Covariant quantum measurements may not be optimal

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Abstract. Quantum particles, such as spins, can be used for communicating spatial directions to observers who share no common coordinate frame. We show that if the emitter’s signals are the orbit of a group, then the optimal detection method may not be a covariant measurement (contrary to widespread belief). It may be advantageous for the receiver to use a different group and an indirect estimation method: first, an ordinary measurement supplies redundant numerical parameters; the latter are then used for a nonlinear optimal identification of the signal.

1. Indiscrete quantum information

Information theory usually deals with the transmission of a sequence of discrete symbols, such as 0 and 1. Even if the information to be transmitted is of continuous nature, such as the position of a particle, it can be represented with arbitrary accuracy by a string of bits. However, there are situations where information cannot be encoded in such a way. For example, the emitter (conventionally called Alice) wants to indicate to the receiver (Bob) a direction in space. If they have a common coordinate system to which they can refer, or if they can create one by observing distant fixed stars, Alice simply communicates to Bob the components of a unit vector \( \mathbf{n} \) along that direction, or its spherical coordinates \( \theta \) and \( \phi \). But if no common coordinate system has been established, all she can do is to send a real physical object, such as a gyroscope, whose orientation is deemed stable.

In the quantum world, the role of the gyroscope is played by a system with large spin. Earlier works [1–4] considered the use of spins for transmitting a single direction. The

\[ \text{Note to printer: this is not a typo. The term we use is indiscrete (meaning not discrete). Do not confuse that with the word indiscreet.} \]
simplest method [1] is to send these spins polarized along the direction that one wishes to indicate. This, however, is not the most efficient procedure: when two spins are transmitted, a higher accuracy is achieved by preparing them with opposite polarizations [2]. If there are more than two spins, optimal results are obtained with entangled states [3, 4].

The fidelity of the transmission is usually defined as

$$F = \langle \cos^2(\chi/2) \rangle = (1 + \langle \cos \chi \rangle)/2,$$

where $\chi$ is the angle between the true $n$ and the direction indicated by Bob’s measurement. The physical meaning of $F$ is that the infidelity,

$$1 - F = \langle \sin^2(\chi/2) \rangle,$$

is the mean square error of the measurement, if the error is defined as $\sin(\chi/2)$ [5]. The experimenter’s aim, minimizing the mean square error, is the same as maximizing fidelity. We can of course define “error” in a different way, and then fidelity becomes a different function of $\chi$ and optimization leads to different results. With the definition in Eq. (2), it can be shown [3, 4] that for a large number $N$ of spins, the infidelity asymptotically tends to

$$1 - F = 5.783/N^2 = 1.446/d,$$

where $d$ is the dimension of the subspace of Hilbert space that is effectively used for the transmission.

A more difficult problem is the transmission of a complete Cartesian frame, if a single quantum messenger is available. In an earlier publication [6], we showed how a hydrogen atom (formally, a spinless particle in a Coulomb potential) can transmit a complete frame. We assumed the hydrogen atom to be in a Rydberg state (an energy eigenstate is needed to ensure the stability of the transmission). The $n$-th energy level of that atom has degeneracy $d = n^2$ because the total angular momentum may take values $j = 0, \cdots, n - 1$, and for each one of them $m = -j, \cdots, j$. A similar calculation was done by Bagan, Baig, and Muñoz-Tapia (hereafter BBM) [7], who were able to reach much higher values of $j$ and to prove that the asymptotic behavior was

$$1 - F \rightarrow 1/\sqrt{d}.$$
Here the infidelity $1 - F$ is the sum of the mean square errors for three orthogonal axes.

There is an essential difference between our work and that of BBM. We considered a single system (a hydrogen atom in a Rydberg state), while BBM took $N$ spins, and one irreducible representation for each value $j$ of the total angular momentum. The maximum value is $j_{\text{max}} = N/2$, and then the mathematics are the same as for our Rydberg state, with $j_{\text{max}} = n - 1$, as explained above. However, if there are $N$ spins that can be sent independently, there is a better method. Alice can use half of them to indicate her $z$ axis, and the other half for her $x$ axis. The two directions found by Bob may not be exactly perpendicular, because separate transmissions have independent errors due to limited angular resolution. Some adjustment will be needed to obtain Bob’s best estimates for the $z$ and $x$ axes, before he can infer from them his guess of Alice’s $y$ direction. Even without this adjustment, this method is far more accurate, especially if $N$ is large. From Eq. (3), the mean square error for each one of the $z$ and $x$ axes is $5.783/(N/2)^2 = 23.13/N^2$, rather than $4/3N$ which is the result with the method used by BBM [7].

