Cubic Algebraic Equations in Gravity Theory, Parametrization with the Weierstrass Function and Non-Arithmetic Theory of Algebraic Equations.

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Abstract

The present paper demonstrates how methods from algebraic geometry and theory of elliptic functions may find further application in the theory of gravity and also brane, string and Rundall-Sundrum theories. A cubic algebraic equation has been obtained from the standard gravitational Lagrangian. The equation does not follow from a variational principle. By means of a linear - fractional transformation it has been brought to a parametrizable form, expressed through the elliptic Weierstrass function. It was shown that the Weierstrass function can satisfy the standard parametrizable form of the cubic equation, but with $g_2$ and $g_3$ functions of a complex variable instead of the definite complex numbers, known from the usual (arithmetic) theory of elliptic functions and curves. The explicit form of all the coefficient functions in the negative- and positive power Loran decomposition of $g_2$ and $g_3$ was found. Several interesting relations about the sums of the inverse powers of the poles were established. In the special case of the infinite point of the linear-fractional transformation it was demonstrated how the "modified" parametrizable form can be used.

INTRODUCTION

The synthesis of algebraic geometry and physics has been known for a long time, beginning from chiral Potts model, algebraic Bethe ansatz (for a review of these aspects see [1]) and ending up with orbifold models of string compactification [2]. In the context of string theories, the application of algebraic curves, related to Fermat’s theorem, has been pointed out also in [3].
Concerning gravitational physics, which is an inherent constituent of any string, brane or ADS - theory, any applications of the theory of algebraic curves are almost absent. In this aspect perhaps one of the most serious attempts was undertaken in the recent paper of Kraniotis and Whitehouse [4]. Based on a suitably chosen metric of an inhomogeneous cosmological model and introducing a pair of complex variables, the authors have succeeded to obtain a nonlinear partial differential equation for a function, entering the trial solution of the field equations. The most peculiar and important feature of the obtained equation is that it can be parametrized by the well-known Weierstrass function (for a classical introduction in the theory of elliptic and Weierstrass functions see [5-7] ). This convenient representation enabled the authors to express important physical quantities such as the Hubble constant and the scale factor through the Weierstrass and the Jacobi theta functions. In fact, an analogy has been used with examples on the motion of a body in the field of a central force, depending on the inverse powers of the radial distance \( r \). The cases of certain inverse powers of \( r \), when the solution of the trajectory equation is expressed in terms of elliptic and Weierstrass functions, have been classified in details in [8].

Three important conclusions immediately follow from the paper of Kraniotis and Whitehouse [4], and they provide an impetus towards further investigations. The first two conclusions are correctly noted by the authors themselves:

1. Other cases may exist, when solutions of nonlinear equations of General Relativity might be expressed in terms of Weierstrass or theta functions [9], associated with Riemann surfaces.
2. The differential equations of General Relativity in a much broader context might be related to the mathematical theory of elliptic curves and modular forms (for an introduction, see [10-12] ) and even to the famous Taniyama-Shimura conjecture, stating that every elliptic curve over the field of rational numbers is a modular one. For a short review of some of the recent developments in the arithmetic theory of elliptic curves, the interested reader may consult also the monograph [13]. In fact, in [4] this eventual connection of General Relativity Theory with Number and Elliptic Functions Theory was formulated even in the form of a conjecture that "all nonlinear exact solutions of General Relativity with a non-zero cosmological constant \( \Lambda \) can be given in terms of the Weierstrass Jacobi Modular Form". Of course, such a conjecture is expressed for the first time, and yet there are no other solutions derived in terms of elliptic functions, not to speak about any classification of the solutions on that bases.

The present paper will not have the purpose to present any new solutions of the Einstein’s equations by applying elliptic functions, nor will give any new physical interpretation, which in principle should be grounded on previously developed mathematical techniques. This is as much as the above mentioned problem is concerned. Rather than that, in this paper an essential algebraic "feature" of the gravitational Lagrangian will be proved, which is inherent in its structure, mostly in its partial derivatives. This 'algebraically inherent structure' represents the third conclusion, which in a sense may be related to the problems, discussed in [4].

