Practical Relativistic Model of Microarcsecond Astrometry in Space

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ABSTRACT

This paper is devoted to a practical model for relativistic reduction of positional observations with an accuracy of 1 µas. All the relativistic effects important in practice for this accuracy are thoroughly discussed and calculated. The model is presented in the form which could be directly implemented into a software system for data processing.

Subject headings: astrometry, reference systems, relativity

1. Introduction

Within the next decade the accuracy of space-based astrometric positional observations is expected to attain a level of 1 microarcsecond (µas). The problem of relativistic modelling of positional observations with a microarcsecond accuracy has become very practical in the recent years when several astrometric space projects have been approved by NASA, ESA and other boards and selected for launch in the next several years (FAME (Horner et al. 1999), SIM (Shao 1998), GAIA (GAIA 2000; Perryman et al. 2001), and DIVA (Bastian et al. 1996)). Following this practical trend this paper describes a practical relativistic model of space-based optical positional observations which is valid at a level of 1 µas and which can be readily implemented in the corresponding software.

Relativistic effects in positional observations have been studied by many authors in many different aspects. Needless to say that the gravitational light deflection in the gravitational field of the Sun being one of the most important constituents of the relativistic model of positional observations was one of the first experimental tests of general relativity. Already in the time of Hipparcos it was realized that relativity must play an important role in the formulation of the transformation between the observed positions of a star and what should be included in the resulting catalogue (Walter et al. 1986). However, it was a relatively uncomplicated task to formulate such a model for Hipparcos which was aimed at the final accuracy of 1 mas. A model of positional observations suitable for an accuracy of 1 µas is much more intricate. It is clear since the typical
relativistic effects here exceed the required accuracy by several orders of magnitude. The whole concept of the modelling should be re-formulated in the framework of general relativity. Besides that at such a high level of accuracy many additional, more subtle effects should be taken into account (e.g., in the light propagation).

The first complete general-relativistic model of positional observations at the microarcsecond level of accuracy was formulated in Klioner & Kopeikin (1992). That work was primarily stimulated by the early project for microarcsecond astrometry in space POINTS (Reasenberg et al. 1988). The model was formulated in a rather general form and was primarily intended for a satellite on a geocentric orbit (geostationary or lower). The model described in this paper is based on the model presented in Klioner & Kopeikin (1992), but differs from the latter in several important aspects: (1) the model takes into account refined treatment of several relativistic effects which has become available in the recent years; (2) the model is optimized and simplified as much as possible, which makes it straightforward to implement the model in the corresponding software; (3) the model is formulated in the framework of the so-called Parametrized Post-Newtonian (PPN) formalism (Will 1993) which makes it possible to use the positional observations to test general theory of relativity.

Since the interest to the microarcsecond astrometry was revived in the recent years, a number of numerical simulations of astrometric missions has appeared. de Felice et al. (1998, 2000, 2001) have used a highly simplified relativistic model (Schwarzschild field of the Sun) to simulate GAIA-like observations and investigate various statistical properties of the solution in different cases. Analogous work based on a more realistic model has been started by another group (Kopeikin et al. 2000). The model represented in this paper is intended to facilitate further investigations of this kind.

Section 2 is devoted to a general scheme of relativistic modelling of astronomical observations of any kind in the framework of general relativity or PPN formalism. The overall structure of the specific modelling scheme for positional observations made from a space station is described in Section 3. Modelling of the motion of the observer (satellite) and its proper time is discussed in Section 4. Section 5 deals with the relativistic description of aberration. Gravitational light deflection is discussed in Section 6. Parallax and proper motion are analyzed in Sections 7 and 8, respectively. Section 9 summarizes the suggested relativistic model. In Section 10 several known relativistic effects beyond the given model are described.

2. General scheme of relativistic modelling of astronomical observations

Let us first outline general principles of relativistic modelling of astronomical observations. General scheme is represented on Fig. 1. Starting from general theory of relativity, any other metric theory of gravity or PPN formalism one should define at least one relativistic 4-dimensional reference system covering the region of space-time where all the processes constituting particular kind of astronomical observations are located. Typical astronomical observation depicted on Fig. 2
Fig. 1.— General principles of relativistic modelling of astronomical observations (see text for further explanations).
Fig. 2.—Four constituents of an astronomical event: 1) motion of the observed object, 2) motion of the observer, 3) propagation of an electromagnetic signal from the observed object to the observer, 4) the process of observation.
consists of four constituents: motion of an observer, motion of an observed object, light propagation and the process of observation. Each of these four constituents should be modelled in the relativistic framework. The equations of motion of both the observed object and the observer relative to the chosen reference system should be derived and a method to solve these equations should be found. Typically the equations of motion are the second-order ordinary differential equations and numerical integration with suitable initial or boundary conditions can be used to solve them. All the information on the object which is available to the observer can be read off the electromagnetic signals propagating from the object to the observer. Therefore, the corresponding equations of light propagation relative to the chosen reference system should be derived and solved. The equations of motion of the object and the observer and the equations of light propagation enable one to compute positions and velocity of the object, observer and the photon (light ray) with respect to the particular reference system at a given moment of coordinate time, provided that the positions and velocities at some initial epoch are known. However, these positions and velocities obviously depend on the used reference system. On the other hand, the results of observations cannot depend on the reference system used to theoretically model the observations. Therefore, it is clear that one more step of the modelling is needed: relativistic description of the process of observation. This part of the model allows one to compute a coordinate-independent theoretical prediction of the observables starting from the coordinate-dependent position and velocity of the observer and, in some cases, the coordinate velocity of the electromagnetic signal at the point of observation.

Mathematical techniques to derive the equations of motion of the observed object and the observer, the equations of light propagation and to find the description of the process of observation in the relativistic framework are well known and will be discussed below. These three parts can now be combined into relativistic models of observables. The models give an expression for each observables under consideration as a function of a set of parameters. These parameters should be fitted to observational data to produce astronomical reference frames, which represent sets of estimates of certain parameters appearing in the relativistic models of observables.

It is very important to understand at this point that the relativistic models contain parameters which are defined only in the chosen reference system(s) and are thus coordinate-dependent. A good example of such coordinate-dependent parameters are the coordinates and velocities of various objects (e.g., major planets or the satellite) at some epoch. On the other hand, from a physical point of view any reference system covering the region of space-time under consideration can be used to describe physical phenomena within that region, and we are free to choose the reference system to be used to model the observations. However, reference systems, in which mathematical description of physical laws is simpler than in others, are more convenient for practical calculations. Therefore, one can use the freedom to choose the reference system to make the parametrization as convenient and reasonable as possible (e.g., one prefers the parameters to have simpler time-dependence).

For modelling of physical phenomena localized in some sufficiently small region of space (e.g., in the vicinity of a massless observer or a gravitating body) one can construct a so-called local reference system where the gravitational influence of the outer world are effaced as much as possible
in accordance with Einstein equivalence principle. In the local reference system of a material system the gravitational field of the outer matter manifests itself in the form of tidal gravitational potential. First, the Solar system as a whole can be considered as one single body, and a reference system can be constructed where the gravitational influence of the matter situated outside of the Solar system can be described by a tidal potential. That tidal potential (mainly due to the influence of the Galaxy) is utterly small and can be then neglected for most purposes. Such a reference system is often called barycentric reference system of the Solar system. It can be used far beyond the Solar system and is suitable to describe the dynamics of the Solar system (motion of planets and spacecraft relative to the barycenter of the Solar system) as well as to model the influence of the gravitational field of the Solar system on the light ray propagating from remote sources to an observer. Second, the corresponding reference system can be constructed for the Earth. Such a geocentric reference system is convenient to model geocentric motion of Earth satellites, rotational motion of the Earth itself, etc. Underlying theory and technical details concerning local reference systems have been discussed in a series of papers by Brumberg and Kopeikin (Kopeikin 1988; Brumberg & Kopeikin 1989a,b; Brumberg 1991, see also Klioner & Voinov (1993)) and Damour, Soffel and Xu (1991, 1992, 1993, 1994). IAU (2001) has recently recommended the use of a particular form of the barycentric and local geocentric reference systems for modelling of astronomical observations. These two standard relativistic reference systems are called Barycentric Celestial Reference System (BCRS) and Geocentric Celestial Reference System (GCRS). Coordinate time $t$ of the BCRS is called Barycentric Coordinate Time (TCB). Coordinate time $T$ of the GCRS is called Geocentric Coordinate Time (TCG). Throughout the paper the spatial coordinates of the BCRS will be designated as $x$ while those of the GCRS as $X$.

Both Brumberg-Kopeikin and Damour-Soffel-Xu theories are based on Einstein’s general relativity theory. It is clear, however, that one of the important goals of the future astrometric missions is to test general relativity. The model presented in this paper will be given in the framework of the PPN formalism (see, e.g., Will 1993) including two main parameters $\beta$ and $\gamma$ which characterize possible deviations of physical reality from general relativity. Both parameters are equal to unity in general relativity. The theory of local reference systems with parameters $\beta$ and $\gamma$ has been constructed in Klioner & Soffel (1998b, 2000). Both that theory of local PPN reference systems and the model given in the present paper are constructed in such a way that for $\beta = \gamma = 1$ all the formulas coincide with those which can be derived directly from the general-relativistic versions of the BCRS and GCRS recommended by the IAU.

3. General structure of the relativistic model of positional observation

The relativistic model of positional observations relates the observed direction to the light source with the coordinate direction toward the source at the moment of observation. A set of the coordinate directions toward the source for different moments of time can be then used to obtain further parameters of the source describing its spatial position and spatial motion with respect
to the BCRS (parallax and proper motion or barycentric orbital parameters). It is convenient to divide the conversion of the observed direction into the coordinate one into several steps. Let us introduce five vectors which will be used below for the intermediate steps of the conversion: \( s \) is the observed direction, \( n \) is the unit tangent vector to the light ray at the moment of observation (the word “unit” means here and below that the formally Euclidean scalar product \( n \cdot n = n^i n^i \) is equal to unity), \( \sigma \) is the unit tangent vector to the light ray at \( t = -\infty \), \( k \) is the unit coordinate vector from the source to the observer, \( l \) is the unit vector from the barycenter of the Solar system to the source (see Fig. 3). Note that the last four vectors are defined formally in the coordinate space of the BCRS and should not be interpreted as “Euclidean” vectors in “Newtonian” physical space. For the same physical situation these vectors are different if different reference systems are used instead of the BCRS. The word “vector” is used here to refer to a set of three real numbers defined in the coordinate space of the BCRS, rather than to a geometric object. A slightly different meaning of vector \( s \) is discussed in Section 5 below.