Similar results hold even for low values of $N$. For example, if Alice has four spins at her disposal, she can do better in this way than with the BBM method: she sends two spins with opposite polarizations along her $z$ axis (the Gisin-Popescu method [2]) and two with opposite polarizations along her $x$ axis. The infidelity for each one of these axes is 0.21132 (this can still be improved by forcing orthogonality on Bob’s axes, as explained in Sect. 4). On the other hand, with a hydrogen atom [6] and $j_{\text{max}} = 2$, if we optimize two axes, the mean square error per axis is 0.23865. Why is there such a discrepancy?

2. Covariant measurements are not always optimal

In all the works that were mentioned above, and in many other similar ones, it was assumed that Holevo’s method of covariant measurements [8] gave optimal results. The only problem was to find optimal quantum states for Alice’s signals and Bob’s detectors. The method of covariant measurements indeed gives good results (but not always optimal ones) if Alice’s signals are the orbit of a group $G$, with elements $g$. Namely, if $|A\rangle$ is one of the signals, the others are $|A_g\rangle = U(g)|A\rangle$, where $U(g)$ is a unitary representation of the group element $g$. Originally, Holevo considered only irreducible representations. It is known now that in some cases reducible representations are preferable [3, 4]. One then never needs to use more than one copy of each irreducible representation in the reducible one. For example, if $|A_g\rangle$ has four spins as in the above example, this state can be written
by using each one of $j = 0, 1, 2$ only once, as shown explicitly in the next section.

We now turn our attention to Bob. The mathematical representation of his apparatus is a positive operator valued measure (POVM) [9], namely a resolution of identity by a set of positive operators:

$$\sum_h E_h = 1,$$

where the label $h$ indicates the outcome of Bob’s experiment. This is true for any type of measurement, provided that the labels $h$ are kept “raw” and not subjected to further classical processing into a new set of labels, as explained in Sect. 4 and 5. In the case of covariant measurements, the labels $h$ run over all the elements of the group $G$ (with a suitable adjustment of the notation in the case of continuous groups). Then the probability that Bob’s apparatus indicates group element $h$ when Alice sent a signal $|A_g\rangle$ is

$$P(h|g) = \langle A_g|E_h|A_g\rangle.$$  

The method of covariant measurements further assumes that $E_h$ can be written as

$$E_h = |B_h\rangle\langle B_h|,$$  

where

$$|B_h\rangle = U(h)|B\rangle.$$  

Here, $|B\rangle$ is a fiducial vector for Bob (which has to be optimized) and $U(h)$ is a representation (possibly a direct sum of irreducible representations) of the same group $G$ that Alice is using.

All this seems quite reasonable (and this indeed usually works well) but, as the above example of four spins shows, this may not be the optimal method. In that example, Alice’s signals $|A_g\rangle$, for all possible positions of her axes, are $SO(3)$ rotations of a fiducial state $|A\rangle$ with $j = 0, 1, 2$ (see next section for details). On the other hand Bob uses two separate POVMs, each one testing only two of the four spins. Each one of these POVMs also involves $SO(3)$, but with $j = 0$ and 1 only. (Strictly speaking, the relevant mathematical structure is $S_2 \otimes S_2$, where $S_2$ is the quotient $SO(3)/SO(2)$, namely the two-dimensional sphere which is not a group. We shall ignore this technical point and informally call it a group, to avoid unnecessarily cumbersome terminology.)
3. Equivalent irreducible representations