However, this algebraic feature will become evident under some special as-
sumptions. While in standard gravitational theory it is usually assumed that the metric tensor has an inverse one, in the so called theories of spaces with covariant and contravariant metrics (and affine connections) [14] instead of an inverse metric tensor one may have another contravariant tensor $g^{jk}$, satisfying the relation $g^{ij}g^{jk} \equiv l^k_i \neq \delta^k_i$. But then, since $l^k_i$ cannot be determined from any physical considerations and at the same time the important mathematical structure from a physical point of view is the Gravitational Lagrangian, a natural question arises: Is it possible that in such a theory with a more general assumption in respect to the contravariant metric tensor, the Gravitational Lagrangian is the same (scalar density) as in the usual case? From a physical point of view, this is the central problem, treated in this paper, and the answer, which is given, is affirmative. Namely, it has been shown that if $e_i$ are the components of the covariant basic vectors, and $dX^j$ are the components of a contravariant vector field (which, however, are not contravariant basic vectors and therefore $e_idX^k \equiv l^k_i \neq \delta^k_i$), then they satisfy a cubic algebraic equation. Of course, if $dX^i$ are to be found from this equation, then it can be shown that $g^{jk}$ will also be known because of the relation $g^{jk} = dX^j dX^k$. Also, it has been assumed that the affine connection $\Gamma^k_{ij}$ and the Ricci tensor $R_{ij}$, determined in the standard way through the inverse metric tensor are known.

The obtained cubic algebraic equation can be expressed in a very simple form, but unfortunately it is not easy at all to solve it, and that is why a mathematical approach for dealing with such an equation has been developed in the paper. The equation has been derived in two cases: when $d^2X^i \equiv 0$ and when $d^2X^i \neq 0$. As will be shown, the first assumption means that $dX$ has zero-vorticity components (and non-zero divergency, however), and in physical considerations this restriction can be imposed. The second assumption would mean that $dX$ has both non-zero divergency and non-zero vorticity components. From the mathematical point of view of manipulating with the cubic equation, the investigation of the two cases will not be different, because in the second case only the algebraic variety from the first case (with $dX^i$) will be supplemented with the components $d^2X^i$. It is worth mentioning also that the derivation of the equation does not presume zero-covariant derivatives of the covariant tensor field, thus leaving an opportunity to investigate the different kinds of transports on the space-time manifold. The algebraic equation may enable one to make a kind of a classification (from an algebraic point of view) of the contravariant tensors, satisfying the same gravitational Lagrangian.

So far, the problem investigated here may seem to be of pure “theoretical” interest, but it is believed that this is not the case. In supergravity theories, ADS/CFT, five-dimensional and brane theories [15-17], one deals with an action, consisting of a gravitational part, added to a (for example) string action of the kind $S_{str.} \equiv -T2 \int d^2\xi \sqrt{-h^\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$, where $X^\mu$ are the string coordinates, $h^\alpha\beta$-the world sheet metric tensor, $T$- the string tension and the partial derivatives are taken in respect to the world sheet coordinates $\xi^\alpha = (\tau, \sigma)$. One can easily guess that the above described methodology can easily be applied
to the string part of the action. More concretely, $h^{\alpha \beta}$ may be expressed as $h^{\alpha \beta} = d\xi^\alpha d\xi^\beta$, the gravitational metric tensor may be assumed to depend on the string coordinates and the derivatives of the string coordinates will be taken in respect to the world sheet coordinates $\xi^\alpha$. Also, the partial derivatives in the gravitational part of the action may also be taken in respect to the coordinates $\xi^\alpha$. As a result, taking the gravitational and the string part of the action together and without applying any variational principle, one would get the same kind of a cubic algebraic equation as the one, which will be proposed in this paper. In a sense, this dependence of $g_{ij}$ on the string coordinates is a sort of a coupling between the gravitational part of the action and the string one, and the resulting cubic equation may be called "an algebraic equation for the effective parametrization of the total Lagrangian in terms of the string coordinates".

