We consider the boundary conditions on \( Z \) and the field strength \( F \). The field strength is \( F = \text{const} \). The boundary condition is the standard one, while the field strength is \( F = \text{const} \).

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where \( \partial_{\mu} \) is the partial derivative.

General Mechanism

In the context of the standard model, we consider the following boundary conditions on \( Z \) and \( F \). The boundary conditions are the standard ones, while the field strength is \( F = \text{const} \).

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Introduction

Quantum cohomology is a field of study in theoretical physics and mathematics. It is concerned with the quantum behavior of fields and the cohomology of the space of fields. The mathematical framework of quantum cohomology is based on the concept of quantum groups and quantum algebras.

We consider the cohomology of the quantum algebra and its quantum group.

\[
\text{Hom}(\text{algebra}, \text{group}) \to \text{Hom}(\text{group}, \text{algebra})
\]

where \( \text{Hom} \) is the set of homomorphisms.

Conformal quantization of higher-dimensional \( \text{SU}(N) \) and \( \text{U}(N) \) theories

We consider the conformal quantization of higher-dimensional \( \text{SU}(N) \) and \( \text{U}(N) \) theories. The conformal quantization is based on the concept of conformal symmetry and the conformal group.

\[
\text{Conf}(\mathbb{R}^4) \cong \text{SO}(1,3) \times \mathbb{R}^4
\]

where \( \text{Conf} \) is the conformal group and \( \mathbb{R}^4 \) is the four-dimensional Euclidean space.

We consider the conformal quantization of the \( \text{SU}(N) \) and \( \text{U}(N) \) theories in four dimensions.

\[
\text{Conf}(\mathbb{R}^4) \to \text{Conf}(\mathbb{R}^4) / \text{SO}(1,3)
\]

where \( \text{Conf}(\mathbb{R}^4) \) is the conformal group and \( \text{SO}(1,3) \) is the Lorentz group.
\( \Psi(y) \) are multi-valued on the circle, it is convenient to define the twist on the real axis:
\[
\Psi(y) = U_\beta \Psi(y + 2\pi R) ,
\]
where the matrix \( U_\beta \) depends on the real parameters \( \beta \), but not on the space-time coordinates. A well-known consistency condition \([3, 4]\) between the twist and the orbifold projection is that
\[
U_\beta Z U_\beta^T = Z .
\] (4)

If we write \( U_\beta = \exp(i\beta_\gamma T) \), where the matrix \( \beta_\gamma T \) is hermitian, we see that eq. (4) is satisfied if \( \beta_\gamma T Z = 0 \).
This implies that the generator \( \beta_\gamma T \) is purely off-diagonal in the basis of eq. (2).

Our theory is defined on the orbifold \( S^1/Z_2 \), so we allow the fields to jump at the orbifold fixed points:
\[
\Psi(y_0 + \xi) = U_q \Psi(y_0 - \xi),
\]
where \( y_q = q\pi R, \; q \in Z, \; 0 < \xi \ll 1 \) and \( U_q \) is a global symmetry transformation. The jumps across points related by a \( 2\pi R \) translation must be the same, so
\[
U_{q+1} = U_0 , \quad U_{q+1} = U_\tau .
\] (6)

A consistency condition identical to (4) holds for each of the jumps:
\[
U_q Z U_q = Z .
\] (7)

The physical spectrum is controlled by the Scherk-Schwarz twist and by the jumps at the orbifold fixed points. The discontinuities are the result of mass terms localized at the fixed points. In the next section, we shall see that the mass terms can be described by more than one brane action. We will also see that the theory with discontinuities is equivalent to a conventional Scherk-Schwarz theory with a modified twist. In particular, it is possible for the discontinuities to completely remove the symmetry breaking induced by the twist!

**EXAMPLE**

To illustrate our mechanism in a simple setting, we consider the equation of motion for a free 5D massless fermion, written in terms of 5D fields with 4D spinor indices
\[
i\sigma^\mu \partial_\mu \Psi - i\sigma^\gamma \partial_\gamma \Psi = 0 ,
\]
valid in each region \( y_1 < y < y_{1+1} \) of the real axis. In the notation of eqs. (1) and (2), we write:
\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \quad \Psi^\dagger = \begin{pmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \end{pmatrix} , \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,
\]
\[
\Psi(y) = U_\beta \Psi(y + 2\pi R) ,
\]
the equation of motion (8) is invariant under global SU(2) transformations of the form \( U = U_\beta \), where \( U \in SU(2) \). We take
\[
U_\beta = \exp \left( i\beta_\gamma \sigma^\gamma \right) = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} ,
\]
\[
U_q = \exp \left( i\delta_q \sigma^\gamma \right) = \begin{pmatrix} \cos \delta_q & \sin \delta_q \\ -\sin \delta_q & \cos \delta_q \end{pmatrix} ,
\]
where \( \delta_q = \delta_0 + \delta_q \) and \( \delta_q = \delta_0 + \delta_{q+1} \) for any \( q \in Z \).

