[ \alpha_2 \left( \tau_{2p}^\dagger \Phi \right) \right]_{\alpha_2=0} = L_{np} \left[ \alpha_2 \left( \tau_{2p}^\dagger \Phi \right) \right]_{\alpha_2=0},
\]

where \( L_{nn} \) and \( L_{np} \) (see eq. (11)) contain, respectively, the singlet neutron-neutron \((a_{nn}, R_{nn}, P_{nn})\) and triplet neutron-proton \((a_{np}, R_{np}, P_{np})\) parameters

\[
L_{nn} = \frac{\rho}{\sqrt{\mu}} \left[ \frac{1}{a_{nn}} + \frac{1}{2} R_{nn} \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^2 + P_{nn} R_{nn}^3 \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^4 \right],
\]
\[
L_{np} = \frac{\rho}{\sqrt{\mu}} \left[ \frac{1}{a_{np}} + \frac{1}{2} R_{np} \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^2 + P_{np} R_{np}^3 \left( \frac{\sqrt{\mu} \nu}{\rho} \right)^4 \right].
\]

Using the explicit form (37) leads to a system of linear equations for the coefficients \( A_i \)

\[
\begin{bmatrix}
A_1 \varphi_0'(0) + A_2 \left( \sigma_{nn}^\dagger \tau_{2p} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}} - A_2 \left( \sigma_{nn}^\dagger \tau_{p3} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}} \\
A_2 \varphi_0'(0) + A_1 \left( \tau_{2p} \sigma_{nn} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}} - A_2 \left( \tau_{2p} \tau_{p3} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}}
\end{bmatrix} = L_{nn} A_1 \varphi_0(0),
\]
\[
\begin{bmatrix}
A_2 \varphi_0'(0) + A_1 \left( \tau_{2p} \sigma_{nn} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}} - A_2 \left( \tau_{2p} \tau_{p3} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}}
\end{bmatrix} = L_{np} A_2 \varphi_0(0).
\]

Again this system of equations has nontrivial solutions only when the determinant of the corresponding matrix \( M \) is zero, which gives the eigenvalue equation

\[
\det M = 0,
\]

\[
\begin{pmatrix}
\varphi_\nu'(0) - L_{nn} \varphi_\nu(0) \\
\tau_{2p} \sigma_{nn}^\dagger \varphi_\nu(\pi/3) - L_{np} \varphi_\nu(0)
\end{pmatrix}
\begin{pmatrix}
\frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}} - \left( \sigma_{nn}^\dagger \tau_{p3} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}} \\
\varphi_\nu'(0) - \left( \tau_{2p} \tau_{p3} \right) \frac{4 \varphi_\nu(\pi/3)}{\sqrt{3}} - L_{np} \varphi_\nu(0)
\end{pmatrix} = 0,
\]

where the cross-products of the spin functions can be easily calculated from (35)

\[
\sigma_{nn}^\dagger \tau_{2p} = \frac{\sqrt{3}}{2}, \ \sigma_{nn}^\dagger \tau_{p3} = \frac{\sqrt{3}}{2}, \ \tau_{2p} \tau_{p3} = -\frac{1}{2}.
\]

Without the regularizing terms, that is when effective ranges and shape parameters are zero, the eigenvalue equation (44) has in the limit \( \rho \ll a \) an imaginary root \( \nu \cong 0.787 i \) which results in the Thomas collapse and the Efimov effect[18].

With non-zero effective ranges and shape parameters the equation (44) has a real root \( \nu(\rho = 0) = 0 \) and the Thomas collapse is therefore removed. The three-body system then has a well defined ground state. The Efimov effect can still exist in the limit \( a \rightarrow \infty \).

The large distance asymptotics is determined by the deuteron binding energy and therefore is the same as for the quartet case, eq. (33).
5.2.3 Adiabatic eigenvalues and results

Contrary to the quartet eigenvalue, which only depends on the \( np \) parameters, the doublet eigenvalue depends upon both \( V_{np} \) and \( V_{nn} \). Fig. 4 shows the solutions of the eigenvalue equation (44) for all possible combinations of \( V_{np} \) and \( V_{nn} \) from Table 1. The doublet effective potential \( \left( \lambda + 4 \right)/\rho^2 \) has an attractive pocket where variation is somewhat larger than in the quartet case. One also notices that all curves have basically the same shape, the only difference being in the depth of the attractive pocket. This property is essential for the discussion of the Phillips line in the next subsection.

