\[ e^\alpha - e^\beta = H, \quad \alpha = V. \]

where \( H \) and \( V \) are functions of

\[ (1) \quad \delta^x \left( \delta^x - \delta^t \right) \rho(x) + \delta^x \left( \delta^x - \delta^t \right) \rho(x) + \delta^x \left( \delta^x - \delta^t \right) \rho(x) = \delta^x \rho(x) \]

\( \{ \delta^x, \delta^t \} \) is the Poisson bracket of \( \{ t, x \} \). The formalism of spinning black holes in vacuum, as in [9] and also in [6], can be extended to include curved space-time (see [8]) and to give new solutions in curved space-time that appear in the literature (see [8]).

1. Introduction and Summary

Boost-Rotation Symmetric Vacuums: Solutions with Spinning Sources
It has two Killing vectors

\[ \xi = \frac{\partial}{\partial \varphi}, \quad \eta = \frac{\partial}{\partial z} + z \frac{\partial}{\partial t} \]  

with norms

\[ \xi^\alpha \xi_\alpha = g_{\varphi \varphi} = -\rho^2 e^{-\theta} + a^2 (z^2 - t^2)e^{\mu} = -Ae^{-\theta} + a^2 Be^\mu, \]

\[ \eta^\beta \eta_\beta = \gamma_{zz} = g_{zz}t^2 + 2 \gamma_{z\varphi} z t = Be^\mu. \]

As in the non-spinning case \cite{10}, two null hyperplanes \( B = 0 \), i.e. \( z = \pm t \) will be called the “roof”, the points with \( A = 0 \) the “axis”, the region of the spacetime with \( B < 0 \) “above the roof”, and finally the region with \( B > 0 \) “below the roof”. Notice that the behaviour of the boost and axial Killing vectors (2) is more complicated in the spinning case. Below the roof (\( B > 0 \)), the boost Killing vector \( \eta \) is mostly timelike as in the non-spinning case but in the vicinity of spinning sources there may also occur ergoregions where it is spacelike. Due to the presence of spinning strings there may be also regions in their neighbourhood with closed timelike curves where the axial Killing vector \( \xi \) is timelike. In order to determine if there exist both timelike and spacelike Killing vectors everywhere below the roof (\( B > 0 \)), we study a general linear combination of the boost and the axial Killing vectors with constant coefficients \( X = c_1 \xi + c_2 \eta \). Its norm may be both positive and negative if the product of eigenvalues of the quadratic form \((c_1^2 g_{\varphi \varphi} + \ldots)\) given by the norm is negative, i.e. if \(-\rho^2 B < 0\), which is satisfied everywhere below the roof, where the spacetime is thus stationary and may be transformed to the stationary Weyl metric (A1) (see e.g. \cite{6}). However, above the roof (\( B < 0 \)), the product is everywhere positive \( \rho^2 B > 0 \) and thus there does not exist a timelike Killing vector and the spacetime is nonstationary.\footnote{It is easier to perform these calculations in coordinates \((\gamma, \rho, \beta, \varphi)\) and \((h, \rho, \chi, \varphi)\), given in App. A, for regions below and above the roof, respectively.}

Vacuum Einstein’s equations for the spinning brs metric (1) are

\[ A \mu, A + B \mu, B + \mu, A + \mu, B = -\frac{B}{A} \left( e^{2\beta} (A a, A^2 + B a, B^2) \right), \]  

\[ 0 = A B \left( e^{2\beta} a, A \right) \cdot a + \left( B e^{2\beta} a, B \right) B, \]  

\[ (A + B) \lambda, A = (A - B) \mu, A - 2 B \mu, B - B (B \mu, B^2 - A \mu, A^2) + 2 A B \mu, A \mu, B \]  

\[ + \frac{B^2}{A} e^{2\beta} (B a, B^2 - A a, A^2 - 2 A a, A a, B) \],  

\[ (A + B) \lambda, B = (A - B) \mu, B + 2 A \mu, A + A (B \mu, B^2 - A \mu, A^2) + 2 A B \mu, A \mu, B \]  

\[ - B e^{2\beta} (B a, B^2 + A a, A^2 + 2 A a, a, B) \].

Notice that Eqs. (3), (4) are integrability conditions for Eqs. (5), (6).