The "coupling" between the two parts of the action provides another interesting possibility, if the first variation of the Lagrangian is performed, even without taking into account any equations of motion. As it was shown in [18], provided that the gravitational Lagrangian depends on the first and second differentials of the metric tensor (which can be proved with similar to the above arguments), the first variation of the Lagrangian can be regarded also as a cubic algebraic equation in respect to the differentials of the vector field. Therefore, applying a variational formalism would mean that three algebraic equations have to be solved simultaneously - the effective parametrization equation, the cubic equation from the first Lagrangian variation and the equation of motion, which will be a quadratic equation. Because of the "coupling", the equation will be only in respect to the string coordinates, and not in respect to both the string coordinates and the gravitational metric tensor, as usually. Of course, other approaches are also possible - for example, considering the equations of motion not as algebraic ones, but as equations in partial derivatives in respect to those coordinates, determined by the algebraic effective parametrization equation.

Until now, only the physical aspects of the implementation of the algebraic approach have been discussed. The mathematical theory of cubic algebraic equations is also worth mentioning, and creating the relevant mathematical methods will be the main purpose of this paper. In principle, the theory of cubic algebraic equations and surfaces has been an widely investigated subject for a long time. In [19] the mathematical theory of cubic hypersurfaces has been presented, putting the emphasis especially on the classification of points on the cubic hypersurface, minimal cubic surfaces, two-dimensional birational geometry and quasi-groups. But no concrete applications of cubic curves are given. In [20] the general theory of affine and projective varieties and algebraic and projective plane curves is exposed, and some examples are considered too. However, the theory of cubic forms is restricted only with the Pascal’s theorem. A more comprehensive introduction to the algebraic theory of second and third-rank curves, their normal forms, turning points (where the second derivatives of the curves’s equation equal to zero), rational transformations and etc. is given in the well-known book of Walker [21]. Of particular relevance to the present research will be the theorem [21] that if $f(x, y) = 0$ is a non-degenerate cubic curve, then by introducing an affine set of coordinates $x_1 = xz, y_1 = yz$ and
choosing the turning point at \((0, 0, 1)\), the curve can be brought to the form 
\[ y^2 = g(x), \]
where \(g(x)\) is a third-rank polynomial with different roots. However,
the situation is much more interesting in the complex plane, where one may
define the lattice \(\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}; \omega_1, \omega_2 \in C, Im\omega_1\omega_2 > 0\}\). Let
a mapping \(f : C/\Lambda \to \mathbb{CP}^2\) is defined of the factorized along the points of the
lattice part of the complex plane into the two-dimensional complex projective
space \(\mathbb{CP}^2\). If under this mapping the complex coordinates \(z\) are mapped as
\[ z \to (\rho(z), \rho'(z), 1) \text{ when } z \neq 0 \text{ and } z \to (0, 1, 0) \text{ when } z = 0 \]
(\(\rho(z)\) is the
Weierstrass elliptic function), then the mapping \(f\) maps the torus \(C/\Lambda\) into
the following affine curve 
\[ y^2 = 4x^3 - g_2x - g_3 \]
where \(g_2\) and \(g_3\) are complex
numbers. The important meaning of this statement is that excluding the points
on the lattice which may be mapped into one point of the torus (where the
Weierstrass function has real values), the mapping \(z \to (x, y) = (\rho(z), \rho'(z))\)
parametrizes the cubic curve. The consequence from that is also essential since
the solution of the resulting differential equation can be obtained in terms of
elliptic functions [5, 13, 22]. In spite of the fact that the parametrization can
be presented in a purely algebraic manner, it is inherently connected to basic
notations from algebraic geometry such as divisors and the Riemann-Roch theo-
rem [23], which reveals the dimension of the vector space of the meromorphic
functions, having a pole of order at most \(n\) at the point \(z = 0\). This should
be kept in mind because some results may be obtained by algebraic methods
only, but the explanation may probably be found in algebraic geometry. In
the present paper, a more general parametrization of a cubic curve is consid-
ered, when \(g_2\) and \(g_3\) are not complex numbers (the so called Eisenstein series
\[ g_2 = 60 \sum_{\omega \subset \Gamma} 1\omega^4 = \sum_{n,m} 1(n + m\tau)^4; \quad g_3 = 140 \sum_{\omega \subset \Gamma} 1\omega^6 = \sum_{n,m} 1(n + m\tau)^6, \]
but complex functions. It has been proved that if the Weierstrass function
parametrizes again the cubic curve, then the infinite sums (in pole
number terms) \(\sum_{\omega \subset \Gamma} 1\omega^n\) for \(n = 1\) and \(n = 2\) turn out to be finite (con-
vergent) ones, in spite of the fact that in the general case they might be infinite
ones (divergent). The explanation of this fact from the point of view of alge-
braic geometry remains an open problem, but it can be supposed that standard
arithmetical theory of elliptic functions and algebraic equations is contained in
some other, more general theory, which may be called non-arithmetical theory,
and from this theory the standard parametrization should also follow. The
considered case of parametrization of a cubic curve with coefficient functions
of a complex variable, although performed in this paper in a trivial algebraic
manner, is the first step towards constructing such a theory. At least, a cer-
tain motivation from a physical point of view is evident for constructing such a
theory.

