Dynamics with unitary phase operator: implications for Wigner’s problem

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Abstract

We show that for general deformations of $SU(2)$ algebra, the dynamics in terms of ladder operators is preserved. This is done for a system of precessing magnetic dipole in magnetic field, using the unitary phase operator which arises in the polar decomposition of $SU(2)$ operators. It is pointed out that there is a single phase operator dynamics underlying the dynamics of usual and deformed ladder operators.
1 Introduction

Wigner’s problem [1] usually formulated for the case of quantum harmonic oscillator states that the equations of motion do not determine a unique set of commutation relations for the observables. In classical mechanics also, it is known [2]-[4] that same dynamical equations may be obtained using alternative hamiltonians and definitions of Poisson brackets. Parastatistics is another example of such a nonuniqueness [5]. Recently, Wigner’s problem for a precessing magnetic dipole (with dynamical algebra $SU(2)$) was discussed [6] and a class of modified commutation relations were shown to be compatible with the same dynamical equations. In this paper, we point out that invariance of dynamics under general deformations of the $SU(2)$ algebra, can be understood in a unified manner as an underlying dynamics in terms of unitary phase operator, that arises in the polar decomposition of the ladder operators.

The polar decomposition procedure referred to above is the operator analogue of factorising a complex number into a real argument and an exponential phase. For an operator the factors should be a hermitian part and a unitary phase operator. The unitary phase operator ($e^{i\phi}$) in turn defines a hermitian phase operator $\phi$. For the purpose of $SU(2)$ algebra, the phase or angle operator is conjugate to angular momentum component, though the canonical conjugacy is modified when, as in this case, the operators are bounded [7, 8].

In section 2, we first review the dynamics and algebraic structure of the precessing magnetic dipole in the presence of magnetic field, in terms of generators of $SU(2)$ algebra. Then we describe the polar decomposition procedure for the ladder operators of this algebra and the dynamics is cast in terms of the unitary phase operator. It is shown that dynamics for standard ladder operators can be derived from the dynamical equation for phase operator. In section 3, we consider general deformations of the $SU(2)$ algebra and show that dynamics for deformed ladder operators also follows from the same dynamical equation for phase operator. Section 4 presents some concluding remarks.
The hamiltonian for a magnetic dipole precessing in a magnetic field is given by
\[ H = -\mu (\vec{J} \cdot \vec{B}). \]
For simplicity, let us choose the magnetic field to be along the z-axis, so that
\[ H = -\mu B J_z, \quad (1) \]
where \( J_z \) is the z component of angular momentum operator \( \vec{J} \). In terms of the ladder operators defined by \( J_{\pm} = (J_x \pm iJ_y)/\sqrt{2} \), and \( J_0 = J_z \), which are generators of \( SU(2) \) algebra
\[ [J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_0, \quad (2) \]
the equations of motion are given as
\[ \frac{dJ_{\pm}}{dt} = \mp i\mu BJ_{\pm}, \quad (3) \]
\[ \frac{dJ_0}{dt} = 0. \quad (4) \]

We choose the set of basis states to be the standard angular momentum states
\{ \ket{j, -j}, \ket{j, -j+1}, \ldots, \ket{j, j-1}, \ket{j, j} \},
which define a \((2j+1)\)-dimensional irreducible representation for \( J_{\pm}, J_0 \):
\[ J_{\pm} = \sum_{m=-j}^{+j} \sqrt{(j \mp m)(j \pm m + 1)} \ket{j, m \pm 1} \bra{jm}, \quad (5) \]
\[ J_0 = \sum_{m=-j}^{+j} m \ket{jm} \bra{jm}. \quad (6) \]
The casimir for this algebra is given by
\[ C \equiv \vec{J}^2 = J_- J_+ + J_0(J_0 + 1) = J_+ J_- + J_0(J_0 - 1). \quad (7) \]
In the following we write the equations of motion using the unitary phase operator, which arises in the polar decomposition procedure [9] of ladder operators
\[ J_+ = \sqrt{J_+ J_-} e^{i\phi} = e^{i\phi} \sqrt{J_- J_+}, \quad (8) \]
\[ J_- = \sqrt{J_- J_+} e^{-i\phi} = e^{-i\phi} \sqrt{J_+ J_-}. \quad (9) \]
The exponential phase operator $e^{i\phi}$ is unitary ($e^{i\phi}e^{-i\phi} = 1$) and is given as
\[ e^{i\phi} = \sum_{m=-j}^{j} |j, m+1\rangle\langle j, m| + e^{i(2j+1)\theta_0} |j, -j\rangle\langle j, j|. \] (10)
In other words, operator $\phi$ is hermitian. Here $\theta_0$ is an arbitrary phase angle, which defines the domain of phase operator $\phi$ to be $[\theta_0, \theta_0 + 2\pi)$. Without loss of generality, we can take here $\theta_0 = 0$.

