Comments on conformal stability of brane-world models

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Abstract

The stability of 5–D brane-world models under conformal perturbations is investigated. The analysis is carried out in the general case and then it is applied to particular solutions. It is shown that models with the Poincaré and the de Sitter branes are unstable because they have negative mass squared of gravexcitons whereas models with the Anti de Sitter branes have positive gravexciton mass squared and are stable. It is also shown that 4–D effective cosmological and gravitational constants on branes as well as gravexciton masses undergo hierarchy: they have different values on different branes.

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1 Introduction

It is well known that part of any realistic multidimensional model should be a mechanism for extra dimension stabilization. This problem was a subject of numerous investigations. In the standard Kaluza-Klein approach cosmological models are taken in the form of warped product of Einstein spaces as internal spaces. Corresponding warp (scale) factors are assumed to be functions of external (our) space-time. If these scale factors are dynamical functions then it results in a variation of the fundamental physical constants. To be in agreement with observations, internal spaces should be compact, static (or nearly static) and less or order of electro-weak scale (the Fermi length). The stability problem of these models with respect to conformal perturbations of the internal spaces was considered in detail in our paper [1]. It was shown that stability can be achieved with the help of an effective potential of a dimensionally reduced effective 4–D theory. Small conformal excitations of the internal spaces near minima of the effective potential have the form of massive minimal scalar fields developing in the external space-time. These particles were called gravitational excitons (gravexcitons).

Recently [2, 3, 4], it was realized that it is not necessary for extra dimensions to be very small. They can be enlarged up to sub-millimeter scales in such a way that the Standard Model fields are localized on a 3–brane with thickness of the electro-weak (or less) length in the extra dimensions whereas gravitational field can propagate in all multidimensional (bulk) space. It gives possibility for lowering of the multidimensional fundamental gravitational constant down to the TeV scales (therefore this approach often call as TeV gravity approach). Cosmological models in this approach are topologically equivalent to the standard Kaluza-Klein one. Problem of their stability against conformal perturbations of additional dimensions was considered in papers [4, 5]. A comparison of old and this new approaches from the point of view of conformal stability was given in paper [6]. In papers [4, 5] conformal excitations of the additional dimensions near minimum position of the effective potential were called radions (to our knowledge, the first time that the term radion appeared it was in [4]). However, we prefer to call such particles gravexcitons because, first, from the point of priority and, second and it is the most important, the term radion is widely used now in the brane-world models in different content.

The brane-world models are motivated by the strongly coupled regime of $E_8 \times E_8$ heterotic string theory which is interpreted as M-theory on an orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ with a set of $E_8$ gauge fields at each ten-dimensional orbifold fixed plane. After compactification on a Calabi-Yau three-fold and dimensional reduction one arrives at effective 5–dimensional solutions which describe a pair of parallel 3–branes with opposite tension, and located at the orbifold planes [7]. For these models the 5–dimensional metric contains
a 4–dimensional metric component multiplied by a warp factor which is a function of the additional dimension. A cosmological solution of this type with flat 4–D branes (which we shall refer as Poincaré branes) was obtained in paper [8]. This model was generalized in numerous publications to the cases of bent branes in models with 5 and more dimensions and with single or many branes. In paper [9] it was stressed the necessity of stabilizing the distance between branes to get conventional cosmology on branes. Here, the radius of the extra dimension was called radion. But this definition is not very precise. We should note that there is a confusion in the literature concerning the term radion: quite different forms of the metric perturbations of the brane-world models were called radions. However, in paper [10] it was clearly shown and strictly emphasized that radions describe relative distance between branes (see also papers [11, 12]). It demonstrates the main difference between gravexcitons and radions: gravexcitons describe conformal excitations of geometry (more particular - conformal excitations of the additional dimensions) whereas radions describe relative motion of branes. Obviously, gravexcitons can exist in model where branes are absent at all or in models with a single brane and vice versa radions can exist in absence of gravexcitons. The latter situation can be realized for example in the TeV scale approach where branes can move relatively with respect each other due to interaction between them "sliding" on background fixed geometry (gravexcitons are absent). Branes are considered here as "probe bodies" moving in the background geometry. Nevertheless, in the brane-world models gravexcitons and radions are closely connected with each other (and this is the main reason for the confusion between them). Here, branes are 4–D surfaces along which different 5–D bulk solutions are gluing with each other. In this case positions of branes fix the shape and size of the geometry\(^1\) and relative motion of branes results in conformal changes of the geometry. Thus, for such models stability against radions automatically means their stability against gravexcitons and vice versa.

The radion stabilization problem was investigated in a number of papers devoted to the brane-world models. It was shown in particular, that Randall-Sundrum solution [8] with two Poincaré branes as well as solution with a pair of the de Sitter branes (bent branes with 4–D effective positive cosmological constants) are unstable whereas solution with a pair of the Anti de Sitter branes (bent branes with 4–D effective negative cosmological constants) is stable [11, 12]. This conclusion coincides with the results of our paper. In all these models matter in bulk as well as on branes is absent (correctly speaking, it is considered there in its simplest form as bulk cosmological constant and "vacuum energies" on branes). It was observed that inclusion of matter can stabilize radions. It can be done with the help of bulk scalar field [9], [13]–[15], perfect fluid on branes [16] and the Casimir effect between branes [17, 18]. Some specific forms of the brane-world solution instability were observed in papers [19, 20]. It was shown that single Poincaré brane is unstable under small perturbations of the brane tension\(^2\) [19] and single de Sitter brane is unstable against thermal radiation [20].

The main goal of our present comments consists in investigation of 5–D brane-world stability against conformal perturbations. First, we elaborate a method to study the stability for a large class of solutions and obtain general expressions for 4–D effective cosmological constants on branes and masses of gravitational excitons. Then we apply this method to a number of well known solutions. In particular, we find that models with the Poincaré and the de Sitter branes are unstable because they have negative mass squared of gravexcitons whereas models with the Anti de Sitter branes have positive gravexciton mass squared and are stable under conformal perturbations. We show also that 4–D effective cosmological and gravitational constants on branes as well as gravexciton masses undergo hierarchy: they have different values on different branes (if different branes have different warp factors).