The above-presented outlook on standard parametrization implied the use
of affine coordinates, which unfortunately exclude from consideration the in-
finiteness point. But the infinity point cannot be ruled out not only from math-
ematical grounds, but also from physical considerations. For example, in the
five-dimensional Randall-Sundrum model [29, 30] one has to assume a compact-
ification into a four-dimensional universe from an infinite extra dimension, containing also the infinity point. From this point of view, the more convenient transformation, chosen in the present paper, which brings the cubic curve into a parametrizable form, is the **linear-fractional transformation**. This transformation allows one to parametrize with the Weierstrass function the ratio of the two of the parameters, entering the linear-fractional transformation and in the case the parameters in this transformation represent complex functions. Of course, the other parameters remain unfixed, leaving the opportunity to determine them in an appropriate way. In a sense, from most general grounds the appearance of the Weierstrass function in the linear-fractional transformation might be expected, since according to a theorem in the well-known monograph of Courant and Hurwitz [24], an algebraic curve of the kind \( w^2 = a_0 v^4 + a_1 v^3 + a_2 v^2 + a_1 v + a_4 \) can be parametrized as \( v = a \rho(z) + b c \rho(z) + d = \varphi(z) \) and \( w = \varphi(z) \) by means of the transformations \( v = av_1 + bcv_1 + d \) and \( w = w_1 ad - bc(cv_1 + d)^2 \). In the case \( a_0 = 0 \) (which is the present case of a third-rank polynomial), \( \varphi(z) \) will be a linear function of \( \rho(z) \).

In the present paper, however, the situation is quite different, since the linear-fractional transformation is applied only in respect to one of the variables \( v \), and in order to get the standard parametrizable form \( w^2 = 4v^3 - g_2 v - g_3 \) (with \( g_2 \) and \( g_3 \)— complex functions), an additional quadratic algebraic equation has to be satisfied. What is more interesting is that after the parametrization is performed, the linear-fractional transformation turns out to be of a more general kind \( v = A(z) \rho(z) + bc + B(z) \cdot d \), where \( A, B, C, D \) are functions of \( z \), and the expression for \( v \) represents a rational transformation of the kind \( v(z) = P(z)Q(z) \). Now from another point of view it can also be understood why it is justifiable to apply the rational transformation only in respect to \( v \) and not in respect to \( w \). The reason is in a well-known theorem [13] from algebraic geometry that "each non-degenerate cubic curve does not admit a rational parametrization". Since each non-degenerate cubic curve can be brought to the form \( w^2 = v(v - 1)(v - \lambda) \), \( (\lambda \neq 0, 1) \), the essence of the above theorem is that this (algebraic) form cannot be satisfied by a rational parametrization of both \( v = P(z)Q(z) \) and \( w = T(z)R(z) \).

In respect to the problem about finiteness of the infinite sums \( \sum 1 \omega^n \) for \( n = 1 \) and \( n = 2 \), the application of the linear-fractional transformation has also turned out to be useful, when the case of poles at infinity is considered. In principle, in this paper two separate cases are distinguished - the first case of an infinite point of the linear-fractional transformation and the second case of poles at infinity, when in the infinite limit \( \omega \to \infty \) the sum \( \sum 1 \omega^n \) tends to the Riemann zeta function \( \xi(n) \). For this partial case and applying a different mathematical method, based on the Tauber’s theorem (see [34] for the formulation of this theorem), one comes again to the fact about the finiteness of \( G_1 \), proved in the general case by performing a Loran function decomposition.