Numerical solution of the hyper-radial equation (18) with these eigenvalues gives in each case one bound state, the triton, the energy of which is shown in Table 3 together with the corresponding doublet \( nd \) scattering length. Only the results for the corresponding potentials are shown in the table. Other combinations give the results which fall somewhere in between the shown results.

One can conclude that the contact interactions adequately describe the triton and the low-energy doublet \( nd \) scattering. The potentials II even provide a good quantitative description similar to finite-range potential models.

5.2.4 Phillips line

The Phillips line [19] is an established relationship between the triton binding energy and the doublet neutron-deuteron scattering length – different models...
Table 3: The triton ground state energy, $E_t$, and the doublet $nd$ scattering length, $^2a_{nd}$, for the nucleon-nucleon interactions from Table 1.

<table>
<thead>
<tr>
<th>$V_{nn}$</th>
<th>$V_{np}$</th>
<th>$E_t$, MeV</th>
<th>$^2a_{nd}$, fm</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>-11.4</td>
<td>2.82</td>
</tr>
<tr>
<td>II</td>
<td>II</td>
<td>-7.25</td>
<td>-0.61</td>
</tr>
<tr>
<td>III</td>
<td>III</td>
<td>-4.98</td>
<td>-3.15</td>
</tr>
<tr>
<td>Exp.</td>
<td></td>
<td>-8.48</td>
<td>-0.65</td>
</tr>
</tbody>
</table>

Figure 5: The Phillips plot of the doublet $nd$ scattering length $^2a_{nd}$ versus the triton binding energy $B_t$. The points indicate the results from different combinations of the contact interactions form Table 1. The solid line represents the finite-range potential models [20]. The dashed line represents the results from the renormalized effective field theory with three-body force [21].

Our Phillips line is reasonably close to but still somewhat different from the effective field theory curve. This indicates that the analytic properties of these models are different.

Now the question to answer is why do the results fall on a one-parameter curve while we in principle vary at least two independent ingredients – the $nn$ and $np$ interactions? In our model the answer can be most easily seen on
Fig. 4 where shown are the eigenvalues for all combinations of the $nn$ and $np$ interactions from Table 1. Due to the analytic properties of the eigenvalue equation (44) all curves start from $\lambda(\rho = 0) = 0$ and end up at $\lambda(\rho \gg R) = -2mB_d \rho^2 \hbar^2 - 4$. The shape of the curves is largely preserved – there are no crossings (with one exception) – and the difference is practically only in the depth of the attractive pocket. Therefore we have got basically a one-parameter family of the adiabatic potentials, the parameter being, for example, the depth of the pocket. The one-parameter family of adiabatic potentials would necessarily result in a single curve in the Phillips plot where this parameter is changed along the curve.

There are in fact two curves on Fig. 4 which, despite being almost identical, do cross. Correspondingly these curves produce two close points (around 6.3 MeV) on the Phillips plot, Fig. 5, which are slightly off the line.

Note that in the effective field theory approach it is the three-body cut-off parameter that is changed along the curve [21]. In our language that corresponds to fixing the two-body parameters, thus fixing the adiabatic potential, and instead varying the strength of the three-body force (one parameter). This would, of course, also lead to a single curve on the Phillips plot.

6 Application to $\Lambda np$ system

The $\Lambda np$ system has a weakly bound ground state, hypertriton, with the spin 1/2 and the binding energy of 0.13±0.05 MeV relative to the deuteron. Many investigations have been carried out with different interaction models (see [22, 23] and references therein) some of them including $\Lambda$-$\Sigma$ conversion [23]. The results exhibited considerable variations mostly due to uncertainties in the lambda-nucleon interaction.

One might expect that a certain correlation exists between the $\Lambda d$ scattering length and the hypertriton binding energy, similar to the Phillips line for the $nnp$ system. The purpose of this chapter is to establish the $\Lambda d$ Phillips line for our contact interaction model and compare it with the finite-range model. We shall ignore the $\Lambda$-$\Sigma$ conversion which is beyond the scope of this paper. The mass of the $\Lambda$ particle is assumed to be $m_\Lambda=1116$ MeV.