The paper is organized as follows. In Section 2 we explain the general setup of our model, perform dimensional reduction of the brane-world models to an effective 4–D theory in general case and apply this procedure to a number of well known solutions. In Section 3 we elaborate a method of the investigation of brane-world solution stability against conformal perturbations and apply it to particular solutions considered in Section 2. Here we show also that physical masses of gravexcitons undergo hierarchy on different branes. The brief Conclusions of the paper are followed by three appendices. In Appendix A we present useful expressions for the Ricci tensor components and scalar curvature in the case of block-diagonal metrics. Some useful formulas of the conformal transformation are summarized in Appendix B. In Appendix C we show that the results of the paper do not change if only additional dimension undergoes conformal perturbations: we arrive here to the same 4–D effective theory and the same gravexciton masses as in the case of total geometry conformal perturbations. This provides an interesting analogy between gravity and an elastic media where the eigen frequencies of an elastic body oscillations do not depend on the manner of excitation.

## 2 Model and general setup: dimensional reduction of brane-world models

We consider 5–D cosmological models on a manifold \(M^{(5)}\) which is divided on \(n\) pieces by \(n–1\) branes: \(M^{(5)} = \bigcup_{i=1}^{n} M_i^{(5)}\). Branes are 4–D hypersurfaces \(r = r_i = \text{const}\), \(i = 1, \ldots, n–1\), where \(r\) is an extra radius of the extra dimension was called radion. But this definition is not very precise. We should note that there is a confusion in the literature concerning the term radion: quite different forms of the metric perturbations of the brane-world models were called radions. However, in paper [10] it was clearly shown and strictly emphasized that radions describe relative distance between branes (see also papers [11, 12]). It demonstrates the main difference between gravexcitons and radions: gravexcitons describe conformal excitations of geometry (more particular - conformal excitations of the additional dimensions) whereas radions describe relative motion of branes. Obviously, gravexcitons can exist in model where branes are absent at all or in models with a single brane and vice versa radions can exist in absence of gravexcitons. The latter situation can be realized for example in the TeV scale approach where branes can move relatively with respect each other due to interaction between them "sliding" on background fixed geometry (gravexcitons are absent). Branes are considered here as "probe bodies" moving in the background geometry. Nevertheless, in the brane-world models gravexcitons and radions are closely connected with each other (and this is the main reason for the confusion between them). Here, branes are 4–D surfaces along which different 5–D bulk solutions are gluing with each other. In this case positions of branes fix the shape and size of the geometry\(^1\) and relative motion of branes results in conformal changes of the geometry. Thus, for such models stability against radions automatically means their stability against gravexcitons and vice versa.

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\(^1\)In our paper we shall consider the case of compact with respect to additional dimension brane-world models.

\(^2\)Here it was mentioned also about instability of the single brane Randall-Sundrum solution under homogeneous gravitational perturbations. This conclusion is in agreement with the results of our paper.
dimension. Each brane is characterized by its own tension $T_i(r_i)$, $i = 1, \ldots, n - 1$. We suppose that a boundary $\partial M^{(5)}$ also corresponds to two hypersurfaces $r = \text{const}$: $r = r_0$ and $r = r_n$, and, either 4-D geometry on $\partial M^{(5)}$ is closed (induced 4-D metric vanishes there), or opposite points $r_0$ and $r_n$ are identified with each other. In the first case boundary terms corresponding to $\partial M^{(5)}$ are equal to zero. In the second case, the boundary $\partial M^{(5)}$ is absent, however, if the geometry is not smoothly matched here, it results in appearance of an additional brane with a tension $T_i(r_0)$. For simplicity, bulk matter is considered in the form of a cosmological constant, in general, different for each of $M_i^{(5)}$. Thus, our model is described by the following action:

$$S^{(5)} = \frac{1}{2k^5} \int d^5x \sqrt{|g^{(5)}|} \left( R[g^{(5)}] - 2\Lambda_5(r) \right) + S_{YGH} - \sum_{i=0}^{n-1} T_i(r_i) \int d^4x \sqrt{|g^{(4)}|} \bigg|_{r_i},$$

(2.1)

where $S_{YGH} = -\kappa_n^{-1} \int_{\partial M^{(5)}} d^4x \sqrt{|g^{(4)}|} K$ is the standard York - Gibbons - Hawking boundary term. The Einstein equation corresponding to action (2.1) reads:

$$R_{MN}[g^{(5)}] - \frac{1}{2}g^{(5)}_{MN}R[g^{(5)}] = -\Lambda_5(r)g^{(5)}_{MN} - \frac{k^2}{\sqrt{|g^{(4)}|}} \sum_{i=0}^{n-1} T_i(r_i) \sqrt{|g^{(4)}(x, r_i)|} g^{(4)}_{\mu\nu}(x, r_i) \delta^\mu_M \delta^\nu_N \delta(r-r_i),$$

(2.2)

In equations (2.1) and (2.2)

$$\Lambda_5(r) := \sum_{i=1}^{n} \Lambda_i \theta_i(r), \quad \Lambda_i = \text{const},$$

(2.3)

with piecewise discontinuous functions

$$\theta_i(r) = \eta(r - r_{i-1}) - \eta(r - r_i) = \begin{cases} 0, & r < r_{i-1} \\ 1, & r_{i-1} < r < r_i \\ 0, & r > r_i \end{cases}$$

(2.4)

where step functions $\eta(r - r_i)$ equal to zero for $r < r_i$ and unity for $r > r_i$.

Now, we suppose that a metric

$$g^{(5)}(X) = g^{(5)}_{MN} dX^M \otimes dX^N = dr \otimes dr + a^2(r)\gamma^{(4)}_{\mu\nu}(x) dx^\mu \otimes dx^\nu,$$

$$a(r) = \sum_{i=1}^{n} a_i(r) \theta_i(r)$$

is the solution of the Einstein equation (2.2) and has following matching conditions: $a_i(r_i) = a_{i+1}(r_i)$, $i = 1, \ldots, n - 1$ and $a_1(r_0) = a_n(r_n)$. Scale factors $a_i(r)$ are supposed to be non-negative smooth functions in intervals $[r_{i-1}, r_i]$. Boundary points $r_0$ and $r_n$ are either identified with each other: $r_0 \leftrightarrow r_n$ or they are not identified and the geometry in latter case is closed: $a_i(r_0) = a_n(r_n) = 0$; i.e. induced metric $g^{(4)}_{\mu\nu}(x, r) = a^2(r)\gamma^{(4)}_{\mu\nu}(x)$ vanishes in this points.

Having at hands solution (2.5), we can perform dimensional reduction of action (2.1). Here, the dimensional reduction means integration over extra dimension in 5-D part of action (2.1) to get 4-D effective action. To do that, let us perform first some preliminary calculations.