Apart from the modelling of the motion of the observer which will be considered in the next Section, the model consists in transforming these five vectors. Namely, the following effects will be subsequently considered:

- aberration (the effects related to the motion of the observer with respect to the barycenter of the Solar system): this converts the observed direction to the source \( s \) into the unit BCRS coordinate velocity of the light ray \( n \) at the point of observation \( x_o \);
- gravitational light deflection for the source at infinity: this step converts \( n \) into the unit direction of propagation \( \sigma \) of the light ray infinitely far from the Solar system for \( t \to -\infty \);
- coupling of the finite distance to the source and the gravitational light deflection in the gravitational field of the Solar system: this step converts \( \sigma \) into a unit BCRS vector \( k \) going from the source to the observer (note that as discussed below this step should be combined with the previous one for the sources situated in the Solar system);
- parallax: this step converts \( k \) into a unit vector \( l \) going from the barycenter of the Solar system to the source;
- proper motion: this step provides a reasonable parametrization of the time dependence of \( l \) caused by the motion of the source with respect to the BCRS.

All these steps will be specified in detail in the following Sections. However, let us first clarify the question of time scales which should be used in the model. There are 4 time scales appearing in the model:

- proper time of the observer (satellite): \( \tau_o \);
- proper time of the \( i \)th tracking station: \( \tau_{\text{station}}^{(i)} \).
Fig. 3.—Five principal vectors used in the model: $s$, $n$, $\sigma$, $k$, $l$. See text for further details.
• coordinate time \( t = \text{TCB} \) of the BCRS (alternatively a scaled version of TCB called TDB can be used: \( \text{TDB} = (1 - L_B) \text{TCB} \) with the current best estimate of the scaling constant \( L_B \approx 1.55051976772 \cdot 10^{-8} \pm 2 \cdot 10^{-17} \) (Irwin & Fukushima 1999)).

• coordinate time \( T = \text{TCG} \) of the GCRS (alternatively a scaled version of TCG called TT can be used: \( \text{TT} = (1 - L_G) \text{TCG} \), \( L_B \equiv 6.969290134 \cdot 10^{-10} \) being a defining constant (IAU 2001)).

It is clear that the observational data (e.g., in case of scanning satellites like GAIA, FAME and DIVA projections of vector \( \mathbf{s} \) on a local reference system of the satellite which rotates together with the satellite) are parametrized by the proper time of the satellite \( \tau_o \). It is also clear that the final catalog containing positions, parallaxes and proper motions of the sources relative to the BCRS should be parametrized by TCB. The other two time scales (proper times of the tracking station(s) \( \tau_{\text{station}}^{(i)} \) and TCG) are used exclusively for orbit determination.

The transformation between proper time of the satellite \( \tau_o \) and TCB can be done by integrating

\[
\frac{d \tau_o}{dt} = 1 - \frac{1}{c^2} \left( \frac{1}{2} \dot{x}_o^2 + w(x_o) \right) + \mathcal{O}(c^{-4}),
\]

where \( x_o \) and \( \dot{x}_o \) are the BCRS position and velocity of the satellite and \( w(x_o) \) is the gravitational potential of the Solar system which can be approximated by

\[
w(x_o) \approx \sum_A \frac{GM_A}{|r_{oA}|},
\]

\( r_{oA} = x_o - x_A, \) \( M_A \) being the mass of body \( A \) and \( x_A \) its barycentric position. Both higher order multipole moments of all the bodies and additional relativistic terms are neglected in (2). The transformation between the proper time of a tracking station and TCG can be performed in a similar way. The transformation between TCG and TCB is given in the IAU Resolutions B1.3 (general post-Newtonian expression) and B1.5 (an expression for the accuracy of \( 5 \cdot 10^{-18} \) in rate and 0.2 ps in amplitude of periodic effects) given in (IAU 2001). There are several analytical and numerical formulas for the position-independent part of the transformation (see, e.g., Fukushima 1995; Irwin & Fukushima 1999, and reference cited therein).

Although the use of the relativistic time scales described above is indispensable from the theoretical and conceptual points of view, from a purely practical point of view considerations of accuracy can be used here to simplify the model. However, this depends on the particular parameters of the mission and will be not analyzed here. In the following it is assumed that the observed direction \( \mathbf{s} \) is given together with the corresponding epoch of observation \( t_o \) in TCB scale.
4. Motion of the satellite

It is well known that in order to compute the Newtonian aberration with a precision of 1 µas one needs to know the velocity of the observer with an accuracy of \( \sim 1 \text{ mm/sec} \) (see, e.g., GAIA 2000). This is a rather stringent requirement and special care must be taken to attain such an accuracy. Modelling of satellite motion with such an accuracy is a complicated task involving complicated equations of motion which take into account various non-relativistic (Newtonian \( N \)-body force, radiation pressure, active satellite thrusters, etc.) as well as relativistic effects. Here some general recipe concerning relativistic part of the modelling will be given. Both the non-relativistic parts of the model and a detailed study of the relativistic effects in the satellite motion are beyond the scope of the present paper.

In the relativistic model of positional observations developed in the following Sections it is assumed that the observations are performed from a space station or an Earth satellite whose position \( \mathbf{x}_o \) relative to the BCRS is known for any moment of barycentric coordinate time \( t \). For those satellites the orbits of which are not located in the vicinity of the Earth (this is the case for both GAIA and SIM) it is advantageous to model its motion directly in the BCRS. Since the mass of the satellite is too small to noticeably affect the motion of other bodies of the Solar system, the equations of geodetic motion in the BCRS can be used as equations of motion of the satellite. It is sufficient (at least in the relativistic part of the equations) to neglect herewith the multipole structure of the gravitating bodies as well as the gravitational field produced by rotational motion of these bodies. Therefore, considering a system of \( N \) bodies, each of which can be characterized by position \( \mathbf{x}_A \), velocity \( \dot{\mathbf{x}}_A \) and mass \( M_A \) (\( A \) being an index numbering the bodies) the equations of motion of the satellite read (Will 1993; Klioner & Soffel 2000)

\[
\frac{d^2}{dt^2} \mathbf{x}_o = - \sum_A G M_A \frac{\mathbf{r}_{oA}}{|\mathbf{r}_{oA}|^3} + \frac{1}{c^2} \sum_A G M_A \frac{\mathbf{r}_{oA}}{|\mathbf{r}_{oA}|^3} \left\{ (2\beta - 1) \sum_{B \neq A} \frac{G M_B}{|\mathbf{r}_{AB}|} + 2(\gamma + \beta) \sum_B \frac{G M_B}{|\mathbf{r}_{oB}|} \right. \\
+ \left. \frac{3}{2} \frac{(\mathbf{r}_{oA} \cdot \dot{\mathbf{x}}_A)^2}{|\mathbf{r}_{oA}|^2} \right. \\
- \frac{1}{2} \sum_{B \neq A} \frac{G M_B}{|\mathbf{r}_{AB}|^3} \frac{\mathbf{r}_{oA} \cdot \mathbf{r}_{AB}}{|\mathbf{r}_{AB}|^3} \\
- (1 + \gamma) \dot{\mathbf{x}}_A \cdot \dot{\mathbf{x}}_A - \gamma \dot{\mathbf{x}}_o \cdot \dot{\mathbf{x}}_o
\]
\[ + 2(1 + \gamma) \dot{x}_o \dot{x}_A \right) \]
\[ + \frac{1}{c^2} \sum_A G M_A \frac{\dot{x}_o - \dot{x}_A}{|r_{oA}|^3} \left\{ 2(1 + \gamma) \dot{x}_o \cdot r_{oA} - (2\gamma + 1) \dot{x}_A \cdot r_{oA} \right\} \]
\[ - \frac{1}{c^2} \left( 2\gamma + \frac{3}{2} \right) \sum_A G M_A \sum_{B \neq A} G M_B \frac{r_{AB}}{|r_{AB}|^3} + \mathcal{O}(c^{-4}), \tag{3} \]

where \( r_{oA} = x_o - x_A \) and \( r_{AB} = x_A - x_B \).

The observations of the satellite itself performed to determine its orbit (typically range and Doppler tracking) should be also modelled in the relativistic framework: (1) the positions of the observing stations must be given in the GCRS and then transformed into the BCRS, (2) the description of the signal propagation between the satellite and the observing stations must take into account the corresponding relativistic effects in the BCRS (the Shapiro effect and its consequences for Doppler observations), (3) the difference between the proper time scales of the satellite and the observing sites and the coordinate time of the BCRS must be also taken into account. A detailed description of these steps is beyond the scope of the present paper. The final result of the orbit determination procedure is the BCRS position \( x_o \) and velocity \( \dot{x}_o \) of the satellite as a function of \( t = TCB \).

If the satellite is close enough to the Earth (geostationary or lower) it is advantageous to model its motion in the GCRS. The equations of motion of a satellite with respect to the GCRS are given in (Brumberg & Kopeikin 1989b; Klioner & Voinov 1993; Damour, Soffel and Xu 1994) in the framework of general relativity and in Section VIII of Kliener & Soffel (2000) in the framework of the PPN formalism with parameters \( \beta \) and \( \gamma \). The structure of the GCRS equations of motion of a satellite can be written in the form:

\[ \frac{d^2}{dT^2} X_o = \Phi_E + \Phi_{el} + \frac{1}{c^2} (\Phi_{coup} + \Phi_{mg} + \Phi_{SEP}) + \mathcal{O}(c^{-4}), \tag{4} \]

where \( \Phi_E \) is the acceleration due to the gravitational field of the Earth, \( \Phi_{coup} \) is the Earth-third body coupling term, \( \Phi_{el} \) is the “gravito-electric” part (independent of the velocity of the satellite) of the tidal influence of external bodies (this formally includes the Newtonian tidal force), \( \Phi_{mg} \) is the purely relativistic “gravito-magnetic” part (depending on the velocity of the satellite) of the tidal influence of external bodies, and, finally, \( \Phi_{SEP} \) is the additional acceleration due to possible violation of the strong equivalence principle in alternative theories of gravity. The main relativistic terms in \( \Phi_E \) come from the spherically symmetric part of the Earth’s gravitational field and can be formally derived in the framework of Schwarzschild solution for the Earth considered to be isolated. This main relativistic effect is recommended to be taken into account in the IERS Conventions (IERS 1996). Explicit formulas for the right hand side of (4) can be found in Section VIII of Kliener &
Soffel (2000) (the same equations in the framework of general relativity were derived in Klioner & Voinov (1993); Damour, Soffel and Xu (1994) and in a simplified form in Brumberg & Kopeikin (1989b)).