The present paper is organized as follows:

Section 1 gives some basic formulae about the so called gravitational theory with contravariant and covariant metric tensors. In Section 2 the third-rank algebraic equation has been derived, starting from the standard gravitational
Lagrangian. Also, the effective parametrization problem has been formulated in an algebraic language. In Section 3 the general mathematical setup for parametrization of the cubic equation has been discussed, and some physical motivation for the application of the linear-fractional transformation from the point of view of Randall-Sundrum theory has been presented. Section 4 shows how the cubic algebraic equation transforms under the action of the linear-fractional transformation. Section 5 shows how from the transformed cubic equation one can get the standard parametrizable form of the cubic equation (with $g_2$ and $g_3$ - complex numbers) and also the quadratic algebraic equation is derived, which has to be fulfilled if the parametrizable form holds. The approach is valid also when $g_2(z)$ and $g_3(z)$ are complex functions. In Section 6 the Loran’s decomposition has been performed of the functions on the both sides of the algebraic equation $(dpdz)^2 = M(z)\rho^3 + N(z)\rho^2 + P(z)\rho + E(z)$, where $\rho(z)$ is the Weierstrass function and $M, N, P, E$ are functions of the complex variable $z$. A system of three iterative (depending on $n$) algebraic equations has been obtained, representing a necessary (but not sufficient!) condition for parametrization of a cubic equation of a general form with the Weierstrass function. It is not occasional that the condition is called ”a necessary, but not sufficient one”, because in principle more algebraic equations have to be solved in order to prove the existence of such a parametrization. In Section 7 the possible parametrization of the more simplified cubic equation $[\rho'(z)]^2 = 4\rho^3 - g_2(z)\rho - g_3(z)$ has been considered, and of course the main motivation for considering such a case is the close analogy with the well-known case, when $g_2$ and $g_3$ are complex numbers. By calculating the coefficients in the negative power Loran expansion and combining them, it has been proved that the sums $\sum 1\omega$ and $\sum 1\omega^2$ represent finite (convergent) quantities. The other equations for the other values of $m=-3, -1$ have been presented in Appendix A, for values of $m=2k$ - in Appendix B and for $m=2k+1$ and $m=-k$ - in Appendix C. These Appendixes in fact complete the proof that all the Loran coefficient functions can be uniquely expressed through a combination of the finite sums $G_n$. These calculations are purely technical but they serve as a strict mathematical motivation and a proof of the new and basic qualitative fact that the Weierstrass function can parametrize the simplified form of the cubic equation with coefficient functions $g_2(z)$ and $g_3(z)$. This fact probably might represent one of the starting points in the creation of the so called non-arithmetical theory of algebraic equations. Section 8 investigates the positive-power decomposition of the above-mentioned equation, and from the convergency radius of the infinite sum the asymptotic behaviour of some of the Loran coefficients was found to be $-n^{l+1}l+1$. Section 9 considers the case of poles at infinity in the positive-power Loran decomposition, and from the requirement to have a certain convergency radius, expressions for some of the Loran coefficient functions are obtained. In Section 10 a split of the original cubic equation into two equations is performed, and based on the fact from Section 7, it has been proved that the parametrization of the first equation leads to a parametrization of the second equation. For the two ”splitted-up” equations, Section 11 presents an algebraic equation, defined on a
Riemann surface, which has to be satisfied if the so called $j$--invariants of the two equations are to be equal. In Section 12 on the base of the Loran function decomposition of $g_2(z)$ an infinite sum is obtained in which the coefficient functions contain the sums $G_n$. In Section 13 this formulae has been combined with a proof that the Tauber’s theorem can be applied, and this combination resulted in an expression for $G_1$ in the limit of poles at infinity. In this partial case, the expression again confirms that the sum $G_1$ is convergent. In Section 14 the case of the infinite point of the linear-fractional transformation is considered, and the approach essentially represents a combination of the "split-up" approach from Section 10 and the approach from Section 5, based on the derivation of the additional quadratic equation. In Section 15 the relation between the two integration constants is found which appear in the process of integration of the two splitted-up equations. A peculiarity of the developed approach is the appropriate "fixing" of some of the functions in the linear-fractional transformation so that the simplest and most trivial form of the quadratic equation from Section 5 is obtained. Section 16 starts with an algebraic equation of a fourth rank, derived from the original equation in the case of an infinite point of the linear-fractional transformation. The main result here is that the constant Weierstrass function can parametrize this equation if it should be fulfilled in the entire complex plane. For the same equation, Section 17 investigates the second case, when the fourth-rank algebraic equation determines a Riemann surface for the pair of variables $(\rho(z) = w_1(z) + iw_2(z); z)$, and six values of $w_1$ are found, satisfying this equation. Section 18 finds the necessary and sufficient condition for parametrization with a constant Weierstrass function, based again on the approach of Riemann surfaces. As a result, an integrable nonlinear equation is obtained for the coefficient functions of the algebraic equation, in the solution of which the coefficient functions appear in powers of non-integer (fractional) numbers.