Applying equation (A.3) to our case we obtain

$$R[g^{(5)}] = a^{-2}(r) R[\gamma^{(4)}] - f_1(r),$$

(2.6)

where

$$f_1(r) := 8a'' + 12 \left( \frac{a'}{a} \right)^2.$$

(2.7)

Using properties of $\theta$-function: $\theta^p_i = \theta_i$, $p > 0$; $\theta_i \theta_j = 0$, $i \neq j \Rightarrow \theta^p = \sum_{i=1}^{n} a_i^p \theta_i$, $\forall p$ and $\theta_i(r - r_{i-1}) - \delta(r - r_i)$, the function $f_1(r)$ can be written in the following form:

$$f_1(r) = 12 \sum_{i=1}^{n} \frac{(a_i')^2}{a_i^2} \theta_i + 8 \sum_{i=1}^{n} \frac{a_i''}{a_i} \theta_i - 2 \left[ K(r_0^+ \delta(r - r_0) - K(r_0^- \delta(r - r_0) + \sum_{i=1}^{n-1} K(r_i) \delta(r - r_i) \right],$$

(2.8)

In compact brane-world models it is worthy to include this term even if boundary $\partial M^{(5)}$ is absent because it is convenient here (as well as for all models with branes) to split manifold $M^{(5)}$ by branes into n submanifolds: $M^{(5)} = \bigcup_{i=1}^{n} M_i^{(5)}$, each of them has boundaries $\partial M_i^{(5)}$ defined by positions of the branes. Such boundary terms at $\partial M_i^{(5)}$ take into account the presence of the branes and are needed in order to satisfy the variational principle and the junction conditions on the branes [21]. These junction conditions coincide with ones following directly from the Einstein equation (2.2). Different parts of the manifold $M^{(5)}$ can be covered by different coordinates charts. We show an explicit example below.
where \( \hat{K}(r_i) = K(r_i^+) - K(r_i^-) \) and \( K(r_i^+) = -4a_i'/a_i |_{r_i^+} \), \( K(r_i^-) = -4a_i'/a_i |_{r_i^-} \) in accordance with equation (B.6). As we can see, function \( f_1 \) contains all information about boundary terms and for correct dimensional reduction of action (2.1) it is not necessary to include additionally boundary term \( S_{YGH} \) because it will lead in this case to its double counting\(^5\). It can be easily seen also that integral

\[
\int_{r_0}^{r_n} \text{d}r^4(r) f_1(r) = -12 \sum_{i=1}^{n} \int_{r_{i-1}}^{r_i} \text{d}a_i^2 \left( a_i' \right)^2 , \tag{2.9}
\]

where we used integration by parts. Thus, dimensional reduction of action (2.1) will result in the following effective 4-D action:

\[
S_{\text{eff}}^{(4)} = \frac{1}{2\kappa_4^2} \int_{M^{(4)}} d^4 x \sqrt{|g^{(4)}|} \left\{ R[\gamma^{(4)}] - 2\Lambda^{(4)}_{\text{eff}} \right\} , \tag{2.10}
\]

where effective 4-D cosmological constant is

\[
\Lambda^{(4)}_{\text{eff}} = \frac{1}{B_0} \left[ B_1 + B_2 + B_3 \right] \tag{2.11}
\]

and

\[
B_0 = \sum_{i=1}^{n} \int_{r_{i-1}}^{r_i} \text{d}a_i^2 , \tag{2.12}
\]

\[
B_1 = -6 \sum_{i=1}^{n} \int_{r_{i-1}}^{r_i} \text{d}a_i^2 \left( a_i' \right)^2 , \tag{2.13}
\]

\[
B_2 = \sum_{i=1}^{n} \Lambda_i \int_{r_{i-1}}^{r_i} \text{d}a_i^4 , \tag{2.14}
\]

\[
B_3 = \kappa_5^2 \sum_{i=0}^{n-1} a_i^4(r_i) T_i(r_i) . \tag{2.15}
\]

Effective 4-D gravitational constant is defined as follows\(^6\): \( \kappa_4^2 = \kappa_5^2/B_0 \). Equation (2.10) shows that solution (2.5) of eq. (2.2) takes place only if 4-D metric \( \gamma^{(4)} \) is the Einstein space metric.

### 2.1 Examples

In this subsection we apply considered above procedure of the dimension reduction to some well known solutions (see e.g. [8, 12, 15, 22, 23]).

a) Poincaré branes

In this model\(^7\) \( r_0 = -L \), \( r_1 = 0 \), \( r_2 = L \),

\[
a_1(r) = \exp(r/l) , \quad -L \leq r \leq 0 , \tag{2.16}
\]

\[
a_2(r) = \exp(-r/l) , \quad 0 \leq r \leq L ,
\]

\(^5\)There are two equivalent ways of the dimensional reduction. First, we can divide action integral (2.1) into \( n \) integrals in accordance with the splitting procedure described in footnote 3 and take into account the boundary terms at \( \partial M^{(5)} \) arising due to the presence of branes. In this case, the scale factors \( a_i(r) \) for each of the submanifold \( M_i^{(5)} \) are smooth functions and their derivatives do not result in \( \delta \)-functions. Here, the brane boundary terms are taken into account directly in the action functional. In second approach, we consider full non-split action (2.1) without the brane boundary terms but take into account that scale factor \( a(r) \) is not a smooth function in points corresponding to the branes location. Thus, its second derivative has \( \delta \)-function terms which completely equivalent to the brane boundary terms (see (2.8)). It can be easily checked that integration over extra dimension in both of these approaches results in the same 4-D effective action. In the present paper we applied second approach.

\(^6\)In this paper we focus on the problem of stability of considered models and not discuss cosmology on branes. It is clear that from the point of an observer on a brane, physical metric is the induced metric on this brane (let it be \( i \)-th brane): \( g_{\mu\nu}^{(ph)} = a_i^2(r_i) \gamma_{\mu\nu}^{(4)} \). It means we should perform evident substitutions \( a_i(r) \rightarrow a_i(r)/a_i(r_i) \) in corresponding formulas. For example, for this observer physical effective 4-D cosmological and gravitational constants read as follows: \( \Lambda^{(4)}_{\text{eff}} \rightarrow \Lambda^{(4)}_{\text{eff}}(\mu\nu)_{\text{brane}} = \Lambda^{(4)}_{\text{eff}}/a_i^2(r_i) \) and \( \kappa_4^2 \rightarrow \kappa_5^2/\kappa_4^2 = \kappa_5^2a_i^2(r_i) \). On proportionality of the effective 4-D Newton’s constant on brane to \( a_i^2(r)|_{\text{brane}} \) was pointed e.g. in [10].