6.1 Contact interactions

For the $np$ subsystem we shall use the interaction II from Table 1. Due to lack of experimental scattering data we cannot use the same procedure for the $\Lambda N$ subsystem. Instead we simply fix the scattering length and the effective range and vary the shape parameter around the values similar to those of the $nn$ interaction. In [22] a set of low-energy parameters was found which gives a reasonable description of the hypertriton within the finite-range three-body model. We assume that $p \Lambda$ and $n \Lambda$ interactions are identical and adopt the scattering lengths and effective ranges in Table 4. They are close to the set $E1r$ in [22]. The singlet $P_s$ and triplet $P_t$ parameters are for simplicity assumed to be equal and are varied in the range 0.08-0.11.
Table 4: Parameters of the $^1S_0$ and $^3S_1$ Λ-nucleon contact interactions. The singlet and triplet shape parameters are assumed to be equal and are varied in the range 0.08-0.11.

<table>
<thead>
<tr>
<th></th>
<th>$a$, fm</th>
<th>$R$, fm</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1S_0$</td>
<td>1.9</td>
<td>3.5</td>
<td>0.08-0.11</td>
</tr>
<tr>
<td>$^3S_1$</td>
<td>1.5</td>
<td>3.7</td>
<td>0.08-0.11</td>
</tr>
</tbody>
</table>

6.2 Adiabatic eigenfunction and eigenvalue equation

We assume that the $np$ pair is in the triplet state while $pΛ$ and $nΛ$ pairs can be both in triplet and singlet states. We shall therefore use the following spin-functions

$$\tau_{np} = \left((n \otimes p)_{1/2} \otimes Λ\right)_+ = \sqrt{2/3} (|n\uparrow p\downarrow Λ\downarrow\Lambda\rangle - \sqrt{1/6} (|n\downarrow p\downarrow + \downarrow n\downarrow p\downarrow\uparrow\Lambda\rangle \downarrow\Lambda\rangle, \quad (46)$$

$$\sigma_{pΛ} = \left((p \otimes Λ)_{1/2} \otimes n\right)_+ = \sqrt{1/2} (|p\downarrow Λ\downarrow\downarrow n\uparrow Λ\rangle \uparrow n\rangle, \quad (47)$$

$$\tau_{pΛ} = \left((p \otimes Λ)_{1/2} \otimes n\right)_+ = \sqrt{2/3} (|p\downarrow Λ\downarrow\downarrow n\uparrow Λ\rangle \downarrow n\rangle - \sqrt{1/6} (|p\downarrow Λ\downarrow + \downarrow p\downarrow Λ\rangle \downarrow n\rangle, \quad (48)$$

$$\sigma_{nΛ} = \left((Λ \otimes n)_{1/2} \otimes p\right)_+ = \sqrt{1/2} (|Λ\downarrow n\downarrow - \downarrow Λ\downarrow n\rangle \uparrow p\rangle, \quad (49)$$

$$\tau_{nΛ} = \left((Λ \otimes n)_{1/2} \otimes p\right)_+ = \sqrt{2/3} (|Λ\downarrow n\downarrow \downarrow p\downarrow - \sqrt{1/6} (|Λ\downarrow n\downarrow + \downarrow Λ\downarrow n\rangle \uparrow p\rangle. \quad (50)$$

We assume that the wave-function is symmetric under exchange of the two nucleons. Since under exchange $n \leftrightarrow p$ the spin functions transform as

$$\sigma_{pΛ} \leftrightarrow -\sigma_{nΛ}, \tau_{pΛ} \leftrightarrow \tau_{nΛ}, \tau_{np} \leftrightarrow \tau_{np},$$

the symmetric hyper-angular function (25) can be written as

$$\Phi = A \frac{\varphi_1(α_1)}{\sin 2α_1} \tau_{np} + \frac{\varphi_2(α_2)}{\sin 2α_2} (Bσ_{pΛ} + Cτ_{pΛ}) + \frac{\varphi_3(α_3)}{\sin 2α_3} (-Bσ_{nΛ} + Cτ_{nΛ}). \quad (52)$$

