Quantum dissipative chaos in the statistics of excitation numbers

Gagik Yu. Kryuchkyan and Suren B. Manvelyan
Yerevan State University, Manookyan 1, Yerevan, 375049, Armenia
and
Institute for Physical Research, National Academy of Sciences,
Ashtarak-2, 378410, Armenia

A quantum manifestation of chaotic classical dynamics is found in the framework of oscillatory numbers statistics for the model of nonlinear dissipative oscillator. It is shown by numerical simulation of an ensemble of quantum trajectories that the probability distributions and variances of oscillatory number states are strongly transformed in the order-to-chaos transition. The nonclassical, sub-Poissonian statistics of oscillatory excitation numbers is established for chaotic dissipative dynamics in the framework of Fano factor and Wigner functions. These results are proposed for testing and experimental studing of quantum dissipative chaos.

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What is the quantum manifestation of classical chaotic dynamics and what kind of macroscopic quantum effects assist to chaotic behavior? These are important but rather difficult questions pertaining to many problems of fundamental interest [1]. The consideration of these questions for an open time-dependent quantum system is the subject of this Letter.

The study of quantum dynamics for isolated or so called Hamiltonian systems, the classical counterparts of which are chaotic has a long history. The majority of studies focus on static properties as spectral statistics of energy levels and the transition probabilities between eigenstates of the system. A variety of studies have also been carried out for understanding the features of time-dependent chaotic systems [2]. By contrast to that, very little works have been done to investigate the quantum chaos for open nonlinear systems. The beginning of study of an open chaotic system can be dated back to the work of Ott et al. [3], and the papers of Graham and Dittrich [4], where the authors have analyzed the kicked rotor and similar systems with discrete time. It is obvious, that the investigations of quantum chaotic systems are connected with the quantum-classical correspondence problem in general and with environment induced decoherence and dissipation in particular. Recently this topic has been the focus of theoretical investigations. As part of these studies it has been recognized by Zurek and Paz [5] and in the later works [6] that the decoherence has rather unique properties for systems the classical analogues of which are chaotic. From the experimental viewpoint, observation of dissipative effects and environment induced decoherence of dynamically localized states in the quantum delta-kicked rotor is carried out with the gas of ultracold cesium atoms in a magneto-optical trap subjected to a pulsed standing wave [7,8]. Recently, new problems of chaotic motion has been studied in the experimental scheme with ultra-cold atoms in magneto-optical double-well potential [9].

Nowadays there is no universally accepted definition of quantum chaos. Among criteria suggested for definition of chaos in open quantum systems we single out ones which are based on entropy production and Wigner functions [4]-[6]. In spite of these important developments in the investigation of chaos for open quantum systems, there are still many open questions, and there is a clear need in new available for experiment models, as well as comparatively more simple physical criterions for testing dissipative quantum chaos.

The first purpose of this Letter is to investigate the order-to-chaos transition at the level of statistics of elementary excitations for the quantum model of nonlinear oscillator. We show below that the distributions of oscillatory occupation numbers can be used to distinguish between the ordered and chaotic quantum dissipative dynamics.

The requirement in realization of this study is to have a proper quantum model showing both the regular and chaotic dynamics in the classical limit. We propose a nonlinear oscillator driven by two forces at different frequencies for this goal. This model was proposed to study the quantum stochastic resonance in our previous paper [10], where it was shown in detail that the model is apt to verification in experiments.

Our second purpose is to identify the kind of statistics of oscillatory number states taking place for quantum chaos. Our central result here is that nonclassical, sub-Poissonian statistics can be realized for chaotic dynamics of the system under consideration.

Open quantum systems are usually studied in the framework of reduced density matrix obtained by tracing over the degrees of freedom of environment. Recently, Spiller and Ralph [11] have described a dissipative chaotic system on individual quantum trajectories in the framework of quantum state diffusion method (QSD) [12]. More detailed studies have been performed in [13]. In addition to these studies, here we describe the quantum dissipative chaos using a statistical ensemble of trajectories.