\(^7\)Here, we follow the original solution [8], where scale factors are dimensionless.
and bulk cosmological constants $\Lambda_1 = \Lambda_2 = -6/l^2$, where $l$ is the AdS radius. The points $r_0$ and $r_2$ are identified with each other. A free parameter $L$ defines the size of models in the additional dimension. The geometry is not smooth at points $r = 0$ and $r = r_0 \equiv r_2$, thus we have two branes with tensions: $-T_0(r_0) = T_1(r_1) = 6/ (\kappa_5^2 l)$. Substituting concrete expressions into formulas (2.12) - (2.15), we obtain respectively:

\[ B_0 = l \left( 1 - e^{-2L/l} \right) > 0 \tag{2.17} \]
\[ B_1 = 3l^3 \left( \frac{1}{2} \sinh \frac{L}{l} - \frac{L}{l} \right) > 0 \tag{2.18} \]
\[ B_2 = 3l^3 \left( \frac{1}{2} \sinh \frac{L}{l} - \frac{L}{l} \right), \tag{2.19} \]
\[ B_3 = 6l^3 \left( 1 - e^{-2L/l} \right). \tag{2.20} \]

Thus, in this model

\[ \Lambda_{(4),eff}^{(4)} \equiv 0 \tag{2.21} \]

and $\gamma_{(4)}^{(4)}$ is flat space-time metric. The Randall–Sundrum one brane solution [22] corresponds to trivial limite $L \rightarrow +\infty$ and also results in eq. (2.21).

b) De Sitter brane (symmetric solution)

In this model $r_0 = 0$, $r_1 = L$, $r_2 = 2L$,

\[ a_1(r) = l \sinh \frac{r}{l}, \quad 0 \leq r \leq L, \]
\[ a_2(r) = l \sinh \frac{2L-r}{l}, \quad L \leq r \leq 2L, \tag{2.22} \]

and bulk cosmological constants $\Lambda_1 = \Lambda_2 = -6/l^2$. In the points $r_0$ and $r_2$ the geometry is closed: $a_1(r_0) = a_2(r_2) = 0$ ($r_0$ and $r_2$ are horizons of AdS5). The geometry is not smooth in $r_1$. Therefore, in this model we have one brane with tensions: $T_1(r_1) = 6/ (\kappa_5^2 l) \coth (L/l)$. Substituting these expressions into formulas (2.12) - (2.15), we obtain respectively:

\[ B_0 = l^3 \left( \frac{1}{2} \sinh \frac{2L}{l} - \frac{L}{l} \right) > 0 \tag{2.23} \]
\[ B_1 = -3l^3 \left( \sinh \frac{L}{l} \cosh \frac{L}{l} + \frac{1}{4} \sinh 2\frac{L}{l} - \frac{1}{2} \frac{L}{l} \right), \tag{2.24} \]
\[ B_2 = -3l^3 \left( \sinh \frac{L}{l} \cosh \frac{L}{l} - \frac{3}{4} \sinh 2\frac{L}{l} + \frac{3}{2} \frac{L}{l} \right), \tag{2.25} \]
\[ B_3 = 6l^3 \sinh \frac{L}{l} \cosh \frac{L}{l}. \tag{2.26} \]

So, in this model

\[ \Lambda_{(4),eff}^{(4)} = 3 \tag{2.27} \]

and $\gamma_{(4)}^{(4)}$ describes either Riemannian 4–sphere with scalar curvature $R[\gamma^{(4)}] = D_0 (D_0 - 1) = 12$ or 4–D de Sitter space-time with cosmological constant $\Lambda = 3$ and scalar curvature $R[\gamma^{(4)}] = [2D_0/(D_0 - 2)] \Lambda = 12$.

c) De Sitter brane (non-symmetric solution)

We obtain this solution gluing together two submanifolds covered by different charts. First submanifold describes truncated Garriga–Sasaki solution [23] and second one describes flat 5–D space:

\[ a_1(r) = l \sinh \frac{r}{l}, \quad 0 \leq r \leq L, \]
\[ a_2(R) = R_0 - R, \quad 0 \leq R \leq R_0, \tag{2.28} \]

where $R_0 = l \sinh (L/l)$. Bulk cosmological constants $\Lambda_1 = -6/l^2$ and $\Lambda_2 \equiv 0$. In the points $r = 0$ and $R = R_0$ the geometry is closed: $a_1(0) = a_2(R_0) = 0$. The geometry is not smooth on the hypersurfaces of gluing $r = L$ and $R = 0$. That why, we have one brane with tensions: $T_1|_{r=L,R=R_0} = 3/\kappa_5^2 [1/R_0 + (1/l) \coth (L/l)]$. Substituting these expressions into formulas (2.12) - (2.15), we obtain respectively:

\[ B_0 = \frac{1}{2} l^3 \left( \frac{1}{2} \sinh \frac{2L}{l} - \frac{L}{l} \right) + \frac{1}{3} R_0^2 > 0 \tag{2.29} \]
\begin{align}
B_1 &= \frac{3}{2} l^3 \left( \sinh \frac{L}{l} \cosh \frac{L}{l} + \frac{1}{4} \sinh 2\frac{L}{l} - \frac{1}{2} \frac{L}{2l} \right) - 2r_0^3, \\
B_2 &= \frac{3}{2} l^3 \left( \sinh \frac{L}{l} \cosh \frac{L}{l} - \frac{3}{4} \sinh 2\frac{L}{l} + \frac{3}{2} \frac{L}{2l} \right), \\
B_3 &= 3l^3 \sinh \frac{L}{l} \cosh \frac{L}{l} + 3r_0^3.
\end{align}

Therefore, as in the symmetric case, in this model
\begin{equation}
\Lambda^{(4)}_{ij} = 3
\end{equation}
and \(\gamma^{(4)}_{\mu \nu}\) describes either 4–sphere or the de Sitter space with cosmological constant \(\Lambda = 3\).