1 Covariant and Contravariant Metric Tensor

Usually in gravitational theory it is assumed that a local coordinate system can be defined so that to each metric tensor $g_{ij}$ an inverse one $g^{jk}$ can be defined

\[ g_{ij} g^{jk} \equiv \delta_i^k = \begin{cases} 0 & \text{if } i \neq k \text{ and } 1 & \text{if } i = k \end{cases} \tag{1} \]

However, the notion of a reference frame can be defined in different ways [25] - coordinate, tetrad and monad. In the last case the contravariant vector field $dx^i$ of an observer, moving along a space-time trajectory, represents a reference system. In such a case one may have instead of (2)
\[
e_i dx^j \equiv f^j_i \neq \delta_i^j = S(e_i, dx^j).
\]

(2)

In the context of the so called dual algebraic spaces [26], \(S(e_i, dx^j)\) is called a contraction operator. Assuming that an inverse operator of contraction \(f_j^i\) exists, it can easily be obtained [14]

\[
e^j \equiv f^j_i dx^i.
\]

(3)

Therefore, the metric tensor field \(g\) can be decomposed in respect to the contravariant basic eigenvectors in the following way

\[
g \equiv g_{ij}(e^i \otimes e^j) \equiv g_{ij} f^k_i f^l_j dx^k dx^l (e_k \otimes e_l) \equiv (dx^k dx^l) (e_k \otimes e_l),
\]

(4)

and the contravariant components \(g^{ij}\) of the tensor field \(g\) are

\[
g^{ij} \equiv dx^i dx^j.
\]

(5)

Also, from the definition of a length interval in Riemannian geometry

\[
ds^2 \equiv l^2(\mathcal{R}) \equiv g_{ij} dx^i dx^j \equiv l^2dx^i dx^j \equiv dx^i dx^j.
\]

(6)

it can be obtained

\[
g^{ij} \equiv 1l^2dx^i dx^j.
\]

(7)

From (3) and (7) it follows

\[
[1l^2 - g_{kl} f^k_i f^l_j] \ dx^i dx^j \equiv 0.
\]

(8)

Clearly, the requirement for existence of an inverse contraction operator is equivalent to putting \(l = 1\), i.e. assuming that there is a unit length interval. But
this is a serious restriction, and it is physically more natural to assume that the length interval is varying. Let us assume that \( l^2 \) and \( f_{ik} \) are known in advance, then it can be be investigated which is the algebraic variety of values of \( dx^i \), satisfying this quadratic form. The main difficulty in this approach is that \( f_{ik} \) cannot be determined from physical considerations. That is why the aim in the next section will be to derive an algebraic equation, in which known physical quantities will enter - the metric tensor \( g_{ij} \), the Christoffel connection \( \Gamma^k_{ij} \) and the Ricci tensor \( R_{ij} \).