Let us examine carefully the meaning and construction of equivalent irreducible representations. Unitary equivalence is *not* equivalence from the point of view of physics [10]. A simple example is a particle of spin \( \frac{3}{2} \), whose state space has four dimensions and is unitarily equivalent to that of a pair of spin \( \frac{1}{2} \) particles. In atomic physics, unitarily equivalent representation naturally arise when we consider different couplings of the various spins, and we use Clebsch-Gordan coefficients in order to construct new states in a systematic way. These “equivalent” unitary representations actually correspond to quite different states. For example, if we have three spins and we wish to construct states of total spin \( \frac{1}{2} \), we may couple two of the spins into a singlet, so as to get a doublet:

\[
|0\rangle \otimes (|01\rangle - |10\rangle)/\sqrt{2} \quad \text{and} \quad |1\rangle \otimes (|01\rangle - |10\rangle)/\sqrt{2},
\]

(9)

where \( |0\rangle \) and \( |1\rangle \) denote the eigenstates of \( \sigma_z \), as usual. The rotations of this doublet generate an irreducible representation of \( SU(2) \).

We can also generate other, equivalent, irreducible representations by starting with different pairs of spins to make a singlet. These equivalent representations have of course different physical meanings. If we used quantum numbers for indicating internal symmetries, they would have different quantum numbers.

It was shown in [3, 4] that the use of more than one equivalent representation does not improve the fidelity of the transmission. In these articles, the choice of that representation was irrelevant (of course Alice and Bob had to use the same one). However, in some cases, that choice may be imposed on us.

As a simple example, consider a given state \(|001\rangle\) of three spins. We want to split it into a spin \( \frac{3}{2} \) component and a *single* spin \( \frac{1}{2} \) component. This can easily be done, but the spin \( \frac{1}{2} \) component will not be of the type represented by Eq. (9), or by any permutation of the three spins in Eq. (9). To see that, we note that spin \( \frac{3}{2} \) states are symmetric under permutations of the particles. Therefore we project \(|001\rangle\) on

\[
|\frac{3}{2}, \frac{1}{2}\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}.
\]

(10)

It follows that the \( j = \frac{3}{2} \) part of \(|001\rangle\) is

\[
|\frac{3}{2}, \frac{1}{2}\rangle \langle 3, \frac{1}{2} |001\rangle = (|001\rangle + |010\rangle + |100\rangle)/3.
\]

(11)
What remains of $|001\rangle$, namely

$$
|001\rangle - (|001\rangle + |010\rangle + |100\rangle)/3 = (2|001\rangle - |010\rangle - |100\rangle)/3,
$$

(12)
is the spin $\frac{1}{2}$ part. This is not the direct product of a singlet and a doublet as in Eq. (9). Still that state generates a perfectly legitimate irreducible representation, with $j = \frac{1}{2}$. (Of course, had we started from a different state, such as $|010\rangle$, we would have ended with a different basis for the irreducible representation with $j = \frac{1}{2}$.)

We have a similar construction, but slightly more complicated, for Alice’s signal having two opposite spins oriented along $z$ and two others along $x$, as proposed at the end of Sect. 1. It should be clear that a pair of signals (or any combination of signals) still are one signal. Alice’s signal thus is, with the same notations as above

$$
|A\rangle = |01\rangle \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)/2 = (|0110\rangle + |0110\rangle - |0101\rangle - |0111\rangle)/2.
$$

(13)

To find the parts of $|A\rangle$ with $j = 0, 1, 2$, we proceed as we did for the case of three spins (we shall not show the explicit calculations, which are quite lengthy, because they are not necessary for the sequel).

Two points should be emphasized. In earlier works [6, 7], it was not necessary to specify the actual construction of the irreducible representations that were used. Their choice was arbitrary and irrelevant. It just had to be the same for Alice and Bob. Now, the situation is different: these representations are uniquely defined by Alice’s signal in Eq. (13). The choice of that particular signal was suggested by the Gisin-Popescu method for two spins [2]. We do not claim that it is the optimal signal for two pairs of spins. To actually investigate optimization, Alice’s signal should be taken as general as possible, namely a sum of 16 terms,

$$
|A\rangle = a_0|0000\rangle + \cdots + a_{15}|1111\rangle,
$$

(14)

and we would have to determine the coefficients $a_n$, subject to normalization $\sum |a_n|^2 = 1$. However, this optimization is a long shot beyond the scope of the present article. Here, we only want to show that covariant measurements are not always optimal.
4. Contravariant quantum measurements