We seek solutions \( \Psi(y) \) to eq. (8), with the boundary conditions of eqs. (10) and (11). Exploiting the fact that \( \sigma^\mu \partial_\mu \Psi = m\Psi \), we find
\[
\Psi(y) = \chi \begin{pmatrix} \cos \left( \alpha(y) - \alpha(y_0) \right) \\ \sin \left( \alpha(y) - \alpha(y_0) \right) \end{pmatrix} ,
\]
where \( \chi \) is a \( y \)-independent 4D spinor,
\[
m = \frac{\beta - \delta_0 - \delta_\tau}{2\pi R} , \quad (n \in Z) ,
\]
and
\[
\alpha(y) = \frac{\delta_0 - \delta_\tau}{4}(\eta(y) + \delta_0 + \delta_\tau) .
\] (14)

Here \( \eta(y) \) is the ‘sign’ function defined on \( S^1 \), and
\[
\eta(y) = 2q + 1 , \quad y_q < y < y_{q+1} , \quad (q \in Z) ,
\]
is the ‘staircase’ function that steps by two units every \( \pi R \) along \( y \). The function \( \alpha(y) \) satisfies
\[
\alpha(y) = \alpha(y + 2\pi R) = \alpha(y) + \delta_0 + \delta_\tau ,
\]
so the solution (12) has the correct Scherk-Schwarz twist. Sample solutions are shown in Fig. 1.

The spectrum (13) is characterized by a uniform shift with respect to a traditional Kaluza-Klein compactification. In contrast to the usual Scherk-Schwarz mechanism, however, the shift depends on the jumps \( \delta_0 \) and \( \delta_\tau \), as well as on the twist \( \beta \). In particular, it is possible to have a vanishing shift for nonvanishing \( \beta \). In the limit \( \delta_0 \to 0 \), our results reduce to the conventional Scherk-Schwarz spectrum. Note that the eigenfunction of eq. (12) is discontinuous: the even part has cusps and the odd part has jumps at \( y = y_q \), as required by the boundary conditions. In the limit \( \delta_0 \to 0 \) the eigenfunction becomes regular everywhere.

For any \( \delta_0 \), the system is equivalent to a conventional Scherk-Schwarz compactification with twist \( \beta = \beta_0 - \delta_0 - \delta_\tau \). The new field variable, \( \Psi_\psi \), is related to the discontinuous variable, \( \Psi \), via the generalized function \( \alpha(y) \),
\[
\begin{pmatrix} \psi_1 \psi_2 \psi_{1c} \psi_{2c} \end{pmatrix} = \begin{pmatrix} \cos \alpha(y) & \sin \alpha(y) \\ -\sin \alpha(y) & \cos \alpha(y) \end{pmatrix} \begin{pmatrix} \psi_1 \psi_2 \end{pmatrix} .
\] (17)
This is reminiscent of strong CP violation, where the physical order parameter is not \( \theta \), but the combination \( \theta - \arg \det m_q \), where \( m_q \) is the quark mass matrix. Similarly, the mass shift of our system is controlled not by \( \beta \) alone, but by the twist \( \beta_c \), which includes contributions from jumps in the fermion fields. As in QCD, where we can eliminate the phase in \( \det m_q \) by a chiral transformation, here we can remove the jumps by a redefinition of the fermion fields. In the new basis, there are no jumps, but the twist acquires an additional contribution.