First let us investigate the regularity of the roof and the axis. From Eq. (5) it follows that on the roof (i.e. for \( B = 0 \))

\[ \lambda, A (A, 0) = \mu, A \Bigl( A(0), 0 \Bigr) \rightarrow \lambda (A, 0) - \mu (A, 0) = K_1 = \text{const}. \]

The roof is regular (i.e. \( g_{zz}, g_{\varphi \varphi}, \) and \( g_{z\varphi} \) in (1) are nonsingular on the roof) if for \( B = 0 \)

\[ \lambda (A, 0) = \mu (A, 0), \quad \text{i.e.} \quad K_1 = 0. \]

From Eqs. (3), (5), and (6) on the axis (\( A = 0 \)) we get

\[ a, B (0, B) = 0 \rightarrow a = \tilde{a}_0 + \tilde{a}_1 (B) A + \mathcal{O} (A^2), \quad \tilde{a}_0 = \text{const}, \]

\[ \lambda, B (0, B) + \mu, B (0, B) = 0 \rightarrow \lambda (0, B) + \mu (0, B) = K_2 = \text{const}. \]

The axis regularity condition

\[ \lim_{r \rightarrow 0} \frac{1}{2 \pi} \int_0^{2\pi} \sqrt{g_{\varphi \varphi}^r_{11} r^2 d \varphi} = 1. \]
(or equivalently $g_{x x}, g_{y y}$ are nonsingular and $g_{x y} = 0$ there, see (A5) in App. A) is satisfied if
\[ a(0, B) = 0 , \quad \Rightarrow \quad a = \tilde{a}_1(B) A + O(A^2) , \quad \text{i.e.} \quad \tilde{a}_0 = 0 , \quad (8) \]
\[ \lambda(0, B) + \mu(0, B) = 0 , \quad \text{i.e.} \quad K_2 = 0 . \quad (9) \]
If $K_2 \neq 0$ there is a conical singularity along the axis and if $\tilde{a}_0 \neq 0$ a torsion singularity is present there. The regularity condition of the roof (7) is the same as for nonspinning brs spacetimes [10], however, a new condition (8) arises for the regularity of the axis except for (9) which also appears in the nonspinning case [10].