We now come to Bob’s measurement method. As explained in Sect. 1, Bob examines separately the spins sent by Alice to indicate her \( z \) axis, and those indicating her \( x \) axis. (The argument below applies to any number of spins, not just two spins for each axis.) In his coordinate frame, Bob thus gets two sets of polar angles, \( \theta_z \phi_z \) and \( \theta_x \phi_x \) respectively, from which he has to infer the Euler angles \( \psi \theta \phi \) that transform Alice’s Cartesian frame into his frame. If Bob’s measurements were perfect, the relations between these angles would be given by equating the Cartesian components of Bob’s results for Alice’s \( z \) and \( x \) axes with the corresponding columns of the orthogonal transformation matrix [11]. This gives

\[
\begin{pmatrix}
\sin \theta_z \cos \phi_z \\
\sin \theta_z \sin \phi_z \\
\cos \theta_z
\end{pmatrix} =
\begin{pmatrix}
\sin \psi \sin \theta \\
\cos \psi \sin \theta \\
\cos \theta
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\sin \theta_x \cos \phi_x \\
\sin \theta_x \sin \phi_x \\
\cos \theta_x
\end{pmatrix} =
\begin{pmatrix}
\cos \psi \cos \phi - \sin \psi \cos \theta \sin \phi \\
- \sin \psi \cos \phi - \cos \psi \cos \theta \sin \phi \\
\sin \theta \sin \phi
\end{pmatrix}.
\]

These are four independent equations (owing to normalization) for three unknowns. If Bob’s experimental data were exact, a simple solution would be to obtain from Eq. (15)

\[
\theta = \theta_z \quad \text{and} \quad \psi = (\pi/2) - \phi_z.
\]

The fidelity of this result, namely for finding the direction of a single axis by using any number of spins, is discussed in [3, 4] where it is shown that, asymptotically, \( (1 - F) \propto N^{-2} \).

Once \( \theta \) is known, \( \phi \) can be obtained from the third line of (16):

\[
\sin \phi = \cos \theta_z / \sin \theta_z,
\]

where use was made of the result in Eq. (17). Now there is a difficulty: if \( N \) is finite, Bob’s estimates are not perfectly accurate and the right hand side of (18) may be larger than 1. In general, it is preferable to solve the four equations (15) and (16) simultaneously and to seek a best fit for the three unknowns. The accuracy of this best fit is of course better than the one given by Eqs. (17) and (18), where one of the four original equations
was ignored. A simple geometric construction of the solution is as follows: first, find the direction perpendicular to the estimated $z$ and $x$ axes; this direction is the best estimate for the $y$ axis, and therefore for the $zx$ plane. Then, in that plane, the angle between the estimated $z$ and $x$ axes (given by $\theta_z \phi_z$ and $\theta_x \phi_x$ respectively) is adjusted so that they become exactly perpendicular. Detailed calculations are under progress (we hope they will appear in a future publication).

5. The dihedral group

A clearer understanding of contravariant measurements may be gained by using a finite group. As a concrete example, let us consider six directions, defined by the polar angles $\theta = 45^\circ$ or $135^\circ$, and $\phi = 0$ or $\pm 120^\circ$. Alice wishes to indicate one of these directions to Bob. Now, these directions are the orbit of the dihedral group $D_3$ with six elements: $E$ (no change), $A$, $B$, $C$ (rotations by $180^\circ$ around the symmetry axes $\phi = 0$ and $\phi = \pm 120^\circ$ in the $xy$ plane), and $D$, $F$ (rotations by $\pm 120^\circ$ around the $z$ axis). Here, we are using the same notations as Wigner [12]. This group has a one-dimensional representation where all the elements are 1, another where $E$, $D$, $F$ are 1, while $A$, $B$, $C$ are $-1$, and there is a two-dimensional representation, explicitly given in [12]. From the characters of these three representations, it is possible to find the contents of any other, reducible one.