We can write the equation of motion for $e^{i\phi}$
\[ \frac{d}{dt}e^{i\phi} = \frac{1}{i\hbar}[e^{i\phi}, H], \] (11)
using the following commutator [9]
\[ [e^{\pm i\phi}, J_0] = \pm \hbar\{ -e^{\pm i\phi} + (2j+1)e^{\pm i(2j+1)\theta_0}\} \pm (-j)\langle \pm j|\langle \pm j|. \] (12)
Now to get equation of motion for $J_+$, we just multiply Eq. (11) with $\sqrt{J_+J_-}$ on the left (or with $\sqrt{J_-J_+}$ on the right) and use the fact that $J_+J_-(J_-J_+)$ commutes with $J_0$ and hence with the Hamiltonian. Also we use $J_+|j, j\rangle = \langle j, j|J_+ = 0$. Similarly, we can obtain Eq. (3) corresponding to $J_-$ starting with equation of motion for $e^{-i\phi}$ and using $J_-|j, -j\rangle = \langle j, -j|J_+ = 0$.

### 3 Dynamics with deformed ladder operators

In this section, we show how starting from the equation of motion for $e^{\pm i\phi}$, we can preserve the linear dynamics in terms of deformed ladder operators. Specifically, we consider general deformations of $SU(2)$ algebra [10]
\[ [\bar{J}_0, \bar{J}_\pm] = \pm \bar{J}_\pm, \quad [\bar{J}_+, \bar{J}_-] = f(\bar{J}_0). \] (13)
$f(z)$ is a real parameter-dependent analytic function of its argument, holomorphic in the neighbourhood of zero and goes to $2z$ for certain limiting value of the parameter. Also define a function $g$ through
\[ f(\bar{J}_0) = g(\bar{J}_0) - g(\bar{J}_0 - 1). \] (14)
The function $g(\bar{J}_0)$ is not unique and is determined up to any periodic function of unit period. The Casimir for this algebra is $\bar{C} = \bar{J}_-\bar{J}_+ + g(\bar{J}_0) = \bar{J}_+\bar{J}_- + g(\bar{J}_0 - 1)$. 

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A generalized map which does not preserve hermitian conjugation between \( \tilde{J}_+ \) and \( \tilde{J}_- \) may be given as

\[
\tilde{J}_+ = J_+ A(C, J_0), \quad \tilde{J}_- = B(C, J_0) J_-, \quad \tilde{J}_0 = J_0.
\] (15)

It is easy to verify the first relation in Eq. (13). To satisfy the second relation, the following condition must hold

\[
A(J_0 - 1) B(J_0 - 1)(C - J_0(J_0 - 1)) - B(J_0) A(J_0)(C - J_0(J_0 + 1)) = f(J_0). \quad (16)
\]

Assuming that \( A \) and \( B \) commute, the above condition implies

\[
A(J_0 - 1) B(J_0 - 1)(C - J_0(J_0 - 1)) = -g(J_0 - 1) + p(J_0), \quad (17)
\]

where \( p(J_0) \) is some periodic function of period unity. Note that this only fixes the product \( A(J_0)B(J_0) \). Different choices of these functions as well as the function \( p \), produce a variety of realizations for the deformed algebra.