Now let us turn our attention to asymptotic behaviour of spinning brs spacetimes at null infinity. For this purpose we transform (1) to null coordinates in two steps: first transforming it to coordinates \{ $b$, $\rho$, $\chi$, $\varphi$ \} by (3.10) in [10]
\[ b = \sqrt{-B} = \sqrt{t^2 - z^2} , \quad \tanh \chi = \pm \frac{z}{t} \]
we obtain the metric
\[ ds^2 = e^{\lambda}(db^2 - d\rho^2) - \rho^2 e^{-\mu} d\varphi^2 \left( 1 - 2b^2 \right) e^{\mu} (d\chi + a d\varphi)^2 . \quad (10) \]
Finally transforming (10) to coordinates \{ $\bar{\tau}$, $\tau$, $\chi$, $\varphi$ \} by (3.15) in [10]
\[ \bar{\tau} = b - \rho , \quad \tau = b + \rho \]
we obtain
\[ d\bar{\tau}^2 - d\tau^2 = e^{\lambda} (d\bar{\tau}^2 - d\tau^2) - \rho^2 e^{-\mu} (d\chi + a d\varphi)^2 . \quad (11) \]
Vacuum Einstein’s equations for (11) read
\[ \mu_{,\sigma} + \frac{1}{\bar{\tau}^2 - \tau^2} (\dot{\varphi}_{,\sigma} - \ddot{\varphi}_{,\varphi}) = \left( \frac{\bar{\tau} + \tau}{\bar{\tau} - \tau} \right)^2 e^{\mu} a_{,\sigma} a_{,\varphi} , \]
\[ 0 = a_{,\sigma} + a_{,\varphi} \left( \mu_{,\sigma} + \frac{\bar{\tau} - \tau}{\bar{\tau} + \tau} \right) + a_{,\varphi} \left( \mu_{,\varphi} + \frac{2(\bar{\tau} - \tau)}{\bar{\tau} + \tau} \right) , \]
\[ -\bar{\tau} \lambda_{,\sigma} = \bar{\tau} \mu_{,\sigma} + \frac{\varphi^2 - \bar{\tau}^2}{4} \mu_{,\varphi}^2 + \frac{(\bar{\tau} + \tau)^3 e^{2\mu} a_{,\sigma}^2}{4} \]
\[ -\tau \lambda_{,\varphi} = \tau \mu_{,\varphi} - \frac{\varphi^2 - \tau^2}{4} \mu_{,\varphi}^2 + \frac{(\bar{\tau} + \tau)^3 e^{2\mu} a_{,\varphi}^2}{4} . \quad (12) \]
Assuming the metric functions $\mu$, $\lambda$, and $a$ to have expansions in $\bar{\tau}^{-1}$ for $\bar{\tau} \to \infty$ ($\mu(\bar{\tau}, \tau) = \mu_0(\bar{\tau}) + \mu_1(\bar{\tau})/\bar{\tau} + \ldots$) and solving Eqs. (12) at null infinity, i.e. for the limit $\bar{\tau} \to \infty$ and $\bar{\tau}$, $\chi$, $\varphi$ constant, we get
\[ \mu = \mu_0 + \frac{\mu_1(\bar{\tau})}{\bar{\tau}} + O(\bar{\tau}^{-2}) , \]
\[ \lambda = \lambda_0(\bar{\tau}) + \frac{\lambda_1(\bar{\tau})}{\bar{\tau}} + O(\bar{\tau}^{-2}) , \quad (13) \]
\[ a = a_0 + \frac{a_1(\bar{\tau})}{\bar{\tau}} + O(\bar{\tau}^{-2}) , \]
where $a_0$ and $\mu_0$ are constants and $\lambda_0(\bar{\tau})$ satisfies
\[ \lambda_0^4 = \frac{1}{\bar{\tau}^2} \left( 4\mu_{1,\sigma} + \mu_{1,\varphi}^2 + e^{2\mu} a_{1,\sigma}^2 \right) . \]
The metric (11) with the metric functions (13) is asymptotically Minkowskian at null infinity as in the limit $\bar{\tau} \to \infty$ and $\bar{\tau}$, $\chi$, $\varphi$ constant, it can be transformed to the Minkowski metric using the transformations (3.23), (3.24) in [10]
\[ \bar{\tau}' = e^{\frac{\mu_0}{2}} \int e^{-\lambda(\bar{\tau})} d\bar{\tau} , \quad \tau' = e^{-\frac{\mu_0}{2}} \tau , \quad \chi' = e^{\mu_0} \chi \]
and
\[ \chi'' = \chi' + a_0 e^{\mu_0} \varphi . \]
III. THE BONDI-SACHS COORDINATES AND NEWS FUNCTIONS FOR SPINNING BRS

In this section we transform the spinning brs metric (1) into the Bondi-Sachs coordinates \( \{ u, r, \theta, \phi \} \), in which the metric, that does not depend on \( \phi \) because of the axial symmetry, has the form [14, 15, 16]

\[
\begin{align*}
d^2x^2 &= g_{uu}du^2 + 2g_{ur}dudr + 2g_{u\theta}du d\theta + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{\theta\phi}d\theta d\phi \\
&= 1 - \frac{2M}{r} + \mathcal{O}(r^{-2}) , \\
g_{ur} &= 1 - \frac{c^2 + d^2}{2r^2} + \mathcal{O}(r^{-4}) , \\
g_{u\theta} &= -(c, \phi + 2c \cot \theta) + \mathcal{O}(r^{-1}) , \\
g_{u\phi} &= -(d, \phi + 2d \cot \theta) \sin \theta + \mathcal{O}(r^{-1}) , \\
g_{\theta\theta} &= -r^2 - 2cr - 2(c^2 + d^2) + \mathcal{O}(r^{-1}) , \\
g_{\theta\phi} &= -2d \sin \theta + \mathcal{O}(r^0) , \\
g_{\phi\phi} &= -r^2 \sin^2 \theta + 2r \sin^2 \theta - 2(c^2 + d^2) \sin^2 \theta + \mathcal{O}(r^{-1}) ,
\end{align*}
\]

with the following expansion for \( r \rightarrow \infty \) and \( u, \theta, \phi \) constant

\[
M_{u, r} = -(c, u^2 + d, u^2) + \frac{1}{2}(c, \phi + 3c \cot \theta - 2c, u) .
\]

\[ \text{If there is nonvanishing new function, gravitational radiation is present and the total Bondi mass at future null infinity is decreasing.} \]