For this system we have three spin-projected boundary conditions (7), the triplet $np$, and singlet and triplet $ΛΛ$

$$\tau_{np}^\dagger \left(\frac{∂}{∂α_1} 2α_1 Φ\right)_{α_1=0} = L_{np}^{(t)} τ_{np}^\dagger (2α_1 Φ)_{α_1=0}, \quad (53)$$

$$\sigma_{pΛ}^\dagger \left(\frac{∂}{∂α_2} 2α_2 Φ\right)_{α_2=0} = L_{ΛΛ}^{(s)} σ_{pΛ}^\dagger (2α_2 Φ)_{α_2=0}, \quad (54)$$

$$\tau_{pΛ}^\dagger \left(\frac{∂}{∂α_2} 2α_2 Φ\right)_{α_2=0} = L_{ΛΛ}^{(t)} τ_{pΛ}^\dagger (2α_2 Φ)_{α_2=0}, \quad (55)$$

where $s$ and $t$ refer to singlet and triplet states and where, as usual,

$$L_{ab}^{(c)} = \frac{ρ}{\sqrt{μ_{ab}}} \left(\frac{1}{a_{ab}} \frac{1}{2} P_{ab}^{(c)} \left(\nu \sqrt{μ_{ab}}/ρ\right)^2 + P_{ab}^{(c)} \left(P_{ab}^{(c)} \nu \sqrt{μ_{ab}}/ρ\right)^4 \right), \quad (56)$$
where \( \mu_{ab} = (1/m)a_b m_b/(m_a + m_b) \) is the reduced mass of particles \( a \) and \( b \) in units of \( m \). The resulting system of linear equations for the coefficients \( A, B, \) and \( C \) is

\[
A\phi'_\nu(0) + 2\frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} (B_{\nu \tau_{np}} \sigma_{pA} + C_{\nu \tau_{np}} \tau_{pA}) = AL^{(t)}_{\nu \tau_{np}} \phi'_0(0),
\]

\[
B\phi'_\nu(0) + A\frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \sigma_{pA} \tau_{np} + \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} (-B\sigma_{pA} \sigma_{nA} + C\sigma_{pA} \tau_{nA})
= BL^{(t)}_{\nu \tau_{np}} \phi'_0(0),
\]

\[
C\phi'_\nu(0) + A\frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \tau_{pA} \tau_{np} + \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} (-B\tau_{pA} \sigma_{nA} + C\tau_{pA} \tau_{nA})
= CL^{(t)}_{\nu \tau_{np}} \phi'_0(0).
\]

And, finally, the eigenvalue equation

\[
det M = 0,
\]

\[
\begin{pmatrix}
\phi'_\nu(0) - \varphi_\nu(0) L^{(t)}_{NN} & \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \tau_{pA} \tau_{np} & \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \tau_{pA} \tau_{nA} \\
\frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \sigma_{pA} \tau_{np} & \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \sigma_{pA} \tau_{np} & \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \sigma_{pA} \tau_{nA} \\
\frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \tau_{pA} \tau_{np} & \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \tau_{pA} \tau_{nA} & \frac{2\varphi_\nu(\phi_{23})}{\sin 2\phi_{23}} \tau_{pA} \tau_{nA}
\end{pmatrix}
\]

where the cross products of the spin-functions are

\[
\tau_{np} \sigma_{pA} = \frac{\sqrt{3}}{2}, \tau_{np} \tau_{pA} = -\frac{1}{2}, \tau_{np} \tau_{nA} = -\frac{\sqrt{3}}{2},
\]

\[
\sigma_{pA} \sigma_{nA} = -\frac{1}{2}, \sigma_{nA} \tau_{nA} = -\frac{\sqrt{3}}{2},
\]

\[
\tau_{pA} \tau_{nA} = \frac{\sqrt{3}}{2}, \tau_{pA} \tau_{nA} = -\frac{1}{2}.
\]

### 6.3 Eigenvalues and results

The solutions of the eigenvalue equation (60) with parameters from Table 4 are shown on Fig. 6. Again the asymptotics at both \( \rho=0 \) and \( \rho \gg R \) are independent on the shape parameter. The agreement between the curves in the asymptotic region is better than in the triton case because here we keep the two-body scattering lengths and effective ranges fixed (Table 4) while in the former case they were slightly varied (Table 1). The depth of the attractive pocket, however, does show some variation with the shape parameter.

The corresponding hypertriton binding energies and doublet \( \Lambda d \) scattering lengths are shown on the Phillips plot, Fig. 7, together with the results from finite-range models [22]. The agreement is even better than in the triton case, although the curve for the contact interactions is still marginally below. Basically the contact interaction model proves to be on a par with the finite-range model.