Suppose that Alice has a single particle of spin $\frac{1}{2}$, and she wants to indicate to Bob which one of the six directions she has chosen. Obviously, she orients her spin along that direction, so that there are six input states,

$$\rho = (\mathbb{1} + \mathbf{n} \cdot \mathbf{\sigma})/2,$$

where $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Likewise Bob has six POVM elements

$$E_m = (\mathbb{1} + \mathbf{m} \cdot \mathbf{\sigma})/6.$$

Note that $\sum E_m = \mathbb{1}$. Then the probability of Bob getting result $m$ is Alice’s input is $\mathbf{n}$ is

$$P(m|\mathbf{n}) = \text{tr}(\rho E_m) = (1 + \mathbf{n} \cdot \mathbf{m})/6.$$

Note that the probability of getting the correct result is always $1/3$.

We must now specify a criterion for the fidelity of the transmission. A simple one is to give a score 1 if Bob guesses the correct result, and 0 for all incorrect results. It could
also be argued that some results are more incorrect than others, just as large errors $\chi$ in Eq. (1) are more heavily penalized than small error angles. Here, for a group of order 6, we could assume that elements belonging to the same class are less wrong than those belonging to different classes of the group, and incur a lesser penalty. However, we shall just assume that all wrong results are equally worthless. Therefore the best that can be achieved with one spin is fidelity $F = 1/3$.

Suppose now that Alice sends several spins. Rather than using individual measurements and classical statistics, Bob may perform joint measurements on all these spins. For example, if there are two spins, their state belongs to the rotation group representations with $j = 0, 1$. However, that group is too rich: we are interested only in the $D_3$ group which is a subgroup of $SO(3)$. Obviously $j = 0$ corresponds to the symmetric one-dimensional representation of $D_3$ (all the elements are 1). As for $j = 1$, we have to find the characters of all the rotations that correspond to elements of $D_3$. This is very easy, because one may use for $SO(3)$ the real orthogonal representation, and then, owing to Euler’s theorem [11], the characters (that is, the traces of the rotation matrices) are equal to $1 + 2 \cos \Phi$, where $\Phi$ is the rotation angle. We thus find that the character of $E$ is 3, those of $D$ and $F$ are $-1$, and those of $A$, $B$, $C$ vanish. This means that the triplet state involves the one-dimension representation with alternating signs and the two-dimensional representation.

Therefore a pair of spin $\frac{1}{2}$ particles generates all the representations of $D_3$, each one once. It is plausible that in this case a covariant measurement is optimal (detailed calculations are given in an Appendix). However, taking more spins will not produce any new irreducible representation, and will not improve the fidelity if Bob is restricted to the use of covariant measurements.

A simple method which gives better results is the following. Alice sends $N$ spins, all aligned in the direction she wants to indicate, as in Ref. [1]. This is an angular momentum coherent state [13] for spin $j = N/2$:

$$n \cdot J \ket{\psi} = j \ket{\psi}. \tag{22}$$

(This is surely not the optimal strategy. In the present paper, we are not seeking optimality. We only want to show that some methods give a better fidelity than a straightforward covariant measurement.) Bob then performs the covariant measurement for a particle of
spin \( j \). His POVM elements are coherent states as above, with directions uniformly distributed over the two-dimensional sphere. The overlap of two such states is \[13\]

\[
\cos^{4j}(\chi/2) = \cos^{2N}(\chi/2),
\]  

(23)

where \( \chi \) is the angle between the true and estimated directions.

Once Bob has found a result \( \theta \phi \) (this is what we call the “raw” result), he infers (guesses) that the true answer for Alice’s signal is the direction \( n \) closest to \( \theta \phi \). It is, as in the preceding example, the best fit for the answer, knowing the approximate value given by \( \theta \) and \( \phi \). As there are finite angles between the six directions \( n \) that Alice can use, it follows from (23) that the probability of error decreases exponentially with \( N \). It is plausible that a truly optimal method would also have such an exponential accuracy, but with a larger coefficient for \( N \) in the exponent.