If the equations (4) are used, the whole process of orbit determination can be performed directly in the GCRS. Again the orbit determination observations should be consistently modelled in the relativistic framework: both the coordinates of the stations and the position of the satellite should be described in the GCRS, the propagation of the electromagnetic signals should be adequately modelled in the GCRS, the proper time scales of the satellite and the tracking stations and the TCG should be properly converted into each other when needed. The result of the orbit determination process is the coordinates of the satellite $X_o$ as a function of TCG. In order to be used in the model of positional observations given below these coordinates must be transformed into the corresponding BCRS coordinates:

$$ X_o = r_oE + \frac{1}{c^2} \left( \frac{1}{2} v_E (v_E \cdot r_oE) + \gamma w_{\text{ext}}(x_E) r_oE + \gamma r_oE (a_E \cdot r_oE) - \frac{1}{2} \gamma a_E r_oE^2 \right) + O(c^{-4}), \quad (5) $$

where $r_oE = x_o - x_E$, and $x_E, v_E$ and $a_E$ are the BCRS position, velocity and acceleration of the Earth, and $w_{\text{ext}}(x_E)$ is the gravitational potential of the Solar system except for that of the Earth evaluated at the geocenter. It is easy to estimate that the relativistic effects in (5) amount to $\sim 1 \text{m}$ for $|x_o| \sim 50000 \text{km}$. The GCRS velocity of the satellite can be transformed into the corresponding BCRS velocity as

$$ V_o = \delta v_o + \frac{1}{c^2} \left( \delta v_o \left( \frac{1}{2} |v_E|^2 + (1 + \gamma) (w_{\text{ext}}(x_E) + a_E \cdot r_oE) + v_E \cdot \delta v_o \right) \right. $$

$$ + \frac{1}{2} v_E (v_E \cdot \delta v_o) + \gamma r_oE (a_E \cdot \delta v_o) - \gamma a_E (r_oE \cdot \delta v_o) $$

$$ \left. + \gamma w_{\text{ext}}(x_E) r_oE + \frac{1}{2} a_E (r_oE \cdot v_E) + \frac{1}{2} v_E (r_oE \cdot a_E) \right) $$

$$ + \gamma r_oE^i (a_E \cdot r_oE) - \frac{1}{2} \gamma a_E |r_oE|^2 \right) + O(c^{-4}), \quad (6) $$

where $V_o = \frac{d}{dT} X_o$, $\delta v_o = \frac{d}{dT} x_o - v_E$ and $a_E = \frac{d}{dT} a_E$. It is easy to see from (6) that for a geostationary satellite numerical difference between $V_o$ and $\delta v_o$ is less than 0.1 mm/s. Therefore, for the model with a final accuracy of $\sim 1 \mu\text{s}$ the relativistic terms in (6) can be neglected.

Note that probably even for the goal accuracy of 1 mm/s in the velocity of the satellite one can significantly simplify both the BCRS equations of motion (3) and those in the GCRS (4). However, this crucially depends on particular orbit of the satellite and such an analysis is beyond the scope of the present paper.
Let us also note that rotational motion of the satellite should also be carefully modelled (for some missions the attitude of the satellite will be determined from the observational data produced by the satellite itself, but nevertheless a kind of theoretical modelling is still necessary). From the theoretical point of view in order to model the rotational motion of the satellite it is convenient to introduce a local kinematically-nonrotating reference system for the satellite with coordinates \((\tau_o, \xi)\), where \(\tau_o\) is the proper time of the satellite related to \(t\) by (1) and \(\xi\) are the spatial coordinates of the reference system related to those of BCRS by a transformation analogous to (5) (see Brumberg & Kopeikin 1989a; Klioner 1993; Klioner & Soffel 1998a, for details on the local reference system of the satellite). Then rotational motion of the satellite with respect to the spatial axes of the satellite is described by the post-Newtonian rotational equations of motion discussed in (Damour, Soffel and Xu 1993) for the case of general relativity and in (Klioner & Soffel 1998b, 2000) in the framework of the PPN formalism. From the practical point of view, it is, however, clear that the largest relativistic effect in the rotational motion of the satellite with respect to remote stars is due to geodetic (de Sitter) precession which amounts to \(\sim 2 \mu\text{as} \text{ per hour (}\sim 2'' \text{ per century)}\) for the satellites on a heliocentric orbit with the semi-major axis close to 1 AU (like GAIA and SIM) and \(\sim 140 \mu\text{as} \text{ per hour (}\sim 1.2'' \text{ per year)}\) for the satellites on geostationary orbit (like FAME and DIVA). Taking into account that the satellites will typically monitor and verify their attitude with the help of onboard gyroscopes and observations of specially selected stars, and that the precise attitude with respect to remote stars will be determined aposteriori from the processing of observational data, it is unlikely that such small relativistic effects could be of practical importance.

The rest of the relativistic model of positional observations is totally independent of the GCRS. It is only the orbit determination process (or at least a part of it) which involves the use of the GCRS coordinates and concepts.

5. Aberration

The first step of the model is to get rid of the aberrational effects induced by the barycentric velocity of the observer. Let \(s\) denote the unit direction \((s \cdot s = 1)\) toward the source as observed by the observer (satellite). Let \(p\) be the BCRS coordinate velocity of the photon in the point of observation. Note that \(p\) is directed from the source to the observer. The unit BCRS coordinate velocity of the light ray \(n = p/|p|\) \((n \cdot n = 1)\) can be then computed as (Klioner 1991b; Klioner & Kopeikin 1992)

\[
s = -n + \frac{1}{c} n \times (\dot{x}_o \times n) + \frac{1}{c^2} \left\{(n \cdot \dot{x}_o) n \times (\dot{x}_o \times n) + \frac{1}{2} \dot{x}_o \times (n \times \dot{x}_o)\right\}
\]
\[
+ \frac{1}{c^3} \left\{ \left( \mathbf{n} \cdot \dot{x}_o \right)^2 + (1 + \gamma) w(x_o) \right\} \mathbf{n} \times (\dot{x}_o \times \mathbf{n}) + \frac{1}{2} \left( \mathbf{n} \cdot \dot{x}_o \right) \dot{x}_o \times (\mathbf{n} \times \dot{x}_o) \right\} \\
+ \mathcal{O}(c^{-4}),
\]

where \( w(x_o) \) is the gravitational potential of the Solar system at the point of observation. This formula contains relativistic aberrational effects up to the third order with respect to \( 1/c \). Because of the first order aberrational terms (classical aberration) the BCRS coordinate velocity of the satellite must be known to an accuracy of \( \sim 10^{-3} \text{ m/s} \) in order to attain an accuracy of \( 1 \mu\text{as} \). For a satellite with the BCRS velocity \( |\dot{x}_o| \sim 40 \text{ km/s} \), the first-order aberration is of the order of \( 28'' \), the second-order effect may amount to \( 3.6 \mu\text{mas} \), and the third-order effects are \( \sim 1 \mu\text{as} \). Note also that the higher-order aberrational effects are nonlinear with respect to the velocity of the satellite and cannot be divided into pieces like “annual” and “diurnal” aberrations as it could be done with the first order aberration for an Earth-bound observer. This is the reason why one needs a precise value of the BCRS velocity of the satellite.

Note that vector \( s \) is defined in the kinematically non-rotating local satellite reference system \((\tau_o, \xi^a)\) which is mentioned in the previous Section. As it was discussed in Section 7.1 of Klioner & Kopeikin (1992) this point of view is equivalent to the standard tetrad approach. Eq. (7) can be derived from both standard tetrad formalism (see, e.g., Brumberg 1986; Klioner 1991b) and the considerations related to the local reference system of the satellite (Klioner & Kopeikin 1992). The actual observations of a scanning astrometric satellite are referred to a reference system, spatial axes of which rigidly rotate with respect to \( \xi \) so that the satellite’s attitude remains fixed in the rotating axes. The rotating axes \( \bar{\xi} \) are related to the kinematically non-rotating axes \( \xi \) as (Einstein’s summation rule is assumed here)

\[
\bar{\xi}^a = P^{ab}(\tau_o) \xi^b,
\]

where \( P^{ab} \) is a time-dependent orthogonal matrix, which can be parametrized, e.g., by three Euler angles. These Euler angles define the attitude of the satellite with respect to the kinematically non-rotating axis and are to be determined from the observations and the corresponding modelling.

To clarify the origin of the terms in Eqs. (7) proportional to the gravitational potential \( w \) let us consider a fictitious observer, whose position coincides with that of the satellite at the moment of observation and whose velocity with respect to the BCRS is zero. The direction toward the source observed by this fictitious observer coincides with \(-\mathbf{n}\) as can be calculated from Eq. (7) for \( \dot{x}_o = 0 \). On the other hand, the transformation between the directions toward the source observed by the two observers located at the same point of space-time can be derived from the usual Lorentz transformation which can be used in its closed form to speed up the practical calculations (Mignard 2000). The Lorentz transformation depends only on the velocity of one observer as seen by the other one \( v \). Hence, using the Lorentz transformation in its closed form one gets
\[
\begin{align*}
\mathbf{s} &= \left( -\mathbf{n} + \left\{ \frac{\Gamma}{c} - [\Gamma - 1] \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v}|^2} \right\} \mathbf{v} \right) \frac{1}{\Gamma (1 - \mathbf{v} \cdot \mathbf{n}/c)}, \\
\Gamma &= \frac{1}{\sqrt{1 - |\mathbf{v}|^2/c^2}}.
\end{align*}
\]

(9)

(10)

The velocity of the satellite measured by the fictitious observer is given by

\[
\mathbf{v} = \dot{x}_o \left( 1 + \frac{1}{c^2} (1 + \gamma) w(x_o) \right) + \mathcal{O}(c^{-3}).
\]

(11)

It is this renormalization of the velocity of the satellite which leads to the \(w\)-dependent terms under consideration. It is easy to check that formulas (9)–(11) are equivalent to (7).