Now we observe that

\[
\tilde{J}_+ = J_+ A(C, J_0) = e^{i\phi} \sqrt{J_- J_+} A(C, J_0) \equiv e^{i\phi} G(C, J_0), \quad (18)
\]

where Eqs. (8) and (7) have been used. Then, multiplying Eq. (11) on the right by \( G(C, J_0) \) and using the fact that \( G \) commutes with the hamiltonian, we obtain the dynamical equation for \( \tilde{J}_+ \)

\[
\frac{d\tilde{J}_+}{dt} = -i\mu B\tilde{J}_+. \quad (20)
\]

Similarly we can write

\[
\tilde{J}_- = B(C, J_0) J_- = B(C, J_0) \sqrt{J_- J_+} e^{-i\phi} \equiv K(C, J_0) e^{-i\phi}. \quad (21)
\]

Again multiplying the equation of motion for \( e^{-i\phi} \) on the left by \( K(C, J_0) \) and using the fact that \( K \) commutes with the hamiltonian, we obtain the dynamical equation for \( \tilde{J}_- \)

\[
\frac{d\tilde{J}_-}{dt} = i\mu B\tilde{J}_-. \quad (23)
\]
Thus we see that equations of motion for $\tilde{J}_\pm$ are identical in form to those for $J_\pm$, Eqs. (3). This can also be proved without using the unitary phase operator, as was done in [6]. But the idea here is to point out that \textit{underlying the identical dynamics of the usual and deformed ladder operators, there is a single unitary phase operator dynamics}. Note that it is not possible to obtain the dynamics of unitary phase operator by going in the opposite fashion, i.e. starting with the equation of motion for ladder operators and using the polar decomposition. This way the second term on the right hand side of Eq. (10) cannot be reproduced.

4 Concluding Remarks

When $\tilde{J}_- = \tilde{J}_0^{\dagger}$ is imposed, we can express the deformed generators in terms of those of $SU(2)$ algebra as the following maps

\[
\tilde{J}_+ = \sqrt{\frac{f(J_0 + j)f(J_0 - 1 - j)}{(J_0 + j)(J_0 - 1 - j)}} J_+ , \quad \tilde{J}_- = \tilde{J}_0^{\dagger}, \quad \tilde{J}_0 = J_0 .
\]

Similarly, deforming maps [11] for well known quantum algebras corresponding to $SU(2)$ can be given, and they can be discussed under this category, e.g. Drinfeld-Jimbo deformation [12], Witten’s first and second deformations [13] and Woronowicz’s deformation [14]. It is clear that adopting the polar decomposition procedure for the ladder operators $J_\pm$ of $SU(2)$ algebra, the dynamical equations for deformed operators can be recovered from dynamics in terms of the unitary phase operator, i.e. Eqs. (20) and (23).

We now argue that realization of deformed ladder operators as proposed in [6] is a special case of the mapping in Eq. (15). Following [6], a non-linear deformation of ladder operators may be defined through an arbitrary function $F(C, J_0)$ as

\[
\tilde{J}_+ = J_+ F, \tilde{J}_- = J_- F, \tilde{J}_0 = J_0 F.
\]

The transformation leads to the following deformed algebra

\[
[\tilde{J}_0, \tilde{J}_\pm] = \left\{ 1 - \frac{F(C, J_0 + 1)}{F(C, J_0)} \right\} \tilde{J}_0 \tilde{J}_\pm \pm F(C, J_0 + 1) \tilde{J}_\pm, \quad \tilde{J}_+ \tilde{J}_- = 2F(C, J_0 + 1) \tilde{J}_0.
\]
Now although the operator $\tilde{J}_0$ is modified as compared to $J_0$ in the above algebra, the Hamiltonian is expressed in terms of the usual operator $J_0$. Thus the commutator $[\tilde{J}_0, \tilde{J}_\pm]$ does not play any role in the dynamics considered in [6]. In fact, if we choose not to deform $J_0$ as considered above in our approach, Eq. (26) just reduces to $[J_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm$. Secondly, the commutator in Eq. (27) follows if we choose $A(C, J_0) = F(C, J_0)$ and $B(C, J_0) = F(C, J_0 + 1)$, in Eqs. (15) and (16).

Concluding, the Wigner problem for a precessing magnetic dipole has been analyzed through the polar decomposition of ladder operators. We have observed that there is a single dynamics related with the unitary phase operator for various types of deformations of ladder operators. Finally, it is known that the procedure of polar decomposition which yields a unitary phase operator does not work for quantum harmonic oscillator defined in infinite dimensional Hilbert space, the system for which Wigner originally formulated his problem.

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**References**