The evolution of the system of interest is governed by the following master equation for the reduced density matrix in the interaction picture

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}\left[H,\rho\right] + \gamma\left(a\rho a^{+} - \frac{1}{2}a^{+}a\rho - \frac{1}{2}\rho a^{+}a\right), \tag{1}$$

where the Hamiltonian

$$H = \hbar\Delta a^{+}a + \hbar\left[\left(\Omega_1 + \Omega_2\exp\left(-i\delta t\right)\right)a^{+} + \left(\Omega_1^{*} + \Omega_2^{*}\exp\left(i\delta t\right)\right)a\right] + \hbar\chi(a^{+}a)^2 \tag{2}$$

describes an anharmonic oscillator with oscillatory frequency ω_0 driven by two periodic forces at frequencies ω_1 and ω_2. The couplings with two driving forces are given by Rabi frequencies Ω_1 and Ω_2, and χ is the strength of anharmonicity. Here $\Delta = \omega_0 - \omega_1$ is the detuning, $\delta = \omega_2 - \omega_1$ is the difference between driving frequencies, that plays the role of modulation frequency in the interaction picture, and a, a^{+} are boson annihilation and creation operators. The last terms in Eq. (1) concern the influence of the environment induced diffusion, where γ is the spontaneous decay rate of the dissipation process.

For $\Omega_2 = 0$ this equation describes the single driven, dissipative anharmonic oscillator, which is a well-known and archetypal model in nonlinear physics [14]. In case of double driven oscillator ($\Omega_2 \neq 0$), the Hamiltonian (2) is explicitly time-dependent and the system exhibits regions of regular and chaotic motion. In the classical limit, the corresponding equation of motion for the dimensionless amplitude $\alpha(t) = \langle a(t)\rangle$ has the form

$$\frac{d}{dt}\alpha = -\frac{1}{2}\gamma\alpha - i\left(\Delta + \chi(1 + 2\left|\alpha\right|^2)\right)\alpha - i\left(\Omega_1 + \Omega_2\exp\left(-i\delta t\right)\right). \tag{3}$$

As is well known, in study of the transition from order-to-chaos in classical systems a useful tool is an examination of a constant phase map in the phase-space. Indeed, our numerical analysis of the classical equation of motion in the (X, Y) plane ($X = \mathrm{Re}\,\alpha$, $Y = \mathrm{Im}\,\alpha$) shows that the classical dynamics of the system is regular in domains of small and large values of modulation frequency, i.e. $\delta \ll \gamma$ and $\delta \gg \gamma$, and also when one of the perturbation forces is much greater than the other: $\Omega_1 \ll \Omega_2$ or $\Omega_2 \ll \Omega_1$. The dynamics is chaotic in the range of parameters $\delta \gtrsim \gamma$ and $\Omega_1 \simeq \Omega_2$, where classical strange attractors for Poincaré section are realized.

The proposed model can not be described analytically. Our numerical analysis is based on QSD approach that represents the reduced density operator by the mean over the projectors onto the stochastic states $|\Psi_\xi\rangle$ of the ensemble: $\rho(t) = M\left(|\Psi_\xi\rangle\langle\Psi_\xi|\right)$, where M denotes the ensemble averaging.

We start by looking at the oscillatory mean excitations number $\langle n\rangle = M(\langle\Psi_\xi|\, a^{+}a\,|\Psi_\xi\rangle)$. We verified that in both cases of regular and chaotic dynamics this quantity exhibits a periodic time-dependent behavior, which is approximately sinusoidal with a period $2\pi/\delta$. Contrary to that, the classical results for $n = |\alpha|^2$ obtained from Eq.(3), shows usual chaotic behavior, corresponding to the classical attractor. Thus, we see that the quantum manifestation of chaotic dissipative dynamics is not evident on the mean oscillatory number. It should be clear that in the quantum statistical theory the distinction between regular and chaotic dynamics will be displayed on the quantities having essentially quantum nature. We will quantitatively demonstrate this point by consideration of both the probability distribution of oscillatory excitation numbers $P_n = \langle n|\,\rho\,|n\rangle$, where $|n\rangle$ is the number states, and the Fano factor which describes the excitation number uncertainty, normalized to the level of fluctuations for coherent states, i. e. $F = \left\langle(\Delta n)^2\right\rangle/\langle n\rangle$, $\left\langle(\Delta n)^2\right\rangle = \left\langle(a^{+}a)^2\right\rangle - \langle a^{+}a\rangle^2$. This investigation will be complemented by testing the quantum chaos on phase-space with the help of the Wigner function. We use for computations the expression for the Wigner function in terms of the density matrix elements $\rho_{nm} = \langle n|\,\rho\,|m\rangle$, and the matrix elements $W_{mn}(r,\theta)$ of the Wigner characterization function [15].