Alternatively, it is easy to calculate a formula relating the observed angular distance \(\varphi_{12}^{\text{obs}}\) (\(\cos \varphi_{12}^{\text{obs}} = s_1 \cdot s_2\)) between two given sources and the coordinate angular distance \(\varphi_{12}^{\text{coord}}\) between them (\(\cos \varphi_{12}^{\text{coord}} = n_1 \cdot n_2\)):

\[
\cos \varphi_{12}^{\text{obs}} = \cos \varphi_{12}^{\text{coord}} + \left( \cos \varphi_{12}^{\text{coord}} - 1 \right) \left[ -\left( 1 + \frac{(1 + \gamma) w(x_o)}{c^2} \right) \frac{|\dot{x}_o|}{c} \left( \cos \psi_1 + \cos \psi_2 \right) \\
+ \frac{|\dot{x}_o|^2}{c^2} (\cos^2 \psi_1 + \cos^2 \psi_2 + \cos \psi_1 \cos \psi_2 - 1) \\
- \frac{|\dot{x}_o|^3}{c^3} (\cos \psi_1 + \cos \psi_2) (\cos^2 \psi_1 + \cos^2 \psi_2 - 1) \right] + \mathcal{O}(c^{-4}),
\]

(12)

where \(\psi_i\) are the angles between the direction toward the \(i\)th source and the direction of the velocity \(\dot{x}_o\):

\[
\cos \psi_i = -\frac{1}{|\dot{x}_o|} n_i \cdot \dot{x}_o, \quad i = 1, 2.
\]

(13)

The terms in Eqs. (7) and (12) proportional to \(w\) are the largest terms of the third order. However, the third order terms are of order \(1 \mu\)s and, therefore, the potential \(w\) can be approximated here by the potential of the spherically symmetric Sun

\[
w(x_o) \approx \frac{GM_{\text{sun}}}{|x_o - x_{\text{Sun}}|}.
\]

(14)
A closed-form expression equivalent to (12) can be derived from (9):

\[ 1 - \cos \varphi^{\text{obs}}_{12} = \left( 1 - \cos \varphi^{\text{coord}}_{12} \right) \frac{1 - \frac{|v|^2}{c^2}}{\left( 1 + \frac{|v|}{c} \cos \psi_1 \right) \left( 1 + \frac{|v|}{c} \cos \psi_2 \right)}, \]  

(15)

where \( v \) is defined by (11).

For order-of-magnitude estimations it is also useful to derive a formula for the angular shift \( \delta \psi \) of the source toward the apex of the satellite's motion:

\[ \delta \psi = \frac{1}{c} \left| \dot{x}_o \right| \sin \psi \left( 1 + \frac{(1 + \gamma) w(x_o)}{c^2} + \frac{1}{4} \frac{\left| \dot{x}_o \right|^2}{c^2} \right) - \frac{1}{4} \frac{\left| \dot{x}_o \right|^2}{c^2} \sin 2\psi + \frac{1}{12} \frac{\left| \dot{x}_o \right|^3}{c^3} \sin 3\psi + O(c^{-4}) \]  

(16)

or in closed form

\[ \cos(\psi - \delta \psi) = \frac{\cos \psi + \frac{|v|}{c}}{1 + \frac{|v|}{c} \cos \psi}. \]  

(17)

Here \( \psi \) is the angle between the direction toward the source and the direction of the satellite’s velocity \( \cos \psi = -\mathbf{n} \cdot \mathbf{v} / |\mathbf{v}| \).

### 6. Gravitational light deflection

The next step is to account for the general-relativistic gravitational light deflection, that is to convert \( \mathbf{n} \) into the corresponding unit coordinate direction \( \mathbf{k} \) from the observed source at the moment of emission to the observer at the moment of observation. Here two different cases should be distinguished: (1) objects outside the Solar system for which their finite distance from the origin of the BCRS plays almost no role for this step of the reduction scheme, and (2) Solar system objects, the finite distance of which must be taken into account here. Let us first relate \( \mathbf{n} \) to the unit coordinate direction \( \mathbf{\sigma} \) of the light propagation infinitely far from the gravitating sources for \( t \to -\infty \) and then consider the influence of the finite distance to the objects on the gravitational light deflection separately for these two classes of objects in Sections 6.1 and 6.2 below.
The Solar system is assumed here to be isolated. This means that the gravitational field produced by the matter outside of the Solar system is neglected. This assumption is well founded if time dependence of the gravitational field produced outside of the Solar system is negligible. Otherwise the external gravitational field must be explicitly taken into account (e.g., for observations of edge-on binary stars, where the gravitational field of the companion can cause an additional time-dependent light deflection, or for astrometric microlensing events). Some of such cases will be discussed below in Section 10.

The equations of light propagation can be derived from the general-relativistic Maxwell equations. It is sufficient, however, to consider only the limit of geometric optics. The relativistic effects depending on wavelength (and thus representing deviations from geometric optics) are much smaller than 1 µas in the Solar system (see, e.g., Mashhoon 1974). In the limit of geometric optics the relativistic equations of light propagation can be written in the form

\[
x_p(t) = x_p(t_o) + c \sigma (t - t_o) + \Delta x_p(t),
\]

(18)

where \( t_o \) is the moment of observation, \( x_p(t_o) \) is the position of the photon at the moment of observation (this position obviously coincides with the position of the satellite at that moment \( x_p(t_o) = x_o(t_o) \)), \( \sigma \) is the unit coordinate direction of the light propagation at the past null infinity (\( \sigma \cdot \sigma = 1 \))

\[
\sigma = \lim_{t \to -\infty} \frac{1}{c} \dot{x}_p(t),
\]

(19)

and \( \Delta x_p \) is the sum of all gravitational effects in the light propagation which satisfies the conditions

\[
\Delta x_p(t_o) = 0,
\]

(20)

\[
\lim_{t \to -\infty} \Delta \dot{x}_p(t) = 0.
\]

(21)

From the BCRS metric one can easily see that \( \Delta x_p(t) \sim \mathcal{O}(c^{-2}) \) and \( \frac{1}{c} \Delta \dot{x}_p(t) \sim \mathcal{O}(c^{-2}) \).

Several effects in \( \Delta x_p \) should be apriori considered at the level of 1 µas: (1) the effects of the spherically symmetric gravitational field of each sufficiently large gravitating body of the Solar system, (2) the effects due to the non-sphericity (mainly due to the quadrupole moment) of the bodies, (3) the effects caused by the gravitomagnetic field due to translational motion of the bodies, (4) the effects caused by the gravitomagnetic field due to rotational motion of the bodies. The reduction formulas for all there effects have been derived and discussed in detail in (Brumberg, Klioner & Kopejkin 1990; Klioner 1991a,b; Klioner & Kopeikin 1992).
Table 1 illustrates the maximal magnitudes of various gravitational effects due to Solar system bodies and the maximal angular distances between the source and the body at which the gravitational light deflection from that body should be accounted for to attain the final accuracy of 1 µas. Note that the values in Table 1 are slightly different from those published in (Brumberg, Klioner & Kopejkin 1990; Klioner 2000) because for Table 1 the best current estimates for physical parameters of the Solar system bodies were taken from Weissman et al. (1999), whereas the IAU (1976) system of astronomical constants was used for the previous publications.

One can see that the post-post-Newtonian terms attain 1 µas only within 53′ from the Sun and can be neglected in case of space astrometry since no of the proposed satellites could observe so close to the Sun. For the same reason the effect due to rotational motion of the Sun amounting to 0.7 µas for a grazing ray is unobservable. The largest observable effect due to rotational motion is 0.2 µas for a ray grazing Jupiter. Therefore, the effects due to rotational motion of the bodies are also too small to be taken into account at the level of 1 µas.

The largest contribution in $\Delta x_p$ for Solar system applications comes from the spherically symmetric components of the gravitational fields of the massive bodies and can be calculated as

$$\Delta_{pN}x_p(t) = -\sum_A (1 + \gamma)GM_A \frac{1}{c^2} \left( d_A I_A + \frac{1}{c} J_A \right), \quad \text{(22)}$$

$$d_A = \sigma \times (r_{Ao} \times \sigma), \quad \text{(23)}$$

$$I_A = \frac{1}{|r_A| - \sigma \cdot r_A} - \frac{1}{|r_{Ao}| - \sigma \cdot r_{Ao}}, \quad \text{(24)}$$

$$J_A = \log \frac{|r_A| + \sigma \cdot r_A}{|r_{Ao}| - \sigma \cdot r_{Ao}}, \quad \text{(25)}$$

$$r_A = x_p(t) - x_A, \quad \text{(26)}$$

$$r_{Ao} = x_p(t_o) - x_A = x_o(t_o) - x_A, \quad \text{(27)}$$

so that

$$\frac{1}{c} \Delta_{pN}x_p(t) = -\sum_A (1 + \gamma)GM_A \frac{1}{c^2} \left( d_A I_A + \sigma \frac{1}{c} J_A \right), \quad \text{(28)}$$

$$\frac{1}{c} I_A = \frac{1}{|r_A|} \left( |r_A| - \sigma \cdot r_A \right), \quad \text{(29)}$$

$$\frac{1}{c} J_A = \frac{1}{|r_A|}, \quad \text{(30)}$$

where $x_A$ is the position and $M_A$ is the mass of body A.

The positions of the bodies $x_A$ are supposed to be constant in (22)–(30). In reality, however, the bodies are moving and this motion cannot be neglected in the calculation of light deflection. It
is hardly possible to calculate analytically the light path in the gravitational field of an arbitrarily moving body without resorting to some approximations. On the other hand, numerical integration which could be used here as remedy is too impractical for massive calculations necessary in astrometry. Let us, therefore, consider a Taylor expansion of $x_A$:

$$x_A(t) = x_A(t_{A0}) + \dot{x}_A(t_{A0})(t - t_{A0}) + \mathcal{O}(\dot{x}_A).$$  \hspace{1cm} (31)$$

One can integrate the equations of motion for a light ray using the linear function (31) for coordinates of the gravitating body. The solution was first derived in Klioner (1989) and describes the light propagation under the assumption that the gravitating bodies move with a constant velocity (only the effects linear with respect to velocity $\dot{x}_A$ are taken into account here, since formally the terms quadratic in $\dot{x}_A$ are of post-post-Newtonian order $\mathcal{O}(c^{-4})$). However, expansion (31) has a free parameter $t_{A0}$ which can be used to minimize the error in the light propagation equations caused by the higher-order terms neglected in (31). In Klioner & Kopeikin (1992) from an analysis of the residual terms in the equations of light propagation proportional to accelerations of the bodies it was shown that one reasonable choice which guarantee the residual effects to be small within the chosen approximation scheme is to set $t_{A0}$ to be the moment of the closest approach of the body and the photon:

$$t_{A0}^{ca} = \max(t_e, t_o - \max\left(0, \frac{g \cdot (x_o(t_o) - x_A(t_o))}{c |g|^2}\right)),$$  \hspace{1cm} (32)$$

$$g = \sigma - \frac{1}{c} \dot{x}_A(t_{A0}),$$  \hspace{1cm} (33)$$

where $t_e$ is the time of emission of the light ray by the source.