In order to find the transformation of spinning brs spacetimes from the coordinates \( \{ t, \rho, z, \varphi \} \) with the metric (1) into the Bondi-Sachs coordinates \( \{ u, r, \theta, \phi \} \) with the metric (14) and its expansions (15) we follow [11] and we first transform the metric (1) to flat-space spherical coordinates \( \{ R, \Theta, \varphi \} \) and a flat-space retarded time \( U \) using the relations

\[
\begin{align*}
t &= U + R , \\
\rho &= R \sin \Theta , \\
z &= R \cos \Theta .
\end{align*}
\]

We assume the metric functions to have the expansions in powers \( R^{-1} \)

\[
\begin{align*}
\lambda(U, \Theta) &= \lambda_0(U, \Theta) + \lambda_1(U, \Theta) + \mathcal{O}(R^{-2}) , \\
\mu(U, \Theta) &= \mu_0 + \mu_1(U, \Theta) + \mathcal{O}(R^{-2}) , \\
\alpha(U, \Theta) &= \alpha_0 + \alpha_1(U, \Theta) + \mathcal{O}(R^{-2}) ,
\end{align*}
\]

where \( \mu_0 \) and \( \alpha_0 \) are constants and thus

\[
\begin{align*}
e^\lambda &= \beta(U, \Theta) \left( 1 + \frac{\lambda_1(U, \Theta)}{R} + \mathcal{O}(R^{-2}) \right) , \\
e^\mu &= \alpha \left( 1 + \frac{\mu_1(U, \Theta)}{R} + \mathcal{O}(R^{-2}) \right) ,
\end{align*}
\]

with

\[
\begin{align*}
\beta(U, \Theta) &= e^{\lambda_0(U, \Theta)} , \\
\alpha &= e^{\mu_0} .
\end{align*}
\]
Now we transform the metric further to the Bondi-Sachs coordinates by an asymptotic transformation

\[ U = \frac{2}{r} (u, \theta) + \frac{1}{r} (u, \theta) + \frac{2}{r^2} (u, \theta) + O(r^{-3}) , \]

\[ R = q(u, \theta) + \frac{1}{r} (u, \theta) + \frac{2}{r^2} (u, \theta) + O(r^{-2}) , \]

\[ \Theta = \frac{1}{r} (u, \theta) + \frac{2}{r^2} (u, \theta) + O(r^{-3}) , \]

\[ \varphi = \phi + f(u, \theta) + \frac{1}{r} (u, \theta) + \frac{2}{r^2} (u, \theta) + O(r^{-3}) . \]  

Comparing the resulting metric expansions with the expansions (19) we obtain differential equations for coefficients entering the asymptotic transformation (20). Since these equations are lengthy we present only their solutions in App. B. Their integrability condition (obtained comparing (B4) and (B5)) turns out to be the same as in the non-spinning case [11]

\[ \beta_\pi^\sigma + \beta_\pi^\theta \tan \frac{\theta}{\pi} = 0 , \quad \text{or equivalently} \quad \lambda_{\pi, u} U + \lambda_{\theta, \theta} \Theta = U \equiv 0 , \]

where we used the relations \( \beta_{\pi, \pi} = \beta_\pi^\sigma \pi_{\pi} \) and \( \beta_{\pi, \theta} = \beta_\pi^\theta \pi_{\theta} + \beta_\pi^\sigma \pi_{\sigma} \). Solving Eqs. (B1), (B2), (B3), and (B6) one may infer the first order coefficients in the expansions (20)

\[ \varphi = 2 \arctan \left[ e^{-\nu} (\tan \frac{\theta}{\pi}) K \right] , \]  

\[ q = \frac{1}{\sqrt{K}} \sin \frac{\theta}{\pi} \cos \frac{\theta}{\pi} \left[ e^{-\nu} (\cot \frac{\theta}{\pi}) K + e^{-\nu} (\tan \frac{\theta}{\pi}) K \right] , \]

\[ f = \frac{a_0 \alpha}{K} \ln \left( \frac{\sin \frac{\theta}{\pi}}{1 + \cos \frac{\theta}{\pi}} \right) , \]  

\[ \beta_{\pi, u} = \frac{1}{\beta q} , \]