2 Cubic Algebraic Equation Not Following From a Variational Principle.

The starting point for the derivation of this equation is the representation (5) of the contravariant metric tensor \( g^{ij} \). If substituted into the formulae for the symmetric Christoffel's connection, one obtains

\[
\Gamma^l_{ik} \equiv 12 g^{ls} (g_{ks,i} + g_{is,k} - g_{ik,s}) = 12 dx^l dx^s g_{ks,i} + 12 dx^l dx^s g_{is,k} - 12 dx^l dg_{ik}.
\]

This formulae shall be used for the calculation of the gravitational Lagrangian, written in its standard form, on the one hand, and also by using formulae (5) for the contravariant metric tensor \( g^{ij} \):

\[
L \equiv -\sqrt{-g}R \equiv -\sqrt{-g}dx^i dx^k (\partial \Gamma^l_{ik} \partial x^i - \partial \Gamma^l_{il} \partial x^k) - \sqrt{-g}dx^i dx^k (\Gamma^l_{ik} \Gamma^m_{lm} - \Gamma^m_{il} \Gamma^l_{km}).
\]

Using (9), the first term in (10) is found to be

\[
\sqrt{-g}dx^i dx^k \partial \Gamma^l_{ik} \partial x^i = 12\sqrt{-g}dx^i dx^k g_{ik} + \sqrt{-g}\partial (dx^i) \partial x^i dg_{ik} dx^i dx^k.
\]

The second term in (10) \(-\sqrt{-g}dx^i dx^k \partial \Gamma^l_{il} \partial x^k\) exactly cancels the first term on the right-hand side of (11). The third term in (10) is

\[
\sqrt{-g}dx^i dx^k \Gamma^m_{lm} \equiv 14\sqrt{-g}dx^i dx^s dx^r \partial g_{ts} \partial g_{kr}
\]

and this term exacly cancels the fourth term. Let us also define the scalar quantity
\[ p \equiv \text{div}(dx) \equiv \partial(dx^i)\partial x^i, \quad (13) \]

which "measures" the divergency of the vector field \( dx \). Further in this paper it shall be assumed that if \( X^i \) are some generalized coordinates, defined on a \( n \)-dimensional manifold with coordinates on it \( (x^1, x^2, \ldots, x^n) \), then the differential \( dX^i \) is defined in the corresponding tangent space \( T_X \) of the generalized coordinates \( X^i \equiv X^i(x^1, x^2, x^3 \ldots, x^n) \). Even if written with a small letter, it shall be understood that \( x^i \) represent generalized coordinates.

Finally, using the representation of the first differential of \( g_{ij} \)

\[ dg_{ij} \equiv \partial g_{ij} dx^a \equiv \Gamma^r_{s(i} g_{j)r} dx^s, \quad (14) \]

the following third-rank (cubic) equation in respect to \( dx^i \) may be obtained

\[ dx^i dx^j dx^k \Gamma^r_{j(i} g_{k)r} - R_{ijk} dx^i dx^j = 0. \quad (15) \]

Note that (15) is not an equality, since it has been derived from two representations of the equation (10), where the left-hand side depends on functions of the space-time coordinates, while the right-hand side (the second representation) depends on the differentials \( dx^i \). In an algebraic language [20, 27, 28], the investigated problem can be formulated in the following way:

**Proposition 1** Let the differentials \( dx^i (i = 1, \ldots, n; n \text{ is the space-time dimension}) \) represent elements of an algebraic variety \( \overline{X} = (dx^1, dx^2, \ldots, dx^n) \). For different metric tensors (and therefore different connections \( \Gamma_{ij}^k \) and Riemannian tensors \( R_{ijk} \)), a set of polynomials \( \text{(cubic algebraic equations)} \ F(\overline{X}) \equiv 0 \) may be obtained, which are defined on the algebraic variety \( \overline{X} \) and belong to the ring \( R[dx^1, dx^2, \ldots, dx^n] \) of all third-rank polynomials. Then finding all the possible parametrizations of some introduced generalized coordinates \( X^i(x^1, x^2, x^3, \ldots, x^n) \) is equivalent to: 1. Finding all the elements \( dX^i \) of the algebraic variety \( \overline{X} \), satisfying the equation \( F(\overline{X}) \equiv 0 \). These elements will be represented in the following way

\[ dx^i = \Phi^i(x^1, \ldots x^n, g_{ij}(x^1, x^2 \ldots x^n), \Gamma_{ij}^k(x^1, x^2, \ldots x^n), R_{ijk}(x^1, x^2, \ldots x^n)). \quad (16) \]