Recently an advanced formalism to integrate the equations of light propagation in the field of arbitrarily moving gravitating bodies has been suggested by Kopeikin & Schäfer (1999). The authors suggest to use the solution of Einstein equations in the form of retarded potentials so that the positions of the gravitating bodies are computed for the retarded moment

$$t_{A0}^{r} = t_o - \frac{1}{c} |x_o(t_o) - x_A(t_{A0}^r)|,$$  \hspace{1cm} (34)$$

and derive rigorous laws of the light propagation within their scheme. The authors prove also that if moment $t_{A0}$ in (31) is taken to be $t_{A0}^{r}$ the effects due to accelerations of the gravitating bodies as well as those proportional to the second and higher orders of velocities $\dot{x}_A$ are much smaller than 1 µas for Solar system applications and thereby completely negligible. On the other hand, the formulas for $\Delta x_p(t)$ derived in (Klioner 1989; Klioner & Kopeikin 1992) using formal expansion (31) with $t_{A0} = t_{A0}^{r}$ and those derived in (Kopeikin & Schäfer 1999) coincide. This means that if the bodies are assumed to move with a constant velocity and the effects quadratic in $\dot{x}_A$ are
neglected the solutions derived in (Klioner 1989; Klioner & Kopeikin 1992) for \( t_{A0} = t_{ca}^{A0} \) (or even for \( t_{A0} = t_o \)) and that derived in (Kopeikin & Schäfer 1999) with \( t_{A0} = t_{r}^{A0} \) are equivalent. Although the approach developed by Kopeikin & Schäfer (1999) allows a rigorous analysis of the residual terms and should be used in those applications where the effects of acceleration are important, this approach does not prove that the choice \( t_{A0} = t_{r}^{A0} \) gives minimal residual terms. It is difficult to judge which moment \( t_{A0} = t_{ca}^{A0} \) or \( t_{A0} = t_{r}^{A0} \) is numerically more advantageous to use in (31). If the effects explicitly proportional to \( \dot{x}_A \) are taken into account to compute the light path, the difference between the solutions for \( t_{A0} = t_{ca}^{A0} \) and for \( t_{A0} = t_{r}^{A0} \) in the gravitational field of the Solar system is several orders of magnitude lower than 1 \( \mu \)as.

The effects explicitly proportional to the velocity of the body \( \dot{x}_A(t_{A0}) \) have been also investigated in detail in (Klioner 1989; Klioner & Kopeikin 1992; Kopeikin & Schäfer 1999). If the position of the body is taken for \( t_{A0} = t_{ca}^{A0} \) or \( t_{A0} = t_{r}^{A0} \) this effect in the light deflection can be estimated as \( \delta T^* \sim \frac{1}{c} |\dot{x}_A| \delta_{pN} \), where \( \delta_{pN} \) is the deflection induced by the spherically symmetric field of the body. According to Table 1 the effect may amount to 0.8 \( \mu \)as for a ray grazing Jupiter and only 0.2 \( \mu \)as in case of the Sun. However, \( \delta_{pN} \) attains its maximal value for the case when the impact parameter of the light ray is much smaller than the distances between the gravitating body and the points of emission and observation. For this case one can prove that \( \delta T^* \sim \frac{1}{c} \sigma \cdot \dot{x}_A \delta_{pN} \). The cosine factor in \( \sigma \cdot \dot{x}_A \) significantly reduces the effect produced by Jupiter or Saturn in their orbital motion for an observer situated near the Earth orbit. Therefore, the effects proportional to velocities of the bodies can be neglected in our model.

If these effects are neglected, we effectively use

\[
x_A(t) = x_A(t_{A0})
\]

for coordinates of the bodies. Numerical simulations show that the difference of the light deflection angle calculated using \( t_{A0} = t_{ca}^{A0} \) and \( t_{A0} = t_{r}^{A0} \) in (35) does not exceed 0.01 \( \mu \)as. Therefore, for practical purposes both \( t_{ca}^{A0} \) and \( t_{r}^{A0} \) can be used and \( x_A \) should be taken to be \( x_A(t_{ca}^{A0}) \) or \( x_A(t_{r}^{A0}) \) in (26)–(27) and all related formulas. Note, however, that, e.g., the use of the moment of reception \( t_o \) instead of \( t_{ca}^{A0} \) or \( t_{r}^{A0} \) in (35) may lead to a significant error in the calculated gravitational light deflection. This error may amount to 6 mas in case of Jupiter and cannot be neglected.

It is to note that a number of smaller bodies should be also taken into account. For a spherical body with mean density \( \rho \), the light deflection is larger than \( \delta \) if its radius

\[
L \geq \left( \frac{\rho}{1 \text{ g/cm}^3} \right)^{-1/2} \left( \frac{\delta}{1 \mu \text{as}} \right)^{1/2} 624 \text{ km}.
\]

Therefore, at the level of 1 \( \mu \)as (and even 10 \( \mu \)as) one should account for several largest satellites of giant planets, Pluto and Charon, and also Ceres. The maximal values of the effects produced by these bodies are evaluated in Table 1. It is clear, however, that the gravitational light deflection
due to these small bodies could be larger than 1 μas only if a source is observed very close to the bodies. The maximal angular distances at which this additional gravitational deflection should be taken into account are also given in Table 1.

Finally the effect of quadrupole field of the giant planets should be taken into account if the angular distance between the planet and the object is smaller than the values given in the sixth column of Table 1. The corresponding reduction formulas for $\Delta Q_p(t)$ and $\Delta \dot{Q}_p(t)$ derived in Klioner (1991a); Klioner & Kopeikin (1992) read

$$
\Delta Q_p(t) = \frac{(1 + \gamma)}{2c^2} G \sum_A \left( \alpha_A \mathcal{U}_A + \beta_A \mathcal{E}_A + \gamma_A \mathcal{F}_A + \delta_A \mathcal{V}_A \right),
$$

(37)

$$
\mathcal{U}_A = \frac{1}{|d_A|} \left( \frac{1}{|r_A|} \left| r_A + \sigma \cdot r_A \right| - \frac{1}{|r_Ao|} \left| r_Ao - \sigma \cdot r_Ao \right| \right),
$$

(38)

$$
\mathcal{E}_A = \frac{\sigma \cdot r_A}{|r_A|^3} - \frac{\sigma \cdot r_Ao}{|r_Ao|^3},
$$

(39)

$$
\mathcal{F}_A = |d_A| \left( \frac{1}{|r_A|^3} - \frac{1}{|r_Ao|^3} \right),
$$

(40)

$$
\mathcal{V}_A = -\frac{1}{|d_A|^2} \left( \frac{\sigma \cdot r_A}{|r_A|} - \frac{\sigma \cdot r_Ao}{|r_Ao|} \right),
$$

(41)

$$
\alpha_A = 2f_A - 2(f_A \cdot \sigma) - (g_A \cdot \sigma + 4f_A \cdot h_A) h_A, \quad \beta_A = 2(f_A \cdot \sigma) h_A + (g_A \cdot \sigma - f_A \cdot h_A) \sigma, \quad \gamma_A = 2(f_A \cdot \sigma) \sigma - (g_A \cdot \sigma - f_A \cdot h_A) h_A, \quad \delta_A = 2g_A - 4(f_A \cdot \sigma) h_A - (g_A \cdot \sigma - 2f_A \cdot h_A) \sigma,
$$

(42-45)

$$
\frac{1}{c} \Delta Q_p(t) = \frac{(1 + \gamma)}{2c^2} G \sum_A \left( \alpha_A \frac{1}{c} \dot{\mathcal{U}}_A + \beta_A \frac{1}{c} \dot{\mathcal{E}}_A + \gamma_A \frac{1}{c} \dot{\mathcal{F}}_A + \delta_A \frac{1}{c} \dot{\mathcal{V}}_A \right),
$$

(46)

$$
\frac{1}{c} \dot{\mathcal{U}}_A = |d_A| \left( \frac{2|r_A| - \sigma \cdot r_A}{|r_A|^3(|r_A| - \sigma \cdot r_A)^2} \right),
$$

(47)

$$
\frac{1}{c} \dot{\mathcal{E}}_A = \frac{|r_A|^2 - 3(\sigma \cdot r_A)^2}{|r_A|^5},
$$

(48)

$$
\frac{1}{c} \dot{\mathcal{F}}_A = -3|d_A| \frac{\sigma \cdot r_A}{|r_A|^5},
$$

(49)

$$
\frac{1}{c} \dot{\mathcal{V}}_A = -\frac{1}{|r_A|^3},
$$

(50)

where $h_A = d_A/|d_A|$, $f_A^j = M_{ij}^A h_A^j$, $g_A^j = M_{ij}^A \sigma^j$ (Einstein’s summation rule is assumed in the last two formulas), and $M_{ij}^A$ is the traceless quadrupole moment of body A. From the point of view of the theory of relativistic local reference systems, the multipole structure of the gravitational field
of a body is defined in the relativistic local reference system of that body. However, for calculation of the gravitational light deflection due to the quadrupole field the relativistic effects in $M_{ij}^A$ can be neglected and one can proceed as if $M_{ij}^A$ is defined in the BCRS. Matrix $M_{ij}^A$ is symmetric and trace-free and has, therefore, five independent components which can be calculated from the second zonal harmonic coefficient $J_2^A$ (other coefficients of the second order are negligible), the mass $M_A$ and the equatorial radius $L_A$ of the planet, and the coordinates $(\alpha_{\text{pole}}^A, \delta_{\text{pole}}^A)$ of the north pole of its figure axis:

$$M_{ij}^A = M_A L_A^2 J_2^A \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & -A - D \end{pmatrix},$$

$$A = \frac{1}{3} - \cos^2 \alpha_{\text{pole}}^A \cos^2 \delta_{\text{pole}}^A,$$

$$B = -\frac{1}{2} \sin 2\alpha_{\text{pole}}^A \cos^2 \delta_{\text{pole}}^A,$$

$$C = -\frac{1}{2} \cos \alpha_{\text{pole}}^A \sin 2\delta_{\text{pole}}^A,$$

$$D = \frac{1}{3} - \sin^2 \alpha_{\text{pole}}^A \cos^2 \delta_{\text{pole}}^A,$$

$$E = -\frac{1}{2} \sin \alpha_{\text{pole}}^A \sin 2\delta_{\text{pole}}^A.$$  

(51)