**d) Anti de Sitter brane**

In this model \(r_0 = -L, \ r_1 = 0, \ r_2 = L\),
\begin{align}
a_1(r) &= l \cosh \frac{L}{l}, \quad -L \leq r \leq 0, \\
a_2(r) &= l \cosh \frac{L}{l}, \quad 0 \leq r \leq L,
\end{align}
and bulk cosmological constants \(\Lambda_1 = \Lambda_2 = -6/l^2\). The points \(r_0\) and \(r_2\) are identified with each other. The geometry is not closed here and can be smoothly glued in this points. The points \(r_0, r_2\) correspond to wormhole throats in the Riemannian space. The geometry is not smooth in \(r_1\). Therefore, in this model we have only one brane\(^8\) with tension: \(T_1(r_1) = \left(6/\left(v_0^2 l\right)\right) \tanh (L/l)\). Substituting these expressions into formulas (2.12) - (2.15), we obtain respectively:
\begin{align}
B_0 &= l^3 \left( \frac{1}{2} \sinh \frac{2L}{l} + \frac{L}{l} \right) > 0, \\
B_1 &= -3l^3 \left( \sinh \frac{3L}{l} \cosh \frac{L}{l} + \frac{1}{4} \sinh 2\frac{L}{l} - \frac{1}{2} \frac{L}{2l} \right), \\
B_2 &= -3l^3 \left( \sinh \frac{3L}{l} \cosh \frac{L}{l} + \frac{5}{4} \sinh 2\frac{L}{l} + \frac{3}{2} \frac{L}{2l} \right), \\
B_3 &= 6l^3 \left( \sinh \frac{3L}{l} \cosh \frac{L}{l} + \frac{1}{2} \sinh 2\frac{L}{l} \right).
\end{align}

Thus, in this model
\begin{equation}
\Lambda^{(4)}_{ij} = -3
\end{equation}
and \(\gamma^{(4)}_{\mu \nu}\) describes either Riemannian 4–hyperboloid with scalar curvature \(R^{(4)}_{\gamma^{(4)}} = -D_0 (D_0 - 1) = -12\) or 4–D Anti de Sitter space-time with cosmological constant \(\Lambda = -3\) and scalar curvature \(R^{(4)}_{\gamma^{(4)}} = \left[2D_0/(D_0 - 2)\right] \Lambda = -12\).

To conclude this section, we consider in more details the Einstein equation (2.2) with the help of formulas (A.1) - (A.3). In addition to equation (2.6) we obtain:
\begin{align}
R_{rr}[g^{(5)}] &= -4 \frac{a''}{a}, \\
R_{\mu\nu}[g^{(5)}] &= R_{\mu\nu}[g^{(5)}] = 0, \\
R_{\mu\nu}[g^{(5)}] &= R_{\mu\nu}[\gamma^{(4)}] - a^2 \gamma^{(4)}_{\mu\nu} \left[ \frac{a''}{a} + 3 \left( \frac{a''}{a} \right)^2 \right].
\end{align}

Then, \(rr\) and \(\mu\nu\)-components of eq. (2.2) are reduced correspondingly to equations:
\begin{equation}
R^{(4)}_{\gamma^{(4)}} = 2 \sum_{i=1}^{n} a_i^2 (r) \Lambda_i (r) \theta_i (r) + 12 \sum_{i=1}^{n} \left( a_i' \right)^2 \theta_i (r) \equiv f_2 (r)
\end{equation}
and
\begin{equation}
R_{\mu\nu}[\gamma^{(4)}] - \frac{1}{2} g^{(4)}_{\mu\nu} R^{(4)}_{\gamma^{(4)}} = -\gamma^{(4)}_{\mu\nu} \left[ 3 \sum_{i=1}^{n} \left( a_i' \right)^2 \theta_i + \sum_{i=1}^{n} \Lambda_i a_i^2 \theta_i + 3 \sum_{i=1}^{n} a_i a_i'' \theta_i \right] \equiv -\gamma^{(4)}_{\mu\nu} f_3 (r).
\end{equation}

\(^8\)This model can be easily generalized to the case of an arbitrary number of parallel branes by gluing one-brane manifolds at throats and identifying the two final opposite throats.
In latter equation δ-function terms originated from $a''$ and tension terms cancel each other. It can be easily seen that for the Poincaré brane model we obtain $f_2(r) \equiv f_3(r) \equiv 0$ in accordance with eq. (2.21). For symmetric de Sitter brane model $f_2(r) \equiv 12$ and $f_3(r) \equiv 3$, which corresponds to eq. (2.27). In non-symmetric de Sitter brane model: $f_2(r) \equiv 12$ and $f_3(r) \equiv 3$ in accordance with eq. (2.33). For the Anti de Sitter brane model $f_2(r) \equiv -12$ and $f_3(r) \equiv -3$, which corresponds to eq. (2.39).

3 Stability under conformal excitations

Let us investigate now stability of metric $g^{(5)}(X)$ in (2.5) with respect to conformal excitations. Action for conformally transformed metric (B.1) reads:

$$S^{(5)} = \frac{1}{2\kappa_5^2} \int M^{(5)} d^5X \sqrt{|\bar{g}^{(5)}|} \left( R[\bar{g}^{(5)}] - 2\Lambda_5(r) \right) - \sum_{i=0}^{n-1} T_i(r_i) \int d^4x \sqrt{|\bar{g}^{(4)}|} r_i ,$$ (3.1)

where we suppose that matter is kept with changes. It means that the bulk cosmological constant as well as the tensions of branes ("vacuum energies" on branes) do not change$^9$.

With the help of equations (B.3) and (2.6) the first term in this action reads:

$$\sqrt{|\bar{g}^{(5)}|} R[\bar{g}^{(5)}] = \Omega^2 a^4 \sqrt{|\gamma^{(4)}|} \left\{ \Omega^{-2} \left[ \kappa^{-2} R[\gamma^{(4)}] - f_1(r) \right] - 8 \Omega^{-3} \Omega_{MN} g^{(5)MN} \right\} .$$ (3.2)

In what follows, we shall consider a particular case when conformal prefactor is a function of 4–D space-time coordinates: $\Omega = \Omega(x) \equiv \exp(\beta(x))$. It is well known that conformal excitations of this form behaves as scalar fields in 4–D space-time (e.g. on branes). Because of the prefactor $\Omega^2(x)$ in front of 4–D scalar curvature $R[\gamma^{(4)}]$ in action, the 4–D metric $\gamma^{(4)}(x)$ is written in the Brans–Dicke frame. However, it is more easy to investigate the conformal perturbation stability in the Einstein frame:

$$\gamma^{(4)}_{\mu\nu}(x) \Rightarrow \bar{\gamma}^{(4)}_{\mu\nu}(x) = \Omega^2(x) \gamma^{(4)}_{\mu\nu}(x) .$$ (3.3)

In this frame dimensionally reduced action (3.1) reads

$$\bar{S}^{(4)}_{eff} = \frac{1}{2\kappa_4^2} \int M^{(4)} d^4x \sqrt{|\bar{\gamma}^{(4)}|} \left( R[\bar{\gamma}^{(4)}] \right) + \frac{1}{2} \int M^{(4)} d^4x \sqrt{|\bar{\gamma}^{(4)}|} \left( -\gamma^{\mu\nu} \bar{\beta}_{\mu\nu} - 2 \bar{U}_{eff} \right) ,$$ (3.4)

where $\beta \equiv \sqrt{3/2(1/\kappa_4)^2}$ and

$$\bar{U}_{eff}(\Omega) \equiv \frac{1}{\kappa_4^2 B_0} \left[ B_1 \Omega^{-3} + B_2 \Omega^{-1} + B_3 \Omega^{-2} \right] .$$ (3.5)

Here, parameters $B_i$ ($i = 0, \ldots, 3$) are defined by equations (2.12) - (2.15).