2. Finding all the solutions of the above system of partial differential equations.
An equivalent formulation in the same spirit was given also in [18], but for an algebraic equation, obtained after the variation of the gravitational Lagrangian and assuming also that the metric tensor $g_{ij}$ depends on two vector fields $\overrightarrow{u}$ and $\overrightarrow{v}$. In the present case, the algebraic equation is obtained before performing the variation of the Lagrangian. However, in principle it is possible to identify the components of one of the vector fields, for example $u^i$, with the differential components $dx^i$ (i.e. $u^i \equiv dx^i$, the components $v^i$ of the other vector fields may be assumed to be zero, i.e. $v^i = 0$). In such a case, for one and the same set of variables, one may obtain two cubic equations - one before and one after the variation of the Lagrangian. An interesting, but rather complicated problem may arise for finding the intersection varieties of the two cubic algebraic equations.

It is important to mention that equation (15) has been derived under one rather important from a physical point of view assumption that $d^2x^i \equiv 0$. Suppose that for the set of generalized coordinates $X^i \equiv X^i(x^1, x^2...x^n)$ one has

$$dX \equiv a_i dx^i$$

and let us assume that the Poincare’s theorem is fulfilled in respect to $dx^i$, i.e. $d^2x^i = 0$. Then

$$d^2X = da_i dx^i + a_i d^2x^i = \partial a_i \partial x^i dx^j \wedge dx^i = (\partial a_i \partial x^j - \partial a_j \partial x^i) dx^i dx^j. \quad (18)$$

Clearly, $d^2X = 0$ only in the following two cases: 1. $a_i = const.$, i.e. $dX^i$ is a full differential.

2. $(rota)_{ij} \equiv \partial a_i \partial x^j - \partial a_j \partial x^i \equiv 0$. The last means that if $dx^i$ are considered to be basic eigenvectors, then $dX^i$ have zero-vorticity components. Throughout the whole paper $dX^i$ shall be considered as vector field’s components in the tangent space $T_X$.

In the present case, the assumption about $d^2x^i = 0$ will not affect the calculation of the third and the fourth term in (10), but it will affect the first and the second term. As a result, one obtains instead of (15) the following modified third-rank algebraic equation

$$dx^i dx^k \Gamma^r_{ij}(g_{kr})(pdx^j - d^2x^j) - 2R_{ik}dx^i dx^j \equiv 0. \quad (19)$$

Remarkably, this equation takes into account two important physical characteristics of the vector field $dx^i$ - the divergency $p$ and the vorticity (through the term
d^2 x^i). It might be required that these characteristics vanish, i.e. \( p = d^2 x^i = 0 \). In such a case one is left only with the equation

\[
R_{ik} dx^i dx^k \equiv 0,
\]

which is valid of course without imposing any additional requirements for existence of an inverse metric tensor. The imposition of such a requirement would mean that the intersection variety of the quadratic form (20) with the quadratic forms (one-when \( \delta^j_i = 0 \), and the other - when \( \delta^j_i = 1 \)).

\[
g_{ik} dx^k dx^j \equiv \delta^j_i.
\]

has to be found. From the two last equations one easily obtains

\[
(R_{ik} - 12g_{ik}R) dx^k dx^j \equiv -12R \delta^j_i,
\]

in which the left-hand side is identically zero for every \( dx^i \) in view of the Einstein’ equations \( R_{ik} - 12g_{ik}R \equiv 0 \), but the right-hand side is zero only for \( i \neq j \), but not also when \( i = j \). The reason for this contradiction is that one should not combine together (in the meaning of substraction) the two equations. Instead, the intersection variety of the two equations should be found, but this is not related at all to any substraction of the two equations and represents a more complicated procedure.

3 A General Mathematical Setup for Treating the Cubic Algebraic Equation (15)

The subsequent investigation of equation (15) will be restricted to the case of a 5-dimensional space-time, although the approach of course may be applicable to any dimensions. The main reason for choosing a 5-d spacetime is related to the widely discussed Randall-Sundrum model [29, 30], in which the process of compactification of the five-dimensional universe to our present four-dimensional universe is related to the existence of a \textit{large extra dimension}. In the original R-S scenario the metric was chosen to be

\[
ds^2 = e^{-2kr_c r_0} \eta^\mu^\nu dx^\mu dx^\nu + r_c^2 dx_5^2,
\]