The effects of $\Delta p_N x_p$ and $\Delta Q x_p$ are additive and one can put

$$\Delta x_p = \Delta p_N x_p + \Delta Q x_p.$$  

(57)

As first discussed in Klioner (1991a), the higher-order multipole moments of Jupiter and Saturn may produce a deflection larger than 1 $\mu$as, provided that the source is observed very close to the surfaces of the planets. It is easy to see that the maximal gravitational light deflection produced by the zonal spherical harmonics $J_n$ (all other harmonics are utterly small for the giant planets) can be estimated as $\delta J_n \sim J_n \delta p_N$. Using modern values for $J_n$ of Jupiter and Saturn (Weissman et al. 1999, p. 342) one can check that it is only the effects of $J_4$ of Jupiter and Saturn which may exceed 1 $\mu$as. Namely, the effect of $J_4$ of Jupiter may amount to 10 $\mu$as and that of Saturn is not greater than 6 $\mu$as. The effect of $J_n$ decreases as $\cot^{n+1} \psi$ with the angular distance $\psi$ between the gravitating body and the source. Therefore, the effect of $J_4$ of Jupiter exceeds 1 $\mu$as only if the angular distance $\psi$ between the center of Jupiter and the source is smaller than 1.6 of the apparent angular radius of Jupiter (i.e., smaller than 24 – 39$''$ depending on the mutual configuration of Jupiter and the satellite). For $J_4$ of Saturn $\psi$ should be less than 1.4 of the apparent radius (i.e., less than 10 – 14$''$ depending on the configuration). The influence of $J_6$ of Jupiter and Saturn also may exceed 1 $\mu$as if the real values of $J_6$ (which are known with a large uncertainty
for both planets) are larger than the values given in Weissman et al. (1999). If the observations so close to Jupiter and Saturn are to be processed, the reduction formulas for the effect of \( J_4 \) can be derived from formulas given in Kopeikin (1997). Let us also note that since the polar radii of the giant planets are significantly smaller than their equatorial radii, impact parameter may be smaller than the equatorial radius of the body. In this case the standard expansion of gravitational potential in terms of spherical harmonics becomes inadequate (the expansion converges only outside the sphere encompassing the body) and some other ways to represent the gravitational potential must be used to calculate the gravitational deflection in such a situation.

Coordinate velocity of the photon can be obtained by taking time derivative of (18):

\[
p \equiv \frac{1}{c} \dot{x}_p(t_o) = \sigma + \frac{1}{c} \Delta \dot{x}_p(t_o).
\]  

The unit coordinate direction of the light propagation at the moment of observation reads

\[
n = \frac{p}{|p|}.
\]  

Eqs. (58)–(59) are more convenient for numerical calculations than an analytical expansion of \( n \) in terms of \( \sigma \) and \( \Delta \dot{x}_p \) which can be derived by substituting (58) into (59) and expanding in powers of \( c^{-1} \). However, the accuracy of 1 \( \mu \)as can be attained with a simplified first-order expansion of (58)–(59)

\[
n = \sigma + \frac{1}{c} \sigma \times (\Delta \dot{x}_p(t_o) \times \sigma)
\]  

if the distance from the observer to the Sun (all other bodies play no role here) is larger than 0.035 AU which is the case for all currently proposed astrometrical missions.

Therefore,

\[
n = \sigma + \delta \sigma_{pN} + \delta \sigma_Q,
\]  

\[
\delta \sigma_{pN} = \frac{1}{c} \sigma \times (\Delta_{pN} \dot{x}_p(t_o) \times \sigma),
\]  

\[
\delta \sigma_Q = \frac{1}{c} \sigma \times (\Delta_Q \dot{x}_p(t_o) \times \sigma).
\]

Using (28) one gets

\[
\delta \sigma_{pN} = - \sum_A \frac{(1 + \gamma)GM_A}{c^2} \frac{d_A}{|d_A|^2} \left( 1 + \sigma \cdot \frac{r_{Ao}}{|r_{Ao}|} \right).
\]
Hence the post-Newtonian deflection angle due to the spherically symmetric part of the gravitational field of body \(A\) reads

\[
\delta_{pN}^A = \frac{(1 + \gamma)GM_A}{c^2 |\mathbf{r}_{Ao}|} \cot \frac{\psi_A}{2},
\]

(65)

where \(\psi_A\) is the angular distance between body \(A\) and the source. It is this formula which was used to compute the data in the third column of Table 1. The effect of the quadrupole field \(\delta\sigma_Q\) can be calculated by substituting (46)–(50) and (42)–(45) with (51)–(56) into (63).

### 6.1. Coupling of finite distance to the source and gravitational deflection: Solar system objects

The next step is to convert \(\sigma\) into the unit vector \(\mathbf{k}\) directed from the source to the observer. Let \(\mathbf{x}_o(t_o)\) be the coordinate of the observer (satellite) at the moment of observation \(t_o\) and \(\mathbf{x}_s(t_e)\) is the position of the source at the moment of emission \(t_e\) of the signal which was observed at moment \(t_o\). The moment of emission \(t_e\) is considered a function of the moment of observation \(t_o\) (see Section 8 for further discussion). Let us denote

\[
\mathbf{R}(t_o) = \mathbf{x}_o(t_o) - \mathbf{x}_s(t_e),
\]

(66)

\[
\mathbf{k}(t_o) = \frac{\mathbf{R}(t_o)}{|\mathbf{R}(t_o)|},
\]

(67)

where \(\mathbf{R}(t_o) = |\mathbf{R}(t_o)|\). It is easy to see that vector \(\mathbf{k}\) is related to \(\sigma\) as (Klioner 1991a):

\[
\sigma = \mathbf{k} + \frac{1}{R} \mathbf{k} \times (\Delta \mathbf{x}_p(t_e) \times \mathbf{k}) + \mathcal{O}(c^{-4}).
\]

(68)

In the case of a Solar system object (68) can be combined with (60) to get

\[
\mathbf{n} = \mathbf{k} + \delta \mathbf{k}_{pN} + \delta \mathbf{k}_Q,
\]

(69)

\[
\delta \mathbf{k}_{pN} = \mathbf{k} \times \left( \left( \frac{1}{R} \Delta \mathbf{p}_N \mathbf{x}_p(t_e) + \frac{1}{c} \Delta \mathbf{p}_N \mathbf{x}_p(t_e) \right) \times \mathbf{k} \right),
\]

(70)

\[
\delta \mathbf{k}_Q = \mathbf{k} \times \left( \left( \frac{1}{R} \Delta \mathbf{Q} \mathbf{x}_p(t_e) + \frac{1}{c} \Delta \mathbf{Q} \mathbf{x}_p(t_e) \right) \times \mathbf{k} \right).
\]

(71)
Table 1: Various gravitational effects in the light propagation in \( \mu \)as: pN and ppN are the post-Newtonian and post-post-Newtonian effects due to the spherically symmetric field of each body, Q are the effects due to the quadrupole gravitational fields, R and T\(^*\) are the effects due to the gravitomagnetic fields caused by rotational and translational motion of the bodies. The estimations of T\(^*\) are given for the case when the coordinates of the gravitating bodies are taken at the moment of the closest approach of the body and the photon (see text). Symbol “—” means that the effect is smaller than 0.1 \( \mu \)as. Physical parameters of the bodies are taken from Weissman et al. (1999). Because of the minimal Sun avoidance angle the influence of some bodies can neglected for certain missions (for GAIA, e.g., Mercury is too close to the Sun and can be neglected). The angle \( \psi_{\text{max}}(\delta) \) is the maximal angular distance between the body and the source at which the corresponding effect still attains \( \delta \) (the smallest possible distance between the observer and each body is taken here; \( \psi_{\text{max}}(\delta) \) is smaller for larger distances). For these estimates the observer is supposed to be within a few million kilometers from the Earth orbit. For the Earth and the Moon two estimates are given: for a geostationary satellite and for a satellite at a distance of \( 10^6 \) km from the Earth.
Hence,

\[
\delta k_{pN} = - \sum_A \frac{(1 + \gamma)GM_A}{c^2} \frac{R \times (r_{Ac} \times r_{Ao})}{|R||r_{Ao}|(|r_{Ac}| - r_{Ao} \cdot r_{Ac})},
\]

(72)

\[r_{Ac} = x_s(t_e) - x_A.\]

(73)

The angle between vectors \(k\) and \(n\) can be calculated as

\[
\frac{(1 + \gamma)GM_A}{c^2 |r_{Ao}|} \tan \frac{\phi}{2},
\]

(74)

where \(\phi\) is the angle between vectors \(r_{Ac}\) and \(r_{Ao}\). Note that the angle (74) depends on the distance between the gravitating body and the point of observation and on angle \(\phi\), but does not depend on the distance between the point of emission and the gravitating body. For \(|r_{Ac}| \rightarrow \infty\) one has \(\phi = \pi - \psi_A\) and (74) coincides with (65).

The effect of quadrupole field \(\delta k_Q\) can be calculated by substituting (37)–(56) into (71).

For an object located in the Solar system a set of vectors \(k\) calculated from the observed directions \(s\) for several different moments of time \(t_o\) allows one to derive the barycentric orbit of that object. Therefore, vector \(k\) is the final result of the relativistic model for the Solar system objects.