Let us first consider the quantities of interest for the region of classically regular behavior with parameters: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, $\Omega_1/\gamma = 27$, $\Omega_2/\gamma = 35$, and $\delta/\gamma = 5$. Figure 1 shows the Wigner function at the fixed moments of time $t_n = [7.13 + (2\pi/\delta)n]\gamma^{-1}$, $(n = 0, 1, 2, ...)$ exceeding transient time. We find that the Wigner function located around the point $X = 0$, $Y = -10$ and its contour-plot has narrow crescent form with the origin of phase space as its centrum. The radial squeezing that represents the known property of the anharmonic oscillator model to produce the excitation number squeezing is also clearly seen in the figure. The important novelty here is that the radial squeezing effect in this model is much stronger, than an analogous one for the model of single driven anharmonic oscillator. Below we will quantitatively demonstrate this by analyzing the Fano factor. Another peculiarity is that the Wigner function is nonstationary. As the calculations show, during the period of modulation $2\pi/\delta$ it is rotated around the origin of the phase-space. In Fig.2 (curve 1) one can see the time-evolution of Fano factor, which shows the formation of nonclassical sub-Poissonian statistics ($\left\langle(\Delta n)^2\right\rangle < \langle n\rangle$) for time intervals exceeding the transient time. The Fano factor reaches its minimum $F_{\min} \simeq 0.12$ and maximum $F_{\max} \simeq 0.65$ values for definite time intervals. One can account for surprisingly high sub-Poissonian statistics only by the quantum nature of oscillatory excitations under the influence of two driving forces. Indeed, for $\Omega_2 = 0$, we have $F \simeq 0.35$ for the same parameters: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, and $\Omega_1/\gamma = 27$ as in Fig.2 (curve 1).

We are now in a position to study the emergence of quantum chaos, which is expected to manifest itself as crucial changes in above results for coming into the classically chaotic operational regime, with parameter values: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, $\Omega_1/\gamma = \Omega_2/\gamma = 27$, and $\delta/\gamma = 5$. In this range the oscillatory mean excitations number oscillates between $\langle n\rangle = 70 \div 130$. Now we pay attention to the Fano factor. Its time evolution is shown in Fig.2 (curve 2). Surprisingly, the excitation-number fluctuations are also squeezed below the coherent level for the considered chaotic regime. However, opposite to the previous regular regime, the excitation number exhibits both the sub-Poissonian ($F < 1$) and super-Poissonian ($F > 1$) statistics, that are alternating in definite time intervals. The minimum and maximum values of F in time intervals during one modulation period are equal to $F_{\min} \simeq 0.30$ and $F_{\max} \simeq 1.98$. Thus, Fig.2 shows the drastic difference between the behavior of Fano factor for regular and chaotic dynamics. It means that the variance of oscillatory number fluctuations may be used for testing of quantum chaos.

It is tempting to explain the emergence of nonclassical sub-Poissonian statistics in the double driven nonlinear oscillator at the transition from regular to chaotic dynamics using the phase-space symmetry properties of the Wigner function. The results of ensemble-averaged numerical calculations of both the Wigner function and of its contour-plot at fixed time intervals $t_n = [6.96 + (2\pi/\delta)n]\gamma^{-1}$, $(n = 0, 1, 2, ...)$ are shown in Figs. 3 (a, b) respectively. One can make sure of that by comparing the contour-plots of Wigner function for sub-Poissonian and super-Poissonian statistics. As we see the contour-plot for chaotic motion still has the radial squeezed form [see Fig.3(b)]. This result takes place for $t_n = [6.96 + (2\pi/\delta)n]\gamma^{-1}$, $(n = 0, 1, 2, ...)$ at which the Fano factor reaches its minimum value $F_{\min} \simeq 0.30$. In the next time intervals during the period of modulation the level of excitation number fluctuations increases, and as a result the radial squeezing in contour-plot decreases.