where \( K \equiv \frac{1 + 2 \alpha^2}{\alpha} \) and \( \nu \) is an arbitrary constant. The axis (which is the same in both coordinates, i.e. \( \Theta = 0, \pi \to \theta = 0, \pi \)) is singular for \( K > 1 \) as \( q \to 0 \) for \( K < 1 \) and to \( \infty \) for \( K > 1 \) there. Since the coordinate system \( \{ t, \rho, z, \varphi \} \) can be chosen in such a way that \( a_0 = 0, \) we present here news functions for \( a_0 = 0 \) and the case \( a_0 \neq 0 \) is given in App. B. From Eqs. (B7) and (B8) we obtain the news functions

\[ c_{\pi, u} = \frac{1}{2} a_1, u - \frac{q, \varphi}{2q^2} - \frac{q, \varphi \cot \frac{\theta}{\pi}}{2q^2} + \frac{1}{2q^2 \beta \sin^2 \varphi} - \frac{1}{2} \frac{\cot \frac{\theta}{\pi}}{2q^2 \alpha} , \]  

\[ d_{\pi, u} = -\frac{1}{2} a_1, u . \]

Having the news functions of the system one can compute the Bondi mass, see [2].

For a special case \( K = \alpha = 1, \) i.e. for a regular axis, we get from (21), (22), and (23)

\[ \varphi = 2 \arctan \left( e^{-\nu} \tan \frac{\theta}{\pi} \right) , \]

\[ q = \cosh \nu + \cos \theta \sinh \nu , \]

\[ f = 0 . \]  

Coordinate systems with different \( \nu \) are connected by Lorentz transformations along the axis belonging to the Bondi-Metzner-Sachs group and thus as in [11] we may without loss of generality put \( \nu = 1 \) which implies \( q = 1 \) and \( \varphi = \theta \).

Then the coefficient \( \pi \) can be computed from the relation

\[ \int e^{\lambda} (\pi, \theta) d \pi = u + \omega(\theta) \]
obtained from Eq. (24). The function \( \omega(\theta) \) in (28) represents a supertranslation also belonging to the Bondi-Metzner-Sachs group and thus it may be again put equal to zero without loss of generality. Finally the news functions (25) and (26) read

\[
\begin{align*}
\eta_{u} &= -\frac{1}{2\sin^2 \theta} + \frac{1}{2\beta \sin \theta} + \frac{1}{\beta \mu_{1} \eta} = \frac{1}{2\sin^2 \theta} (1 - \beta + \mu_{1} \eta \sin^2 \theta), \\
\eta_{u} &= -\frac{1}{2\sin^2 \theta} + \frac{1}{2\beta \sin \theta} + \frac{1}{\beta \mu_{1} \eta} \cdot \\
\eta_{u} &= -\frac{1}{2\sin^2 \theta} + \frac{1}{2\beta \sin \theta} + \frac{1}{\beta \mu_{1} \eta} \cdot
\end{align*}
\]

(29)

For \( a_{1} = 0 \) (29) and (30) reduce to news functions as given in [11, 12] for the nonrotating case.

**APPENDIX A: COORDINATE SYSTEMS ADAPTED TO THE BOOST AND ROTATION SYMMETRIES**

In the nonradiative stationary region below the roof, spinning brs metric can be transformed to the stationary Weyl coordinates \([\hat{\mathbf{r}}, \hat{\mathbf{p}}, \hat{\mathbf{z}}, \hat{\varphi}]\) with the Killing vectors \( \hat{\xi} = \partial \varphi, \eta = \partial \gamma \) and the metric

\[
ds^2 = -e^{2U} \left[ d\hat{r}^2 + d\hat{\varphi}^2 + \beta^2 d\hat{\varphi}^2 + e^{2U}(d\hat{f} + \alpha d\hat{\varphi})^2 \right].
\]

(A1)

Vacuum Einstein’s equations have the form [17]

\[
U_{,\tau\tau} + U_{,\tau z} + \frac{U_{,\tau}}{\hat{p}} = -\frac{\epsilon U}{2\hat{p}}(a_{,\tau}^2 + a_{,z}^2),
\]

\[
0 = \left(\frac{\epsilon U a_{,\tau}}{\hat{p}}, \gamma \right) + \epsilon(U a_{,\tau} + \hat{p} a_{,\gamma}),
\]

\[
\frac{\nu_{,\tau}}{\hat{p}} = U_{,\tau}^2 - U_{,z}^2 - \frac{\epsilon U}{4\beta^2}(a_{,\tau}^2 - a_{,z}^2),
\]

\[
\frac{\nu_{,\varphi}}{\hat{p}} = 2U_{,\varphi} U_{,\tau} - \frac{\epsilon U}{2\beta^2} a_{,\tau} a_{,\varphi}.
\]