6.2. Coupling of finite distance to the source and gravitational deflection: objects outside the Solar system

The only effect in \(\Delta x_p\) which should be taken here into account for the objects located outside the Solar system (with \(|x_s| > 1000\) AU) is the post-Newtonian gravitational deflection from the spherically symmetric part of the gravitational field of the Sun. In this case from (68) one gets

\[
\sigma = k - \frac{(1 + \gamma)GM_{Sun}}{c^2 |R|} \frac{R}{|r_{So}|} - \frac{|r_{Se}|}{r_{So} \times r_{Se}} R \times (|r_{So}| \times r_{Se}) + O(c^{-4}),
\]

(75)

\[r_{Se} = x_s(t_e) - x_{Sun},\]

(76)

\[r_{So} = x_o(t_o) - x_{Sun}.\]

(77)

The angle between \(\sigma\) and \(k\) can be calculated as

\[
\frac{(1 + \gamma)GM_{Sun}}{c^2 |r_{So}| \sin \psi_{Sun}} \left(1 + a - \sqrt{1 - 2a \cos \psi_{Sun} + a^2}\right) \approx \frac{(1 + \gamma)GM_{Sun}}{c^2 |R|} \left(\cot^2 \frac{\psi_{Sun}}{2} + O(a)\right),
\]

(78)
where \( a = |r_{So}|/|R| \) and \( \psi_{\text{Sun}} \) is the angular distance between the source and the Sun. The effect attain 8.5 \( \mu \)as for a source situated at a distance of 1 pc and observed at the limb of the Sun. One can check that the effect (that is the angle between \( k \) and \( \sigma \)) is larger than 1 \( \mu \)as if \( |X| \leq 8.5 \)pc and the source is observed within 2.3\(^\circ\) from the Sun. If at least one of these conditions is violated (which is really the case for all currently proposed astrometric missions since no observations can be done so close to the Sun) one can put

\[
\sigma = k. \tag{79}
\]

Let us note that the requirement to calculate the gravitational effects with an accuracy of 1 \( \mu \)as puts a constrain on the required accuracy of the planetary ephemerides (simply speaking, one has to be able to calculate the impact parameter of the light ray with respect to each gravitating body with a sufficient accuracy). The required accuracy of the ephemerides depends on the minimal allowed angular distance between the observed star and the gravitating body. Thus, for a grazing ray the barycentric position of Jupiter should be known with an accuracy of 4 km and other planets with slightly lower accuracy. The barycentric position of the Sun should be known with an accuracy of about 400 m for a grazing ray, and with an accuracy of \( \sim 6000 \) km for the minimal allowed angular distance of 35\(^\circ\) as adopted for GAIA. Note that the barycentric position of the satellite must be known with at least the same accuracy (see Section 5.4 of GAIA (2000) for other accuracy constrains).

7. Parallax

Only sources situated outside of the Solar system are considered below. For objects situated outside of the Solar system the next step in the model is to get rid of the parallax, that is to transform \( k \) into the unit vector \( l \) directed from the barycenter of the Solar system to the source

\[
l(t) = \frac{x_s(t_e)}{|x_s(t_e)|}. \tag{80}
\]

Note that starting from this point further parametrization of vectors \( k \) and \( l \) formally coincides with what one could expect in the Newtonian framework. From the formal mathematical point of view these vectors may be considered as Euclidean vectors in 3-dimensional space formed formally by the spatial coordinates of the BCRS. It is important to understand, however, that this interpretation is only formal and that those vectors are not Euclidean vectors in some “underlying Euclidean physical space”, but rather integration constants for the equations of light propagations in the BCRS. These vectors are defined by the whole previous model of relativistic reduction and would change if the model is changed (e.g., if another relativistic reference system is used instead of the BCRS).
Here, the definitions of parallax, proper motion and radial velocity compatible with general relativity at a level of 1 \( \mu \)as (or better) are suggested. Although the definitions are quite simple and straightforward, their interpretation at such a high level of accuracy is rather unusual from the point of view of Newtonian astrometry. As it will be clear below parallax and proper motion are no longer two separate effects which can be considered independently of each other as it was in classical astrometry. Thus, second order parallaxes and proper motions as well as the effects resulting from the interaction between these two effects are important. Moreover, parallax, proper motion and other astrometric parameters are coordinate-dependent parameters defined in the BCRS, which is used as the relativistic reference system where the position and motion of sources are described. Therefore these parameters have some meaning only within a particular chosen model of relativistic reductions. That is why the whole relativistic model of positional observations must be considered to define these parameters and clarify their meaning.

Let us define several parameters. The parallax of the source is defined as

\[
\pi(t_o) = \frac{1 \text{ AU}}{|x_s(t_o)|},
\]

(81)

the parallactic parameter \( \Pi \) is given by

\[
\Pi(t_o) = \pi(t_o) \frac{x_o(t_o)}{1 \text{ AU}},
\]

(82)

and finally the observed parallactic shift of the source is defined as

\[
\pi(t_o) = l(t_o) \times (\Pi(t_o) \times l(t_o)).
\]

(83)

With these definitions to sufficient accuracy one has

\[
k = -l \left( 1 - \frac{1}{2} |\pi|^2 \right) + \pi \left( 1 + l \cdot \Pi \right) + \mathcal{O}(\pi^3).
\]

(84)

The second order effects in (84) proportional to \( \pi^2 \) are less 3 \( \mu \)as if \( |x_s| \geq 1 \) pc. The second order terms can be safely neglected if \( |x_s| \geq 2 \) pc.

8. Proper motion

The last step of the algorithm is to provide a reasonable parametrization of the time dependence of \( l \) and \( \pi \) caused by the motion of the source relative to the Solar system barycenter.

It is commonly known that in order to convert the observed proper motion and the observed radial velocity into true tangential and radial velocities of the observed object additional information
is required. Since that information is not always available, the concepts of “apparent proper motion”, “apparent tangential velocity” and “apparent radial velocity” are suggested below. These concepts represent useful information about the observed object and should be distinguished from “true tangential velocity” and “true radial velocity”. Definitions of all these concepts are discussed below.

In the present paper the following simple model for the coordinates of the source is adopted:

\[
x_s(t_e) = x_s(t_0^e) + V \Delta t_e + \frac{1}{2} A \Delta t_e^2 + O(\Delta t_e^3),
\]

(85)

Here, \( \Delta t_e = t_e - t_0^e \), and \( V \) and \( A \) are the BCRS velocity and acceleration of the source evaluated at moment of emission \( t_0^e \) corresponding initial epoch of observations \( t_0^e \). This model allows one to consider single stars or components of gravitationally bounded systems, periods of which are much larger than the time span covered by observations. It is also clear that in more complicated cases special solutions for binary stars, etc. should be considered. Depending on particular properties of the source and on the time span of observations higher-order terms in (85) can also be considered. In this paper we confine ourselves to the linear and second order terms only.

For objects in double or multiple systems for which it is possible to determine the orbit, Eq. (85) gives the coordinates of the center of mass of such a system. The obvious correction should be added to the right-hand side of (85) in order to account for the orbital motion of the object. This case will not be considered here.

Substituting (85) into definitions of \( l \) and \( \pi \) one gets

\[
\pi(t_o) = \pi_0 + \dot{\pi}_0 \Delta t_e + \frac{1}{2} \ddot{\pi}_0 \Delta t_e^2 + O(\Delta t_e^3),
\]

(86)

\[
\dot{\pi}_0 = -\pi_0 \frac{l_0 \cdot V}{|x_s(t_0^e)|},
\]

(87)

\[
\ddot{\pi}_0 = -\pi_0 \frac{l_0 \cdot A}{|x_s(t_0^e)|} - \pi_0 |\dot{l}_0|^2 + 2 \frac{\ddot{\pi}_0}{\pi_0},
\]

(88)

\[
l(t_o) = l_0 + \dot{l}_0 \Delta t_e + \frac{1}{2} \ddot{l}_0 \Delta t_e^2 + O(\Delta t_e^3),
\]

(89)

\[
\dot{l}_0 = \frac{1}{|x_s(t_0^e)|} l_0 \times (V \times l_0),
\]

(90)

\[
\ddot{l}_0 = \frac{1}{|x_s(t_0^e)|} l_0 \times (A \times l_0) - |\dot{l}_0|^2 l_0 + 2 \frac{\ddot{\pi}_0}{\pi_0} \dot{l}_0,
\]

(91)

where \( \pi_0 = \pi(t_0^e) = 1 \, AU/|x_s(t_0^e)| \) and \( l_0 = l(t_0^e) = x_s(t_0^e)/|x_s(t_0^e)| \) are the parameters at the initial epoch of observation \( t_0^e \).
The signals emitted at moments \( t_e^0 \) and \( t_e \) are received by the observer at moments \( t_o^0 \) and \( t_o \), respectively. The corresponding moments of emission and reception are related by

\[
c(t_o - t_e) = |x_o(t_o) - x_s(t_e)| - k(t_o) \cdot \Delta x_p(t_o),
\]

(92)

and a similar equation for the moments \( t_o^0 \) and \( t_o^0 \). The term proportional to \( \Delta x_p \) represents the gravitational signal retardation (the Shapiro effect) due to the gravitational field of the Solar system. For any source inside the Galaxy it is less than \( 3 \cdot 10^{-4} \) s and can be safely neglected. Let us denote \( \Delta t_o = t_o - t_o^0 \) the time span of observations corresponding the time span \( \Delta t_e = t_e - t_e^0 \) of emission. These two time intervals are related as

\[
\Delta t_e = \left(1 + \frac{1}{c} l_0 \cdot V\right)^{-1} \Delta t_o + \frac{1}{c} l_0 \cdot \left(x_o(t_o) - x_o(t_o^0)\right) \left(1 + \frac{1}{c} l_0 \cdot V\right)^{-1}
- \frac{1}{2c} \left(l_0 \cdot A + \frac{|l_0 \times V|^2}{|x_s(t_e^0)|}\right) \Delta t_o^2 + \ldots
\]

(93)

Eq. (93) results from a double Taylor expansion of the first term on the right-hand side of (92) in powers of parallax \( \pi \) and \( \Delta t_e \). Many terms have been neglected here since they were estimated to produce negligible observable effects. Which terms of such an expansion are important depends on many factors. The constrains used here to derive (93) are: \( \pi_0 \leq 1'' \), proper motion \( |l_0| \leq 1 \mu \text{as/s} \approx 32''/\text{yr} \), accuracy of position determination for a single observation is 1 \( \mu \text{as} \), lifetime of the mission is \( \Delta t_o \geq 5 \text{yr} \), the effects of acceleration \( A \) of the object were supposed to be smaller than those of velocity \( V \). For other values of these parameters other terms in the expansion may become important.

The second term represents a quasi-periodic effect with an amplitude \( \sim |x_s|/c \approx 500 \text{ s} \) for a satellite located not too far from the Earth orbit. Below it will be shown that this term gives a significant periodic term in the apparent proper motion of the sources with large proper motion.