Now, the problem of the background solution (2.5) stability against the conformal excitations is reduced to existence of a minimum of the effective potential $U_{eff}$ at point $\Omega = 1 \leftrightarrow \beta = 0$. In Appendix B we explain why minimum should take place namely at $\Omega = 1$. Because only in this case zero order solution (solution corresponding to a minimum of $U_{eff}$) satisfies the same Einstein equation as the original solution (2.5). From other hand, the effective cosmological constant (2.11) coincide with $U_{eff}$ at $\Omega = 1$: $\Lambda^{(4)}_{eff} = U_{eff}(\Omega = 1)$ which is natural in zero order approximation ( where considered system is in position of minimum and conformal excitations are absent ). The extremum existence condition reads:

$$\frac{\partial U_{eff}}{\partial \Omega} \Big|_{\Omega=1} = 0 \implies 3B_1 + 2B_2 + 2B_3 = 0 .$$ (3.6)

Small excitations near a minimum position can be observed on branes as massive scalar fields - gravitational excitons with mass squared:

$$m^2 = \frac{\partial^2 U_{eff}}{\partial \beta^2} \Big|_{\beta=0} = \frac{2}{3} \Omega^2 \frac{\partial^2 U_{eff}}{\partial \Omega^2} \Big|_{\Omega=1} = \frac{2}{3B_0} \left( 12B_1 + 2B_2 + 6B_3 \right) .$$ (3.7)

$^9$It is clear that for conformally transformed metric the Lanczos-Israel junction conditions will change (see e.g. eq. (8.8)). But, at the moment, we do not consider a back reaction of the conformal excitations on the metric; i.e. on the behaviour of $a(r)$, and on the junction condition.
Obviously, the original solution (2.5) is stable under these conformal excitations if \( m^2 > 0 \). As it can be easily seen, all four models considered in previous section satisfy equation (3.6). It means that all these solutions are stationary points of \( U_{\text{eff}} \) if the effective potential is considered as a functional of \( a(r) \). For masses squared in the case of the Poincaré, the de Sitter (symmetric solution), the de Sitter (non-symmetric solution) and the Anti de Sitter branes we obtain respectively:

\[
\begin{align*}
    m^2 &= \frac{4}{l B_0} \left( e^{-4L/l} - 1 \right) < 0, \\
    m^2 &= \frac{4L^3}{B_0} \left( -\sinh^2 \frac{L}{l} \cosh \frac{L}{l} - 3 \frac{3}{4} \sinh \frac{2L}{l} - \frac{3L}{2} \frac{2L}{E} \right) < 0, \\
    m^2 &= \frac{2L^3}{B_0} \left( -\sinh^2 \frac{L}{l} \cosh \frac{L}{l} - 3 \frac{3}{4} \sinh \frac{2L}{l} + \frac{3L}{2} \frac{2L}{E} - 2 \frac{R_B^2}{E} \right) < 0, \\
    m^2 &= \frac{4L^3}{B_0} \left( -\sinh^2 \frac{L}{l} \cosh \frac{L}{l} + \frac{1}{4} \sinh \frac{2L}{l} + \frac{3L}{2} \frac{2L}{E} \right) \leq 0.
\end{align*}
\]

Thus, three first solutions are unstable under considered conformal excitations: \( \Omega = 1 \) corresponds to maximum but not to minimum of the potential (3.5). However, in the AdS brane case, mass squared is positive and decreases from \(^{10}4\) for \( L/l \rightarrow 0 \) to zero for \( L/l \rightarrow 1 \) (more precisely, numerical calculations show that \( m^2 \rightarrow 0 \) for \( L/l \rightarrow 0, 988 \)). So, the AdS brane solution is stable with respect to the conformal excitations if the distance between brane and throats of wormholes is less than the AdS radius.

For small fluctuations near minimum of \( U_{\text{eff}} \) action (3.4) reads:

\[
\begin{align*}
    S_{\text{eff}}^{(4)} &= \frac{1}{2\kappa^2} \int d^4x \sqrt{|\gamma^{(4)}|} \left\{ R[\gamma^{(4)}] - 2\Lambda_{\text{eff}}^{(4)} \right\} + \frac{1}{2} \int d^4x \sqrt{|\gamma^{(4)}|} \left( -\gamma^{\mu\nu} \bar{\beta}_{\mu\nu} \bar{\beta}_{\mu\nu} - m^2 \bar{\beta}^2 \right),
\end{align*}
\]

where first integral corresponds to zero order theory (2.10) and second integral describes gravitational excitons.

If we put in action (3.1) conformally transformed brane tensions \( \bar{T}(r_i) = (1/\Omega) T(r_i) \) (see (B.7) and (B.8)) instead of \( T(r_i) \), the effective potential reads:

\[
U_{\text{eff}}(\Omega) = \frac{1}{B_0} \left[ B_1 \Omega^{-3} + B_2 \Omega^{-1} + B_3 \Omega^{-3} \right].
\]

In this case, \( \Omega = 1 \) is not the extremum of the effective potential (3.13) for all of four considered solutions: they are not stationary points of this potential.

