I. INTRODUCTION

The new generation of cosmic microwave background (CMB) experiments has been performed assuming Gaussian fluctuations. The Edgeworth expansion, however, allows for small deviations from Gaussianity, and the extra parameters introduced by the Edgeworth expansion are expected to be detectable. This paper presents the likelihood of the Edgeworth expansion parameters and the sensitivity of the CMB experiments to the deviations from Gaussianity:

II. EDGEWORTH LIKELIHOOD
log \Delta + Z'_\ell, \delta \text{ so that, neglecting the factors independent of the variables and putting } w = \log \Delta, \text{ we obtain}

\[ L \propto \int \exp \left[ -\sum \frac{[w + \Delta'_\ell + \sigma'_\ell \delta]^2}{2 \sigma'_\ell} \right] dw \propto \epsilon^{-\frac{N}{2}} \left( \gamma - \frac{\alpha^2}{\alpha^2} \right), \quad (2) \]

where

\[ \Delta'_\ell = Z'_\ell, \delta (\ell; \alpha, \beta) - Z'_\ell, \delta (\ell), \]
\[ \alpha = \sum 1/\sigma'_\ell, \quad \beta = \sum \Delta'_\ell/\sigma'_\ell, \]
\[ \gamma = \sum \Delta'_\ell/\sigma'_\ell. \]

Let us now introduce the Edgeworth expansion. Denoting with \( x_\ell = (w + \Delta'_\ell)/\sigma'_\ell \) the normal variable in the Gaussian function, the Edgeworth expansion is [17]

\[ P(x_\ell) = \exp \left[ -\frac{x_\ell^2}{2} \right] \left[ 1 + \frac{k_3, \ell}{6} H_3(x_\ell) + \frac{k_4, \ell}{24} H_4(x_\ell) + \frac{k_5, \ell}{72} H_5(x_\ell) + \ldots \right], \quad (3) \]

where \( k_n, \ell \) is the \( n \)-th cumulant of \( x_\ell \) and \( H_n \) is the Hermite polynomial of \( n \)-th order. Notice that the EE has the same norm, mean and variance as the Gaussian, but different mode (the peak of the distribution). Here, as a first step, we limit ourselves to the first non-Gaussian term containing the skewness \( \kappa_{3, \ell} \).

Assuming that \( x_\ell \) is distributed according to the Edgeworth expansion, we can build the truncated Edgeworth likelihood function [14] to first order in \( \kappa_{3, \ell} \):

\[ L = e^{-\sum \frac{x_\ell^2}{2}} \left[ 1 + \frac{k_3, \ell}{6} \sum \kappa_{3, \ell} H_3(x_\ell) \right], \]

with \( H_3(x_\ell) = x_\ell^3 - 3x_\ell. \) Now, integrating over \( w \) we obtain

\[ L = \int e^{-\sum \frac{x_\ell^2}{2}} \left[ 1 + \frac{1}{6} \sum k_3, \ell H_3(x_\ell) \right] dw = \]
\[ \sqrt{2\pi} e^{-\frac{\gamma - \frac{\alpha^2}{\alpha^2}}{2}} \left[ 1 + \frac{1}{6} g(k_3, \ell, \Delta, \sigma) \right], \quad (4) \]

where

\[\begin{align*}
\alpha^3 g(k_3, \ell, \Delta, \sigma) &= (-3 \alpha \beta - \beta^3) \sum \frac{k_3, \ell}{\sigma'_\ell} \\
+ (3 \alpha^2 + 3 \alpha \beta^2) \sum \frac{k_3, \ell \Delta}{\sigma'_\ell} - 3 \alpha^2 \beta \sum k_3, \ell \left( \frac{\Delta^2}{\sigma^2} - \frac{1}{\sigma'_\ell} \right) \\
+ \alpha^3 \sum k_3, \ell \left( \frac{\Delta_3^3}{\sigma^3} - 3 \frac{\Delta_\ell}{\sigma'_\ell} \right).
\end{align*}\]