In the search for more promising and easily attainable in experiments criterion of quantum chaos, we consider the probability distribution of oscillatory excitation number $P_n = \langle n|\rho|n\rangle$. We present in Fig. 4 the results for both regular (a) and chaotic (b) regimes at two time moments corresponding to $F_{\min}$ (curve 1) and $F_{\max}$ (curve 2). One can conclude comparing these figures that the probability distributions P_n are strongly transformed in the order-to-chaos transition. While P_n for regular dynamics is clearly bell-shaped and localized on narrow intervals of oscillatory numbers, the distribution for chaotic dynamics is flat-topped with oscillatory numbers from $n = 0$ to $n_{\max} \simeq 200$. Moreover, the shape of distributions changes irregularly in time during the period $2\pi/\delta$. Especially typical for chaotic motion is the result shown in Fig. 4(b) (curve 2), where the probability distribution is almost equally probable.

We stress, that the number of possible experimental schemes demonstrating the proposed model can be achieved. One of those is a single relativistic electron in a Penning trap, which is a realization of anharmonic oscillator as was predicted theoretically by Kaplan [16] and experimentally realized by Gabrielse and co-workers [17]. In the presence of two microwave electromagnetic fields this system gives an example of double driven anharmonic oscillator and may be used for demonstration of quantum dissipative chaos. This system can be governed by Eq. (1), where the operators a and a^+ describe the cyclotron quantized motion, Rabi frequencies Ω_1 and Ω_2 characterize the amplitudes of the microwave driving fields, χ is the strength of the anharmonicity due to relativistic effects, and γ is the spontaneous decay rate of the cyclotron motion. In this case the testing of chaos may be carried out by quantitative measurement of the statistics of cyclotron excitations.

In conclusion, we have found the quantum-statistical effects that accompany the chaotic dynamics. These results were obtained for the model of dissipative anharmonic oscillator, which has been proposed for studies of quantum chaos. The model is different from that of kicked rotor and similar models with discrete time and seems suitable for experimental configurations in quantum optics with continuous cw laser. It is demonstrated that the oscillatory excitation numbers statistics could be used as a diagnostic for quantum chaos. Indeed, we have shown that such measurable quantities as the Fano factor and probability distributions of number states are drastically changed in the order-to-chaos transition. But perhaps even more intriguing are the results that nonclassical, sub-Poissonian statistics of oscillatory number states is realized for chaotic dissipative dynamics. One of the consequences of this behavior are the symmetry property of Wigner functions in the phase space for chaotic dynamics. The results of our numerical work were obtained under conditions of strong anharmonicity $\chi/\gamma \lesssim 1$, for the value $\chi/\gamma = 0.1$, which is close to those actually achieved in the experiments with trapped relativistic electron. We believe that the results obtained are applicable to more general systems in quantum regime corresponding to classical chaotic dynamics.

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FIGURE CAPTIONS

Fig. 1. The Wigner function for the regular regime.This and all other numerical results are averaged over 3000 trajectories.

Fig. 2. The Fano factor versus dimensionless time for the regular (curve 1) and chaotic (curve 2) regimes. The parameters are: $\chi/\gamma = 0.1$, $\Delta/\gamma = -15$, $\Omega_1/\gamma = 27$, $\delta/\gamma = 5$,and $\Omega_2/\gamma = 35$ (curve 1), $\Omega_2/\gamma = 27$ (curve 2)

Fig. 3. The Wigner function (a) and its contour-plot (b) in the chaotic regime.

Fig. 4. Probability P_n of finding the system in the state $|n\rangle$ at different time intervals (curves 1 and 2) and for regular (a) and chaotic (b) regimes. The parameters for (a) and (b) coincides with ones for Fig. 2 curve1 and curve 2 respectively.