(A2)

Another appropriate coordinate system in the stationary region below the roof is \([\gamma, \rho, \beta, \varphi]\) with Killing vectors \( \xi = \partial \varphi, \eta = \partial \gamma \) and the metric

\[
ds^2 = -e^{2\lambda}(d\hat{r}^2 + d\beta^2) - \rho^2 e^{-\lambda} d\gamma^2 + \beta^2 e^{2\lambda}(d\hat{f} + \alpha d\hat{\varphi})^2.
\]

(A3)

connected with the stationary Weyl coordinates by (see (5.4), (5.6) in [10])

\[
\hat{\mathbf{r}} = \gamma, \quad \hat{\mathbf{p}} = \rho \beta, \quad \hat{\mathbf{z}} - \tilde{z}_{0} = -\frac{\beta^2 - \rho^2}{2}, \quad \tilde{\varphi} = \varphi, \quad \tilde{z}_{0} = \text{const}, \quad e^{2U} = \beta^2 e^{\lambda}, \quad e^{2u} = \frac{\beta^2}{\rho^2 + \beta^2} e^{\lambda}.
\]

Vacuum Einstein’s equations read

\[
\mu_{,\tau\tau} + \mu_{,\beta \beta} + \frac{\mu_{,\beta}}{\rho} + \frac{\mu_{,\gamma}}{\beta} = -\frac{\beta^2}{\rho} e^{2\lambda}(a_{,\tau}^2 + a_{,\beta}^2),
\]

\[
0 = \left(\frac{\beta^2}{\rho} e^{2\lambda} a_{,\gamma}, \gamma \right) + \left(\frac{\beta^2}{\rho} e^{2\lambda} a_{,\beta}, \beta \right),
\]

\[
(\rho^2 + \beta^2) \lambda_{,\tau} = \left(\rho^2 - \beta^2\right) \rho_{,\tau} - 2\rho \beta \mu_{,\beta} - \frac{\beta^2}{\rho^2} \beta^2 (\mu_{,\beta}^2 - \mu_{,\tau}^2) + \rho^2 \beta \mu_{,\varphi} \mu_{,\beta},
\]

\[
+ \frac{\beta^4}{2\rho} e^{2\lambda} \left( a_{,\beta}^2 - a_{,\tau}^2 - 2\frac{\beta}{\rho} a_{,\gamma} a_{,\beta} \right),
\]

\[
(\rho^2 + \beta^2) \lambda_{,\beta} = \left(\rho^2 - \beta^2\right) \rho_{,\beta} - 2\rho \beta \mu_{,\beta} - \frac{\beta^2}{\rho^2} \beta^2 (\mu_{,\beta}^2 - \mu_{,\tau}^2) + \rho^2 \beta \mu_{,\varphi} \mu_{,\beta},
\]

\[
- \frac{\beta^4}{2\rho} e^{2\lambda} \left( a_{,\beta}^2 - a_{,\tau}^2 + 2\frac{\beta}{\rho} a_{,\gamma} a_{,\beta} \right).
\]

(A4)

The stationary region of a brs spacetime, under the roof, is composed of two identical regions \((z > 0, z < |\hat{f}|)\) and \(z < 0, z < -|\hat{f}|\) and each of them can be transformed to coordinates \([\hat{\mathbf{r}}, \hat{\mathbf{p}}, \hat{\varphi}, \hat{\mathbf{z}}]\) or \([\gamma, \rho, \beta, \varphi]\).

By further transformation (3.5) in [10] to coordinates \([t, \rho, z, \varphi]\)

\[
tanh \gamma = \pm \frac{t}{z}, \quad \beta = \sqrt{z^2 - t^2}, \quad B \equiv \beta^2, \quad A \equiv \rho^2.
\]
we arrive at the metric (1) where nonstationary region above the roof (again composed of two identical regions) appears as in the nonspinning case [10].