4 Conclusions

In the present paper we investigated the stability of 5–D brane-world solutions against conformal perturbations. For these models the 5-dimensional metric contains a 4-dimensional metric components multiplied by a warp (scale) factor \( a(r) \) which is a function of the additional dimension. Models contain \( n \) parallel branes "transversal" to the additional coordinate. As a matter we consider bulk cosmological constants between branes and tensions ("vacuum energies") on the branes. Scale factor is continuous piecewise function while its derivative has jumps on the branes. There are a number of well known exact solutions which belong to this class of models (e.g. [8, 12, 15], [22]–[26]). We investigated stability of some of these models under the conformal excitations which are functions of 4-D space-time. Such excitations are of special interest because they behave as massive minimal scalar fields in 4-D space-time, for example can be observed as massive scalar particles - gravitational excitons on branes, as it takes place in the Kaluza-Klein approach [1, 6]. We performed dimensional reduction of considered model to the effective 4–D theories. After such reduction, dynamical behaviour of the conformal excitations is defined by the form of the effective potential. Obviously, considered models are stable under these fluctuations if the effective potential has minima at points corresponding to the original solutions. Small excitations around these minima are observed as gravitational excitons on branes. We have shown that in the case of the Poincaré, the de Sitter (symmetric solution) and the de Sitter (non-symmetric solution) branes, all these solutions are unstable with respect to these excitations because the effective potential have maxima but not minima at the points corresponding to the original solutions. In these models 4–D effective cosmological constant is non-negative (see (2.21), (2.27) and (2.33)). However, the AdS brane solution is stable if the distance between brane and throats of wormholes is less than the AdS radius. The effective 4–D cosmological constant is negative in this model (see (2.39)). These results are in agreement with conclusions of papers [11, 12, 19]. The situation with these

\(^{10}\text{Masses squared of gravexcitons (3.8) - (3.11) are written in dimensionless units. If we take into account footnote 6, then physical gravexciton mass for an observer on i-th brane is: } m \rightarrow m_{(\rho_k)} = m/a_i(r_i).\)
four solutions is similar to one we have in pure geometrical case in the standard Kaluza-Klein approach [1].
Here, the stability also takes place when 4-D effective cosmological constant is negative. If the effective
cosmological constant is positive we have maximum of the effective potential instead of minimum. To shift
the minimum of the effective potential to positive values, we should include matter into the model.

We found also that 4-D effective cosmological and gravitational constants on branes as well as gravexciton
masses undergo hierarchy. It was shown that for observers on different branes with different warp factors
these parameters have different values. Similar result with respect to the effective 4-D Newton’s constant
was obtained in [10].

There are a number of possible generalizations which are worth to investigate. First, it is of interest
to include more rich types of matter in the model, e.g. perfect fluid in bulk as well as on branes, which
simulates different forms of matter in the Universe. The presence of matter can stabilize radions in the
brane-world models with non-negative 4-D effective cosmological constant on branes, as it was shown in
[9], [13]–[18]. As we mentioned above, stabilization of gravexcitons in models with 4-D positive effective
cosmological constant takes place also in the standard Kaluza-Klein approach if we include matter here [1].
Thus, we expect similar stabilization effect for gravexcitons in the brane-world models. Second possibility
consists in generalization of the model to a multidimensional case with $D > 5$. It will give opportunity
to include into consideration already obtained exact brane-world solutions with

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**A Appendix: Block-diagonal metrics**

In this appendix we present some useful formulas (see also [27]) for curvature tensors in the case of a
block-diagonal metric of the form:

$$g_{MN}^{(D_0 + D_1)}(x, y) = \begin{pmatrix} e^{2\sigma(y)} g_{\mu \nu}^{(D_0)}(x) & 0 \\
0 & g_{\mu \nu}^{(D_1)}(y) \end{pmatrix}. \quad (A.1)$$

For this metric the Ricci tensor (everywhere in this paper we use the Misner-Thorne-Wheeler book conventions [28]) reads

$$R_{\mu \nu}[g^{(D)}] = R_{\mu \nu}[\gamma^{(D_0)}] - e^{2\sigma} \gamma^{(D_0)} \left[ D_0 g^{(D_1)mn} (\partial_\sigma \partial_\nu \partial_\mu \partial_n \sigma + g^{(D_1)mn} \nabla_m^{(D_1)} (\partial_\rho \sigma) \right],$$

$$R_{\mu \nu}[g^{(D)}] = R_{\mu \nu}[g^{(D)}] = 0,$$

$$R_{\mu \nu}[g^{(D)}] = R_{\mu \nu}[g^{(D_1)}] = D_0 \left[ (\partial_\sigma \partial_\nu \partial_\mu \partial_\rho \sigma) + \nabla_m^{(D_1)} (\partial_\rho \sigma) \right], \quad (A.2)$$

where $D = D_0 + D_1$ and $\nabla_m^{(D_1)}$ is a covariant derivative with respect to the metric $g^{(D_1)}$. The scalar curvature reads correspondingly:

$$R[g^{(D)}] = e^{-2\sigma} R[\gamma^{(D_0)}] + R[g^{(D_1)}] -$$

$$- D_0 \left[ D_0 (\partial_\sigma \partial_\nu \partial_\mu \partial_\rho \sigma + 2 g^{(D_1)mn} \nabla_m^{(D_1)} (\partial_\rho \sigma) \right]. \quad (A.3)$$

**B Appendix: Conformal transformation**

For conformally transformed metric

$$\tilde{g}_{MN}^{(D)}(X) = \Omega^2(X) g_{MN}^{(D)}(X) \equiv e^{2\beta(X)} g_{MN}^{(D)}(X) \quad (B.1)$$

$$\text{four solutions is similar to one we have in pure geometrical case in the standard Kaluza-Klein approach [1]. Here, the stability also takes place when 4-D effective cosmological constant is negative. If the}$$

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$$R_{\mu \nu}[g^{(D)}] = R_{\mu \nu}[\gamma^{(D_0)}] - e^{2\sigma} \gamma^{(D_0)} \left[ D_0 g^{(D_1)mn} (\partial_\sigma \partial_\nu \partial_\mu \partial_n \sigma + g^{(D_1)mn} \nabla_m^{(D_1)} (\partial_\rho \sigma) \right],$$

$$R_{\mu \nu}[g^{(D)}] = R_{\mu \nu}[g^{(D)}] = 0,$$

$$R_{\mu \nu}[g^{(D)}] = R_{\mu \nu}[g^{(D_1)}] = D_0 \left[ (\partial_\sigma \partial_\nu \partial_\mu \partial_\rho \sigma) + \nabla_m^{(D_1)} (\partial_\rho \sigma) \right], \quad (A.2)$$

where $D = D_0 + D_1$ and $\nabla_m^{(D_1)}$ is a covariant derivative with respect to the metric $g^{(D_1)}$. The scalar curvature reads correspondingly:

$$R[g^{(D)}] = e^{-2\sigma} R[\gamma^{(D_0)}] + R[g^{(D_1)}] -$$

$$- D_0 \left[ D_0 (\partial_\sigma \partial_\nu \partial_\mu \partial_\rho \sigma + 2 g^{(D_1)mn} \nabla_m^{(D_1)} (\partial_\rho \sigma) \right]. \quad (A.3)$$