This is the likelihood function that we study below. The effect of the extra terms is to shift the peak (or mode) of the distribution of each \( C_\ell \) while leaving the mean unperturbed. Since the shift depends on \( \Delta, \sigma_\ell \) and \( \kappa_{3, \ell} \), the resulting mode spectrum will be distorted with respect to the mean spectrum. Therefore, the likelihood maximization will produce in general results that depend on \( \kappa_{3, \ell} \). In Fig. 1 we show the peak shift introduced in the simplified case in which the skewness is independent of \( \ell \): if \( k_3 \) is negative, the spectrum is shifted upward by a larger amount at the very small and very large multipoles, and by a smaller amount around \( \ell = 200 \), where the relative errors are the smallest; if \( k_3 \) is positive the shift is downward. As a consequence of the distortion, we expect that a constant negative skewness favours spectra which are tilted downward with respect to the Gaussian case, and the contrary for a positive skewness. In general, the cosmological parameters will depend on the multipole dependence of \( \kappa_{3, \ell} \). For small \( \kappa_{3} \sigma_\ell \), the shift can be approximated by

\[ C_{\ell(\text{mode})} = C_{\ell(\text{mean})}(1 - \kappa_3 \sigma_\ell/2). \quad (5) \]

Clearly, if the peak shift introduced by the EE were independent of \( \ell \), the integration over the amplitude \( w \) would erase the non-gaussian effect on the likelihood. That is, putting \( \sigma_\ell \) and \( \Delta_\ell \) equal to a constant independent of \( \ell \) we obtain \( g(k_3, \ell, \Delta, \sigma) = 0 \).

III. DEPENDENCE ON THE SKEWNESS

To evaluate the likelihood, a library of CMB spectra is generated using CMBFAST [18]. Following [1] I adopt the following uniform priors: \( n \in (0.7, 1.3), \quad \omega_b \in (0.0025, 0.08), \quad \omega_c \in (0.05, 0.4), \quad \Omega_M \in (0, 0.9) \). As extra priors, the value of \( h \) is confined in the range \( (0.45, 0.9) \) and the universe age is limited to \( > 10 \) Gyr. The remaining input parameters requested by the CMBFAST code are set as follows: \( T_{\text{cmb}} = 2.726 K, Y_{\text{He}} = 0.24, N_0 = 3.04, \tau_\ell = 0 \). In the analysis of [1] \( \tau_\ell \), the optical depth to Thomson scattering, was also included in

![Figure 1: The squares mark the mode of the C_\ell distribution for k_3 = -2 (light squares), and for k_3 = 2 (dark squares). The data are from Cobe and Boomerang.](image-url)
the general likelihood and, in the flat case, was found to be compatible with zero at slightly more than 1σ. Therefore here, to reduce the parameter space, I assume κ to vanish. The theoretical spectra are compared to the data from COBE [16] and Boomerang [1].

To specify the skewness k3, there are three cases to study: in the first one ("constant skewness"), k3,ℓ = k3 is assumed independent of the multipole ℓ; in the second ("gaussian skewness"), the skewness is assumed to be generated by some process only in a particular range of multipoles:

\[ k_{3,\ell} = k_3 e^{-(\ell-\ell^*)^2/(2\Delta_\ell^2)}, \]  

where, in the numerical examples below, I put \( \ell^* = 200 \) and \( \Delta_\ell^* = 20 \). In the third case ("hierarchical skewness"), the "hierarchical" ansatz is assumed [19], in which the skewness of the temperature field is proportional to the square of its variance. At the first order, we can assume that the skewness of the \( C_\ell \) distribution is proportional to the skewness of the fluctuation field, so I put

\[ k_{3,\ell} = k_3^{\ast} (C_\ell/C_\ell^{\ast})^2, \]  

where, for instance, \( C_\ell^{\ast} = C_{100} \). In all three cases \( k_3^{\ast} \) is left as a free parameter. These three choices are of course purely an illustration of what a real physical mechanism might possibly produce.