6. Concluding remarks and apologies

Due to the pressure of a deadline, we did not attempt to find the optimal strategy in the two examples given above. In both cases, the same pattern emerges. Bob first executes a POVM which uses a group that is not the one for Alice’s signals. This POVM gives redundant raw data to Bob, from which he infers, by a classical statistical analysis, the best estimate for identifying the signal. This final best estimate is nonlinear and it cannot be obtained directly by a POVM of rank one, as in Eq. (7). Indeed, if it could, then it would have to be a covariant POVM, and we have just shown that this is not the best method. Explicitly, the POVM elements that give the best guess are \( E_g = \sum E_h \), where the sum runs over all the raw outcomes \( h \) that lead to the same guess \( g \).

Note that there is no contradiction with Gleason’s theorem [14] because the proof of the latter refers only to the outcome of the measuring process, without a further best fit or other classical statistical analysis. We hope that further research on this problem will clarify the missing details and give a complete prescription for the optimal procedure.

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Appendix. Dihedral group signals with one or two spins

In this Appendix we analyze covariant measurements for the detection of signals belonging to the $D_3$ group. If only one particle of spin $\frac{1}{2}$ is sent by Alice, its state is given as usual by

$$|2_g\rangle = \left(\begin{array}{c} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{array}\right),$$  \hspace{1cm} (A 1)

where $\theta$ and $\phi$ are the angles that correspond to group element $g$. Bob’s fiducial state is

$$|B\rangle = |2_E\rangle/\sqrt{3}.$$  \hspace{1cm} (A 2)

(The factor $\sqrt{3}$ comes from the order of the group divided by the number of dimensions [13].) Then

$$\sum U(g) |B\rangle\langle B| U^\dagger(g) = 1,$$  \hspace{1cm} (A 3)

is Bob’s POVM. The fidelity, which is defined in this problem as the probability of a correct result, is

$$F = |\langle 2_E|B\rangle|^2 = \frac{1}{3}.$$  \hspace{1cm} (A 4)

Suppose now that Alice has two spins. She sends them both in state $|2_g\rangle$, and Bob tests them separately. The probability that he gets twice the correct answer is $\frac{1}{9}$. The probability that he gets the correct answer once, together with one wrong answer, is $\frac{4}{9}$. In the latter case, faced with an ambiguous result, Bob will make a random choice. Then the final probability for a correct guess is $\frac{1}{9} + \frac{2}{9} = \frac{1}{3}$, exactly as for a single signal!

If Alice sent more than two spins in this way and Bob tested them separately, the result, given by a multinomial distribution, would slowly improve. However, Alice and Bob can do much better. Returning to the case of two spins, Alice can send an entangled signal. The latter has singlet and triplet components. Using $D_3$ notations, that state is

$$|A_g\rangle = a_0 |0_g\rangle + a_1 |1_g\rangle + a_2 |2_g\rangle,$$  \hspace{1cm} (A 5)

where $|0_g\rangle$ corresponds to the symmetric representation (all the $|0_g\rangle$ are equal), $|1_g\rangle$ to the antisymmetric one, and $|2_g\rangle$ is still given by Eq. (A 1). The coefficients $a_m$ are normalized:

$$|a_0|^2 + |a_1|^2 + |a_2|^2 = 1.$$  \hspace{1cm} (A 6)
Bob’s fiducial vector now is

\[ |B\rangle = \sqrt{1/6} |0_E\rangle + \sqrt{1/6} |1_E\rangle + \sqrt{1/3} |2_E\rangle, \]  
(A 7)

so that Eq. (A 3) is still valid.

The probability for a correct result thus is

\[ F = |\langle B|A_E\rangle|^2 = |(a_0 + a_1)/\sqrt{6} + a_2/\sqrt{3}|^2. \]  
(A 8)

This is a quadratic expression for the coefficients \(a_m\), subject to the normalization (A 6). Its maximum is easily found to be \(2/3\), when

\[ a_0 = a_1 = 1/2 \quad \text{and} \quad a_2 = 1/\sqrt{2}. \]  
(A 9)

We see that this detection method, which actually is a covariant measurement, dramatically improves the fidelity from \(1/3\) to \(2/3\).

However, testing more than two spins in this way leads to no further improvement, because all the irreducible representations of \(D_3\) have been exhausted. Only a contravariant measurement, such as the one described in Sect. 4, can give better results: the infidelity \((1 - F)\) then decreases exponentially with the number of spins, owing to Eq. (23).

References