It is easy to see from (86)–(91) that time dependence of parallax and proper motion characterized by parameters \( \dot{\pi}_0 \) and \( \ddot{l}_0 \) can be used to determine radial velocity of the source. This question has been recently investigated in full detail and applied to Hipparcos data in (Dravins, Lindegren & Madsen 1999). The tangential and radial components of barycentric velocity \( V \) of the source can be defined by

\[
V_{\text{tan}} = l_0 \times (V \times l_0),
\]

(94)

\[
V_{\text{rad}} = l_0 \cdot V.
\]

(95)
Eqs. (86)–(91) can be combined with (93) to get the time dependence of $l$ and $\pi$ as seen by the observer. Collecting terms linear with respect to $\Delta t_o$ we get the definition of apparent proper motion $\mu_{ap}$ and corresponding apparent tangential velocity $V_{tan}^{ap}$ as appeared in the linear term in $l(t_o)$, and the definition of apparent radial velocity $V_{rad}^{ap}$ as appeared in the linear term in $\pi(t_o)$:

$$V_{tan}^{ap} = l_0 \times (V \times l_0) \left(1 + \frac{1}{c} l_0 \cdot V \right)^{-1} = V_{tan} (1 + c^{-1} V_{rad})^{-1}, \quad (96)$$

$$\mu_{ap} = \frac{1}{|x_s(t_e^0)|} l_0 \times (V \times l_0) \left(1 + \frac{1}{c} l_0 \cdot V \right)^{-1} = \pi_0 \frac{V_{tan}^{ap}}{1 \text{ AU}}. \quad (97)$$

$$V_{rad}^{ap} = l_0 \cdot V \left(1 + \frac{1}{c} l_0 \cdot V \right)^{-1} = V_{rad} (1 + c^{-1} V_{rad})^{-1}. \quad (98)$$

With these definition the simplest possible models for $\pi(t_o)$ and $l(t_o)$ as seen by the observer read (the higher-order terms are neglected here in order to make this example transparent):

$$\pi(t_o) = \pi_0 - \pi_0^2 \frac{V_{rad}^{ap}}{1 \text{ AU}} \Delta t_o + \ldots, \quad (99)$$

$$l(t_o) = l_0 + \mu_{ap} \Delta t_o + \mu_{ap} \frac{1}{c} \left([x_o(t) - x_o(t_0)] \cdot l_0\right) + \ldots. \quad (100)$$

Factor $(1 + c^{-1} l_0 \cdot V)^{-1}$ in (96)–(98) has been discussed in, e.g., (Stumpff 1985; Klioner & Kopeikin 1992). If the vectors $l_0$ and $V$ are nearly antiparallel and $|V|$ is large, this factor may become very large and apparent tangential and radial velocities may exceed light velocity. This phenomenon is one of the well-known possible explanations of apparent superluminal motions.

The amplitude of the third term in (100) is about $170 \mu\text{as}$ for the Barnard’s star with its proper motion of $10 \mu\text{as}$ (see Brumberg, Klioner & Kopejkin 1990; Klioner & Kopeikin 1992). In general the amplitude of this effect is approximately equal to the proper motion of the source during the time interval $\sim |x_o|/c$ required for the light to propagate from the observer to the barycenter of the Solar system multiplied by cosine of the ecliptical latitude of the source. Therefore, for a satellite not too far from the Earth this effect exceeds $1 \mu\text{as}$ for all stars with proper motion larger than $\sim 50 \text{mas/yr}$. This effect is closely related to the Roemer effect used in the 17th century to measure the light velocity. Its potential importance for astrometry was recognized by Schwarzschild (1894).
and later discussed in detail by Stumpff (1985). The Roemer effect is a standard part of typical relativistic models for pulsar timing (see, e.g., Doroshenko & Kopeikin 1990).

The apparent radial velocity $V_{\text{rad}}^{\text{ap}}$ can be immediately used to calculate the true radial velocity $V_{\text{rad}}$. Therefore, if both apparent tangential and apparent radial velocity are determined from observations one can immediately restore the true tangential velocity $V_{\text{tan}}$. However, even if it is not the case the apparent velocity $V_{\text{tan}}^{\text{ap}}$ is useful by itself. Note that the radial velocities measured by Doppler (spectral) observations are neither true nor apparent radial velocity in our terminology. The shift of spectral lines is affected by a number of factors not appearing in positional observations (various gravitational red shifts and Doppler effects; see, a detailed discussion in Kopeikin & Ozernoy (1999)). Note, however, that the relativistic effects induced by the motion of the satellite and by the gravitational field of the Solar system in the Doppler measurements are typically of the order of a few cm/s which is probably too small to be detectable by space astrometry missions (for example, GAIA will measure the Doppler shift of spectral lines with an accuracy of about 1 km/s while the relativistic effects in question are at the level of a few 1 cm/s at most). Therefore, it is only the intrinsic redshift due to local physics of the object which should be considered at this level of accuracy (see, e.g., Neill 1996).

9. Summary of the model

Practical implementation of the model can be summarized as follows:

A. determine the orbit (position $\mathbf{x}_o$ and velocity of $\dot{\mathbf{x}}_o$) of the satellite with respect to the BCRS (Section 4);

B. re-parametrize the observed directions $\mathbf{s}$ by the coordinate time $t$ of the BCRS (to this end Eq. (1) should be numerically integrated along the orbit of the satellite);

C. use Eqs. (9)–(11) to convert the observed direction $\mathbf{s}$ to the source into the unit BCRS direction $\mathbf{n}$ of the light ray at the point of observation;

D. for the objects situated outside of the Solar system use Eqs. (61), (64), (63) with (46)–(50), (42)–(45) and (51)–(56), and (79) to convert $\mathbf{n}$ into the unit BCRS direction $\mathbf{k}$ from the source to the observer at the moment of observation; in order to judge if the light deflection due to a particular gravitating body should be considered here one can use Eq. (65) and/or the maximal angular distances given in Table 1;

D. for the Solar system objects use Eqs. (69), (72) and (71) with (37)–(56) to convert $\mathbf{n}$ into $\mathbf{k}$ (a set of vectors $\mathbf{k}$ for several moments of time can be used to determine the BCRS orbit of the object which represents the final outcome for a Solar system body); Eq. (65) and/or the maximal angular distances given in Table 1 can be again used to judge if a particular gravitating body should be taken into account here;
E. use Eqs. (84) with (81)–(83) to take into account the parallax of the object and to convert $k$ into unit BCRS direction $l$ from the barycenter of the Solar system to the source;

F. use appropriate model for time dependence of $l$ and possibly parallactic parameter $\pi$ to account for possible proper motion of the source (a reasonable model for single sources is represented in Section 8).

The sequence of the steps of the model is depicted on Fig. 4. Let us also note that the model can be in principle further simplified for scanning satellites since they normally deliver high-precision information only in one dimension (along the scan).

10. What is beyond the model

The relativistic model proposed above can be considered as a “standard” model suitable for all sources. This model allows one to reduce the observational data with an accuracy of 1 $\mu$as and restore positions and other parameters of the objects (e.g., their velocities) defined in the BCRS. The model properly takes into account the gravitational field of the Solar system, but ignores a number of possible effects which may be caused by the gravitational fields produced outside of the Solar system. Let us review these effects.

The first additional effect to mention here is the so-called weak microlensing which is simply an additional gravitational deflection of the light coming from a distant source which is produced by the gravitational field of a visible or invisible object situated between the observed source and the observer near the light path. For applications in high-precision astrometry one should distinguish between microlensing events and microlensing noise. Microlensing event is a time dependent change of source’s position (and possibly its brightness) which is large and clear enough to be identified as such. Microlensing events can be used to determine physical properties of the lens so that the unperturbed path of the source can be restored at the end (e.g. Hosokawa et al. 1993, 1995; Høg, Novikov & Polnarev 1995). In this sense microlensing events represent no fundamental problem for the future astrometric missions. On the other hand, microlensing noise comes from unidentified

Fig. 4.— The structure of the model for different kinds of objects (see text).
microlensing events (which are too weak or too fast to be detected as such). The number of such unidentified microlensing events will be clearly much higher than the number of identified ones. Microlensing noise results in stochastic changes of positions of the observed sources with unpredictable (but generally small) amplitude and to unpredictable moments of time. Therefore, microlensing noise can spoil the determination of positions, parallaxes and proper motions of the objects (Zhdanov 1995; Zhdanov & Zhdanova 1995; Hosokawa et al. 1997; Sazhin et al. 1998, 2001). It is currently not quite clear to what extent the microlensing noise produced by the objects of the Galaxy can deteriorate the resulting catalogs of the future astrometric missions. To clarify this question data simulations similar to those described in de Felice et al. (2000), but involving a model for microlensing with a realistic model for the Galaxy would be of much help. The accuracy level of $\sim 1\ \mu\text{as}$ seems to be close to a fundamental limit of astrometric accuracy, since at much higher accuracies the stochastic influence of microlensing becomes too strong so that in too many cases the relativistic deflection effects cannot be properly taken into account (Sazhin et al. 1998, 2001).

In edge-on binary (or multiple) systems gravitational light deflection due to the gravitational field of the companion may be observable. In case of pulsar timing observations of binary pulsars this question has been thoroughly investigated in (Doroshenko & Kopeikin 1995). It is clear that for companions with stellar masses the inclination of the orbit should be very close to 90° for the effect to be observable at the level of 1 $\mu$as. The formulas of Section 6.1 can be directly used here to calculate the effect in the first post-Newtonian approximation. If the lensing companion is a neutron star or a black hole it is necessary to investigate the lensing effect in the strong field regime and also consider secondary images (e.g., a detailed study of the strong-field-regime appearance of a star orbiting a Kerr black hole is given Cunningham & Bardeen (1972, 1973)). Further investigation is necessary to estimate the probability of observing a binary system for which the gravitational lensing of the companion is important at the level of 1 $\mu$as.

Gravitational waves can in principle produce gravitational light deflection. Two cases should be distinguished here: 1) gravitational waves from binary stars and other compact sources and 2) stochastic primordial gravitational waves from the early universe. Gravitational waves from compact sources were shown to produce an utterly small deflection which is hardly observable at the level of 1 $\mu$as (Kopeikin et al. 1999). The influence of primordial gravitational waves was analyzed in (Pyne et al. 1996; Gwinn et al. 1997). Although initially applied for VLBI, the method of these papers can be directly used for optical astrometry.

Finally, cosmological effects should be accounted for to interpret the derived parameters of the objects (e.g., the accuracy of parallaxes $\sigma_\pi = 1\ \mu\text{as}$ allows one to measure the distance to the objects as far as 1 Mpc away from the Solar system; see, e.g., Kristian & Sachs (1965) for a discussion of astrometric consequences of cosmology). It may be interesting here to construct metric tensor of the BCRS with a cosmological solution as a background and analyze the effects of the background cosmology in such a reference system.
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