Note that this is not a simple repetition of the triton results as the hypertriton has a very different structure – the \( \Lambda \) particle is not identical to the neutron,
**Figure 6:** Doublet adiabatic eigenvalue $\lambda$ as function of $\rho$ for $\Lambda np$ system with contact interactions from Table 4.

**Figure 7:** The Phillips plot of the doublet $\Lambda d$ scattering length $2a_{\Lambda d}$ versus the triton binding energy relative to deuteron ($B_d=2.2$ MeV). The curve is obtained from the contact interaction model. The circles represent finite-range models [22].
which results in quite different angular functions (see (37) and (52)), and also does not form a bound state with the proton.

The better agreement reflects also the fundamental property of weakly bound systems – halos [24] – of which the hypertriton is a remarkable example, to be insensitive to the properties of the underlying interaction models.

7 Conclusion

We have applied the recently formulated regularized zero-range model to fermionic three-body systems \( nnp \) and \( \Lambda n p \). The model employs low-energy two-body parameters, scattering length and effective range, and the shape parameter. The three-body problem is solved within the hyper-spheric adiabatic approach where the zero-range model allows analytic solutions for the adiabatic hyper-angular functions.

The large-distance asymptotics of the adiabatic potential is completely determined by the scattering length and effective range and therefore is model-independent. The shape parameter mostly determines the depth of the attractive pocket and is thus important for the actual value of the binding energy.

Our regularization approach does not require an additional three-body parameter in contrast to the alternative momentum space regularizations. The finite triton binding energy is obtained using only two-body data, much like with the finite-range models.

Once the two-body parameters are fit to the nucleon-nucleon phase shifts the model provides an adequate quantitative description of the \( nnp \) system – the bound state, triton, and the low-energy \( nd \) scattering. The established correlation between the doublet \( nd \) scattering length and the triton binding energy, the Phillips line, finds a natural explanation within the model.

For the \( \Lambda n p \) system, with its weakly bound state, hypertriton, the Phillips lines for zero-range and finite-range models are almost identical, reflecting the fact that the topical weakly bound systems, halos, are residing mostly at large distances, where the adiabatic potential is independent upon the underlying potential model. The regularized zero-range model is therefore ideally suited for such systems.

The simple analytic functions of the model might serve as a convenient Sturmian basis in other three-body applications.

In conclusion the regularized zero-range potential model can be effectively used for quantitative descriptions of various three- and many-body systems in discrete and continuum spectra.

8 Acknowledgements

We thank E.Nielsen and J.H.Macek for fruitful discussions.
9 Hyper-spheric coordinates

If $m_i$ and $r_i$ refer to the $i$-th particle then the hyper-radius $\rho$ and the hyper-angles $\alpha_i$ are defined in terms of the Jacobi coordinates $x_i$ and $y_i$ as [25]

\[ x_i = \sqrt{\mu_i} (r_j - r_k), \quad y_i = \sqrt{\mu_{(jk)i}} \left( r_i - \frac{m_j r_j + m_k r_k}{m_j + m_k} \right), \]

\[ \mu_i = \frac{m_j m_k}{m_j + m_k}, \quad \mu_{(jk)i} = \frac{1}{m} \frac{m_i (m_j + m_k)}{m_j + m_k} \]

\[ \rho \sin(\alpha_i) = x_i, \quad \rho \cos(\alpha_i) = y_i, \]

where $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$ and $m$ is an arbitrary mass.

The set of angles $\Omega_i$ consists of the hyper-angle $\alpha_i$ and the four angles $x_i/|x_i|$ and $y_i/|y_i|$. The kinetic energy operator $T$ is then given as

\[ T = T_\rho + \hbar^2 \frac{\Lambda}{2m \rho^2}, \quad T_\rho = -\frac{\hbar^2}{2m} \left( \rho^{-5/2} \frac{\partial^2}{\partial \rho^{5/2}} \rho^{5/2} - \frac{1}{\rho^2} \frac{15}{4} \right), \]

\[ \Lambda = -\frac{1}{\sin(2\alpha_i)} \frac{\partial^2}{\partial \alpha_i^2} \sin(2\alpha_i) - 4 + \frac{l_{x_i}^2}{\sin^2(\alpha_i)} + \frac{l_{y_i}^2}{\cos^2(\alpha_i)}, \]

where $l_{x_i}$ and $l_{y_i}$ are the angular momentum operators related to $x_i$ and $y_i$.

References