Discontinuous field variables arise from mass terms localized at the fixed points. This can be seen by starting with the Lagrangian \( \mathcal{L} \) for the fermions \( \psi_1, \psi_2(y) \), characterized by a twist \( \beta_c = \beta - \delta \theta - \delta \tau \):

\[
\mathcal{L}(\psi) = \bar{\psi} \gamma^\mu \partial_\mu \psi + \bar{\psi} \gamma^5 \partial_\mu \sigma^a \partial_\mu \psi^a + \left[ \frac{1}{2} (\bar{\psi} \gamma^5 \partial_\mu \psi - \psi^a \bar{\psi} \gamma^5 \partial_\mu \psi^a) \right] + \text{h.c.}.
\]

If we perform the field redefinition of eq. (17), the 5D Lagrangian becomes:

\[
\mathcal{L}(\psi_c) = \mathcal{L}(\psi) + \mathcal{L}_{\text{brane}}(\psi),
\]

where

\[
\mathcal{L}_{\text{brane}}(\psi) = - \frac{1}{2} \sigma'(y) (\psi_1 \psi_1 + \psi_2 \psi_2) + \text{h.c.},
\]

and

\[
\sigma'(y) = \sum_{q=0}^{+\infty} \left[ \delta_0 \delta(y - y_{2q}) + \delta_\tau \delta(y - y_{2q+1}) \right].
\]

We see that the jumps \( \delta_\tau \) arise from fermion mass terms localized at the orbifold fixed points.

The discontinuities of the fields can be recovered by integrating the equations of motion. The trick is to find the correct equations. We avoid all subtleties associated with discontinuous field variables by defining the term that appears in the brane action to be continuous across the orbifold fixed points. For the case at hand, this means one must choose the field variables so that the combination \( \psi_1 \psi_1 + \psi_2 \psi_2 \) is continuous. Alternatively, one can obtain the equations of motion by first regularizing the delta functions, so that \( \psi_1 \) and \( \psi_2 \) are continuous, and then taking the singular limit.

It is interesting to note that the same physical system can be obtained from another brane Lagrangian, one in which we treat the even field \( \psi_1(y) \) as continuous. The discontinuity of the odd field \( \psi_2(y) \) is then

\[
\psi_2(y + \xi) - \psi_2(y - \xi) = -2 \tan \frac{\delta_\tau}{2} \psi_1(y),
\]

This jump is reproduced by the brane Lagrangian

\[
\mathcal{L}_{\text{brane}}(\psi) = - \frac{1}{2} f(y) \psi_1 \psi_1 + \text{h.c.},
\]

where

\[
f(y) = 2 \sum_{q \in \mathbb{Z}} \left[ \tan \frac{\delta_0}{2} \delta(y - y_{2q}) + \tan \frac{\delta_\tau}{2} \delta(y - y_{2q+1}) \right].
\]

In this case, we vary with respect to \( \psi_1(y) \) and \( \psi_2(y) \); the discontinuous field \( \psi_2(y) \) does not appear in the brane Lagrangian.

In summary, the brane Lagrangians (20) and (23) give rise to equivalent theories in the absence of brane interactions, provided we use an appropriate procedure to derive the equations of motion.

**CONCLUSIONS**

In this letter we have studied coordinate dependent compactifications of field theories on orbifolds. We have seen that the mass spectrum depends on an overall twist of the fields, together with the jumps of the fields at the orbifold fixed points. Such compactifications can break the symmetries of a theory, either global and local. The order parameter is nonlocal, in the sense that it is determined by a combination of the twist and the discontinuities.

In a supersymmetric Yang-Mills theory, for example, the twist and the jumps are defined by a \( U(1)_R \) subgroup.
of $SU(2)_R$. From a 4D point of view, this typically breaks the $N = 1$ supersymmetry that survives the orbifold projection. Note, though, that it is possible for supersymmetry to remain unbroken. For instance, when $\beta = 0$, supersymmetry is preserved in the presence of opposite, nonvanishing jumps at $y = y_0$ and $y = y_{2q+1}$, in analogy with a phenomenon first discussed in M-theory [9]. This example can be readily extended to the case where the fermions $\psi_1$ and $\psi_2$ come in a distinct copy, in which case flavor symmetry is broken if the matrices $U_\beta$ and $U_\eta$ have a non-trivial structure in flavor space.

It is important to note that our mechanism provides a self-consistent way of introducing other interaction terms such as Yukawa couplings or even kinetic terms, that are localized at the fixed points. Such terms will always occur in non-renormalizable theories, including supergravity, where the kinetic terms typically have a non-canonical (and non-renormalizable) form.

It would be interesting to find string realizations of our mechanism, which so far are missing. These would give rise to models where mass terms for the un twisted fields are localized at the fixed points of a non-free acting orbifold.

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