Fig. 2 shows the one-dimensional Edgeworth likelihood functions marginalized in turn over the other three parameters. For the "constant skewness", \( k_3^{\ast} \) varies from -1.8 to 1.2 (light to dark curves): below and above these values the likelihood begins to show pronounced negative wings, which signals that the first order Edgeworth expansion is no longer acceptable. While the likelihood for \( \omega_c \) is almost independent of \( k_3 \), it turns out that the other likelihoods move toward higher values for higher skewness. As anticipated, this can be explained by observing that a higher skewness implies smaller \( C_\ell \) at small multipoles: a tilt toward higher \( n \) and higher \( \omega_b \) gives therefore a better fit. The effect is of the order of 10% for \( k_3^{\ast} \approx 1 \).

In the "gaussian skewness" case the trend is qualitatively the opposite, as can be seen in Fig. 3, where \( k_3^{\ast} \) ranges from -4 to 4 (light to dark). Here the cosmological parameters decrease for an increasing skewness. The reason is that now the effect is concentrated around the intermediate multipoles \( n \approx 200; \) a positive skewness induces smaller \( C_\ell \) at these multipoles, and therefore a smaller \( n \) and \( \omega_b \) helps the fit. The third case, the "hierarchical skewness", is not shown because is qualitatively similar to the previous case: the region around \( \ell = 200 \) is in fact also the region where \( C_\ell \) is larger and therefore \( k_{3,\ell} \), given by Eq. (7), is larger.

Fig. 4 summarizes the results: the trend of the estimated parameters (mean and standard deviation) versus \( k_3^{\ast} \) in the "constant skewness" case. The constant plateau that is reached for \( k_3^{\ast} < 0 \) depends on the fact that for large and negative skewness the peak shift is independent of \( k_3 \). The cosmological parameters can be well fitted by the following expressions:

\[ n = 0.90(1 + 0.03e^{h^2}), \]  
\[ \omega_b = 0.021(1 + 0.05e^{h^2}), \]  
\[ \omega_c = 0.115(1 + 0.025e^{h^2}), \]  
\[ \Omega_\Lambda = 0.67(1 + 0.06e^{h^2}). \]

Figure 3: Likelihood functions for the four cosmological parameters in the "gaussian skewness" case. The skewness increases from -4 to 4, light to dark.

For \( h \) the fit is \( h^2 = 0.42 \pm 0.05 \). Notice that the trend for \( h \) is stronger than for the other variables: \( h \) goes from 0.65 to 0.85 when \( k_3 \) increases from -1.6 to 1.2. Similar relations can be found for the other cases as well.
IV. CONCLUSIONS

This paper illustrates quantitatively a basic and obvious fact about cosmological parameter estimation, namely the dependence on the underlying statistics. Although the gaussianity of the CMB data is still to be proved, almost all the previous works estimated the cosmological parameters assuming vanishing higher order cumulants. Here it has been shown that a non-zero skewness distorts the mode spectrum with respect to the mean spectrum, inducing a considerable variation to the best fit cosmological parameters.

The Edgeworth expansion we used in this paper is convenient for analytical purposes but its use is limited to relatively small deviations from gaussianity. In fact, the peak shift displayed in Fig.1 is always smaller than the errors, and as a result the parameters, although varying with $k_3$, remain always within one sigma from the zero-skewness case. This, however, does not mean that the dependence on the higher order moments can be neglected, first because it is a systematic effect, and second because more general probability distributions which are not small deviations from gaussianity might introduce much larger shifts.

We have shown that, to first order, the peak shift $\Delta C_\ell / C_\ell$ is proportional to $k_3 \sigma_\ell$. The error $\sigma_\ell$ includes cosmic variance and experimental errors. In the future, the main source of error will be cosmic variance, at least below $\ell = 2000$ or so. A skewness of order unity will therefore introduce an additional “skewness bias” that will limit the knowledge of the cosmological parameters by an amount similar to the cosmic variance itself. At this point it will become necessary to estimate $k_3 \ell$ along with the other parameters. The first order EDG, however inadequate, since it is linear in $k_3 \ell$, and it will be necessary to extend the expansion to higher orders [6, 13], or to adopt a non-perturbative non-gaussian distribution.