For examining regularity of the axis it is convenient to transform (1) to coordinates \( \{ t, x, y, z \} \), where \( x = \rho \cos \varphi, \ y = \rho \sin \varphi \):

\[
    ds^2 = -\frac{1}{\rho^2} \left[ (e^\lambda x^2 + e^{-\lambda} y^2) dx^2 + (e^\lambda y^2 + e^{-\lambda} x^2) dy^2 + 2xy(e^\lambda - e^{-\lambda}) dxdy \\
    - \frac{z^2 - t^2}{\rho^2} a^2 e^\lambda \left( -y dx + x dy \right)^2 - 2ae^\lambda (-y dx + x dy) + y dxdz + x dydz - xtdydz \right] \tag{A5}
\]

\[
    - \frac{1}{\rho^2} \left[ (e^\lambda z^2 - e^{-\lambda} t^2) dz^2 - 2zt(e^\lambda - e^{-\lambda}) dtdz + \frac{e^\lambda e^\lambda}{\rho^2} (e^\lambda z^2 - e^{-\lambda} t^2) dt^2 \right].
\]

Let us finally write down vacuum Einstein’s equations for the metric (10) with the Killing vectors \( \xi = \partial_\varphi, \ \eta = \partial_\chi \)

\[
    \mu_{\varphi}\xi - \mu_{\varphi}\xi + \frac{\mu_{\varphi}\eta}{\rho} - \frac{\mu_{\varphi}\eta}{b} = \frac{b^2}{\rho^2} e^\lambda (a_{\varphi}^2 - a_b^2),
\]

\[
    0 = \left( \frac{b^3}{\rho^2} a_{\varphi} e^\lambda \right)_{,\varphi} - \left( \frac{b^3}{\rho^2} a_b e^\lambda \right)_{,b},
\]

\[
    \left( \rho^2 - b^2 \right) \lambda_{,\eta} = \left( \rho^2 + b^2 \right) \mu_{,\eta} - 2pb \mu_{,\varphi} + 2\rho^2 (\mu_{,\varphi}^2 - \mu_{,\varphi} - b^2 \mu_{,\varphi} + \mu_{,b}^2) + \rho^2 b \mu_{,\varphi} + \mu_{,b}
\]

\[
    + \frac{b^2}{\rho^2} e^\lambda \left( -a_{,\varphi}^2 - a_b^2 + \frac{2b}{2} a_{,\varphi} a_b \right),
\]

\[
    \left( \rho^2 - b^2 \right) \lambda_{,b} = -\left( \rho^2 + b^2 \right) \mu_{,b} + 2pb \mu_{,\varphi} - \rho^2 (b \mu_{,\varphi}^2 + \mu_{,b}^2) + \rho^2 b \mu_{,\varphi} + \mu_{,b}
\]

\[
    + \frac{b^2}{\rho^2} e^\lambda \left( -a_{,\varphi}^2 - a_b^2 + 2 \frac{b}{\rho} a_{,\varphi} a_b \right).
\]

The coordinates \( \{ b, \rho, \chi, \varphi \} \) for the nonstationary region above the roof are analogous to coordinates \( \{ \gamma, \rho, \beta, \varphi \} \) (A3) in the stationary region below the roof.

As for (3)-(6), in each set of Einstein’s equations (A2), (A4), and (A6), the first two are integrability conditions for the other two.

APPENDIX B: TRANSFORMATION OF THE SPINNING BRS METRIC TO THE BONDI-SACHS COORDINATES

The spinning brs metric (13) with expansions (17)-(19) being transformed to the Bondi-Sachs coordinates with the metric (14) using transformations (16), (20) and compared with (15) leads to lengthy equations for coefficients of the asymptotic transformation (20) and metric functions from (15). We present here only their solutions:

\[
    \begin{align*}
            (g_{\varphi \rho}, \ r^2) & = 0 \rightarrow \varphi = \pi \frac{a_{\varphi \rho}}{K \sin \tau}, \\
            (g_{\varphi \varphi}, \ r^2) & = 0 \rightarrow \varphi = \pi \frac{a_{\varphi \varphi}}{K \sin \tau}, \\
            (g_{\rho \rho}, \ r^2) & = 0 \rightarrow \rho = \pi \frac{a_{\rho \rho}}{K \sin \tau}, \\
            (g_{\theta \theta}, \ r^2) & = -1 \rightarrow \tau = \pi \frac{a_{\theta \theta}}{K \sin \tau} (\text{we will use the sign +}), \\
            (g_{ur}, \ r^2) & = 1 \rightarrow \pi \frac{a_{ur}}{1 \beta}, \\
            (g_{r \rho}, \ r^2) & = 0 \rightarrow \frac{1}{K} \frac{a_{\rho \rho}}{q \sin^2 \tau}, \\
            (g_{r \theta}, \ r^2) & = 0 \rightarrow \frac{1}{K} \frac{a_{\theta \theta}}{q \sin^2 \tau} \left[ \pi_{,\theta} \beta K \sin \tau + \pi \cos \tau \left( 1 - \beta K \right) \right], \\
            (g_{r \phi}, \ r^2) & = 0 \rightarrow \frac{1}{K} \frac{a_{\varphi \phi}}{q \sin^2 \tau} \left[ \pi_{,\phi} \beta K \sin \tau - \pi \cos \tau \left( 1 - \beta K \right) \right].
    \end{align*}
\]
(g_{\theta\theta}, \ r^1) &= 0 \rightarrow \rho_{u} = -\frac{1}{\sqrt{q \sqrt{K}}} \left[ \frac{q, \beta K \sin^2 \theta + \rho_{,\theta} q \cos \theta (-1 + \beta K)}{\beta^q \rho K \tau^2 + \rho_{,\theta} q \cos \theta} \right], \quad (B35) \\
(g_{\phi\phi}, \ r^3) &= 0 \rightarrow \sigma = \frac{1}{2 \tau_{,\theta} \sin^2 \theta \tau} \left( \frac{\rho_{,\theta} (1 - 2 \sin^2 \theta \tau) (1 - \beta K)}{K \sin \theta} \right) + \frac{\rho_{,\phi} q \cos \theta \tau (1 - \beta K)}{\beta K \tau^2 + \rho_{,\theta} q \cos \theta} \\
&+ \frac{\rho_{,\phi} (\rho_{,\theta} \sin \theta - \rho_{,\theta} \sin \theta \cos \theta) + \beta (\rho_{,\phi} \sin \theta)}{\beta K \tau^2 + \rho_{,\theta} q \cos \theta}, \quad (B36) \\
(g_{\phi\phi}, \ r^2) &= -\sin^2 \theta \rightarrow \sin \tau = \frac{\sin \theta}{q \sqrt{K}}, \quad (B36) \\
(g_{\phi\phi}, \ r^1) &= 2 \sin^2 \theta \rightarrow c = \frac{1}{2 q \sin^2 \theta (1 + a_0^2 \alpha^2)} \left[ 2 \sin \theta \tau (1 + a_0^2 \alpha^2) \rho_{,\phi} q \sin \theta + \sqrt{q} \cos \theta \right] \\
&+ \mu_1 q \sin^2 \theta \left( -1 + a_0^2 \alpha^2 \right) + 2 a_0 \alpha \left( a_0 \rho_{,\phi} q + a_1 \sin^2 \theta \right) \\
&\rightarrow c_{,u} = \frac{a_0 \alpha}{K} a_{1,u} + \frac{1 - a_0^2 \alpha^2}{2 (1 + a_0^2 \alpha^2) \mu_{1,u} - \frac{q}{q^2}} \left( 1 + a_0^2 \alpha^2 \right) \\
&- \frac{q \cos \theta \sqrt{K}}{q^2} \tau + \frac{1}{2 q^2 \beta \sin^2 \theta (1 + a_0^2 \alpha^2)} - \frac{K \cos \theta \sqrt{K}}{2 q^2}, \quad (B37) \\
(g_{\phi\phi}, \ r^1) &= -2 \sin \theta \rightarrow d = \frac{1}{2 q K \beta \sin^2 \theta \tau} \left[ 1 + a_0^2 \alpha^2 \right] q \sin^2 \theta \tau a_1 + 2 q a_0 \mu_1 \sin^2 \theta \tau + 2 a_0^2 \rho_{,\phi} \tau \\
&\rightarrow d_{,u} = \frac{a_0}{q^2 K \beta \sin^2 \theta \tau} - \frac{a_0}{K \mu_{1,u}} - \frac{1 - a_0^2 \alpha^2}{2 K} a_{1,u}, \quad (B38) \\
\text{Equations } (g_{uu}, \ r^0) = 1, \ (g_{\theta\theta}, \ r^2) = 0, \ (g_{uu}, \ r^1) = 0, \ (g_{\theta\theta}, \ r^{-1}) = 0, \ (g_{r\phi}, \ r^1) = 0, \ (g_{r\phi}, \ r^1) = 0, \ \text{and } (g_{\phi\phi}, \ r^1) = -2c \text{ are satisfied identically.}

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