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$$\tilde{g}_{MN}^{(D)}(X) = \Omega^2(X) g_{MN}^{(D)}(X) \equiv e^{2\beta(X)} g_{MN}^{(D)}(X) \quad (B.1)$$
conformal perturbations in metric (2.5):

In this appendix we shall show that our results do not change if only additional dimension undergoes
the Ricci tensor and the scalar curvature read correspondingly:

\[
\hat{\text{hypersurface}}: \quad \text{the unit vector field orthogonal to } \Sigma \text{ is } \bar{\mathbf{x}}(y_i)\]

where \(
\bar{\mathbf{x}}(y_i)
\) is the Lanczos-Israel junction condition:

\[
R[g^{(D)}] = \Omega^{-2}R[g^{(D)}] - 2(D - 1)\Omega^{-3} \Omega_{MN} g^{(D)MN} - (D - 1)(D - 4)\Omega^{-4} \Omega_{MN} \Omega_{KL} g^{(D)MN},
\]

where in equations (B.2) and (B.3) covariant derivatives are taken with respect to the metric \(g^{(D)}\).

We suppose now that the metric \(\bar{g}^{(D)}\) is a solution of the Einstein equation

\[
R_{MN}[\bar{g}^{(D)}] - \frac{1}{2} g_{MN}^{(D)} R[\bar{g}^{(D)}] = - \Lambda \delta_{MN},
\]

which describe a model with the bulk cosmological constant \(\Lambda_D\) and \(n\) branes of tension \(T_i(y_i)\). Let us consider a particular case of the constant conformal transformation \(\Omega \equiv \text{const}\). Then, with the help of eqs. (B.2) and (B.3) for the conformally transformed metric \(g^{(D)}\) we obtain:

\[
R_{MN}[\bar{g}^{(D)}] - \frac{1}{2} |g^{(D)}| R[\bar{g}^{(D)}] = - \Lambda_D \Omega^2 \bar{g}^{(D)}_{MN} - \frac{\kappa^2_D}{\sqrt{|\bar{g}^{(D)}|}} \sum_{i=0}^{n-1} T_i(y_i) \Omega^{D+D_0+2} \sqrt{|g^{(D_0)}(x,y_i)|} g^{(D_0)}_{\mu
u}(x,y_i) \delta^\mu_\nu \delta^\delta_\delta(y - y_i).
\]

This equation shows that conformally transformed metric \(g^{(D)}\) is the solution for the model with the cosmological constant \(\Lambda_D \equiv \Omega \Lambda_D\) and the brane tensions \(T_i(y_i) \equiv \Omega^{(D+D_0+2)}T_i(y_i)\). The latter one is invariant for \(D = D_0 + 2\) which certainly is not the case for \(D = 5\) if \(D_0 = 4\). Thus, if we want that solution corresponding to a minimum of the effective potential for the conformal excitations describes the model with the original cosmological constant and the brane tensions, this minimum should take place at \(\Omega = 1\).

Let us consider transformation of the trace of the extrinsic curvature evoked by conformal transformation of the metric. In the case of 5-D metric (2.5) written in Gaussian normal coordinates the trace of the extrinsic curvature of the hypersurface \(\Sigma: r = r_i \equiv \text{const}\) reads

\[
K(r_i) = - \nabla_M n^M \bigg|_{r_i} = - \frac{1}{2} g^{(4)\mu\nu} \frac{\partial g^{(4)}_{\mu\nu}}{\partial r} \bigg|_{r_i} = - \frac{4}{\Omega} \frac{da}{a} \bigg|_{r_i},
\]

where \(n^M = \delta^M_i\) is the unit vector field orthogonal to \(\Sigma\). If the extrinsic curvature has a jump at this hypersurface: \(K(r_i \pm) \neq 0\) then it results in the Lanczos-Israel junction condition:

\[
T(r_i) = \frac{1}{\kappa^2_D} \frac{3}{4} K(r_i),
\]

where \(T(r_i)\) is the tension of the brane which causes the jump of the extrinsic curvature.

For the metric \(g^{(D)}\), obtained with the help of the conformal transformation (B.1) of the metric (2.5), the unit vector field orthogonal to \(\Sigma\) is \(\bar{n}^M = \Omega^{-1} \delta^M_i\) \(\Rightarrow \bar{n}_M = \Omega \delta^M_i\). Here, we consider the case when \(\Omega = \Omega(x)\) does not depend on the extra dimension \(r\). Then, we obtain for the trace of the extrinsic curvature of the conformal space-time:

\[
\bar{K}(r_i) = - \nabla_M \bar{n}^M \bigg|_{r_i} = - \frac{4}{\Omega} \frac{da}{a} \bigg|_{r_i}.
\]

Correspondingly, the tensions of the brane in conformal and original space-times are connected with each other as follows: \(\bar{T}(r_i) = \Omega^{-1} T(r_i)\) in accordance with equation (B.5).

**C Appendix: Truncated conformal transformation**

In this appendix we shall show that our results do not change if only additional dimension undergoes conformal perturbations in metric (2.5):

\[
g^{(5)}(X) \Rightarrow \bar{g}^{(5)}(X) = \Omega^2(x) dr \otimes dr + a^2(r) \gamma^{(4)}_{\mu\nu}(x) dx^\mu \otimes dx^\nu.
\]
Subsequent application of appropriate formulas from Appendices A and B yields
\[
\sqrt{|g^{(5)}|} R[g^{(5)}] = \Omega a^4 \sqrt{|\gamma^{(4)}|} \left\{ e^{-2 \left[ R[\gamma^{(4)}] - 2 \Omega^{-1} \Omega_{\mu\nu} \gamma^{(4)\mu\nu} \right] - \Omega^{-2} f_1(r) \} \right\},
\]
where \( f_1(r) \) is defined in (2.7). To get this expression, it is useful to go first to a new coordinate \( R : dR = a^{-1}(r)dr \) and then, after using conformal transformation formulas, come back to \( r \) again. It can be easily seen that after conformal transformation to the Einstein frame:
\[
\gamma^{(4)}_{\mu\nu}(x) \Rightarrow \tilde{\gamma}^{(4)}_{\mu\nu}(x) = \Omega(x) \gamma^{(4)}_{\mu\nu}(x)
\]
and the dimensional reduction, action (3.1) is exactly reduced to effective action (3.4). Thus, gravexcitons have exactly the same masses (3.7) - (3.11). This result shows that geometry (gravitational field) under conformal transformations behaves as an elastic media. For an elastic body the eigen frequencies of its oscillations do not depend on the manner of excitation.

References