UNIVERSE MODELS WITH A VARIABLE COSMOLOGICAL “CONSTANT” AND A “BIG BOUNCE”

Hongya Liu

Department of Physics, Dalian University of Technology
Dalian 116024, P.R. China
hyliu@dlut.edu.cn

Paul S. Wesson

Department of Physics, University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
wesson@astro.uwaterloo.ca

ABSTRACT

We present a rich class of exact solutions which contains radiation-dominated and matter-dominated models for the early and late universe. They include a variable cosmological “constant” which is derived from a higher dimension and manifests itself in spacetime as an energy density for the vacuum. This is in agreement with observational data and is compatible with extensions of general relativity to string and membrane theory. Our solutions are also typified by a non-singular “big bounce” (as opposed to a singular big bang), where matter is created as in inflationary cosmology.

Subject headings: cosmology, theory-relativity, strings and membranes

1. INTRODUCTION

In general relativity, the cosmological constant Λ may be regarded as a measure of the energy density of the vacuum, and can in principle lead to the avoidance of the big-bang singularity which is characteristic of other Friedmann-Robertson-Walker (FRW) models.

1 Corresponding author.

2 Fax: (519) 746 8115.
However, the rather simplistic properties of the vacuum that follow from the usual form of Einstein’s equations can be made more realistic if that theory is extended, which in general leads to a variable $\Lambda$. Recently, Overduin (1999) has given an account of variable-$\Lambda$ models which have a non-singular origin. Here we wish to approach this subject from a different perspective, and give a class of exact solutions wherein $\Lambda$ can change in a manner that is in agreement with observation but where the singular big bang is replaced by a non-singular “big bounce”.

There are several motives for this. In Einstein’s theory, the density and pressure of the vacuum are given in terms of the gravitational constant and the speed of light by $\rho = -p/c^2 = \Lambda/8\pi G$ (see e.g. Adler, Bazin and Schiffer 1975 or Wesson 1999). However, astrophysical data constrain the values of these to be many orders smaller than the corresponding quantities inferred from particle physics, leading to the cosmological constant problem (Weinberg 1989; Ng 1992; Adler, Casey and Jacob 1995). This can in principle be resolved by introducing a variable scalar field to the right-hand side of Einstein’s equations. There are constraints on this approach, notably from galaxy number counts, gravitational lens statistics and cosmic microwave background anisotropies (for references see Overduin 1999; Overduin and Cooperstock 1998). Also, while it is possible to modify 4D relativity by introducing a scalar field, the latter is most naturally a part of $N(>4)$D theory. For example, in 5D Kaluza-Klein theory the 10 potentials for the gravitational field are augmented by the 4 potentials of electromagnetism and a scalar potential which is related to mass (See Wesson 1999: even if the electromagnetic potentials are absent and the scalar potential is a constant, the 5D theory implies significant differences from the 4D theory). Further, the horizon problem posed by the isotropy of the microwave background in the context of the standard FRW models can be resolved by a period of inflation in 4D which is most naturally driven by the dynamics of higher dimensions (Linde 1990, 1991). Indeed, the most promising route to a unification of classical gravity and electromagnetism with the quantum theory of particle interactions is via $N$D manifolds, as in superstrings, supergravity and membrane theory (West 1986; Green, Schwarz and Witten 1987; Youm 2000). The latter theories, however, have yet to be developed to the stage where they can be meaningfully tested. By comparison, the 5D theory is a modest extension of 4D general relativity and is known to be in agreement with the classical tests in the solar system (Kalligas, Wesson and Everitt 1995; Liu and Overduin 2000), as well as the less exact data from cosmology (Wesson 1992). We will therefore in what follows go with the minimal extension of general relativity, using previous work on 5D manifolds (Liu and Mashhoon 1995, 2000) to see what effects an extra dimension has on $\Lambda$ and the big bang.

The plan of this paper is as follows. In Section 2 we will give a broad class of 5D solutions and derive their associated 4D properties of matter. In Section 3 we will focus for
practical reasons on the sub-class of solutions which is flat in 3D; and then in Sections 4 and 5 study this when the equation of state is that of dust and radiation, respectively. Section 6 is a conclusion.

2. 5D SOLUTIONS AND THEIR 4D MATTER

We choose units such that $8\pi G = c = 1$, and let upper-case Latin indices run 0-4 and lower-case Greek indices run 0-3. Our coordinates are $(t, r, \theta, \phi, y)$ with $d\Omega^2 \equiv (d\theta^2 + \sin^2 \theta d\phi^2)$. The 5D line element $dS^2 = g_{AB}dx^A dx^B$ contains the 4D one $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. The 5D field equations in terms of the Ricci tensor are $R_{AB} = 0$. These contain the 4D field equations, which in terms of the Einstein tensor and an induced energy-momentum tensor are $G_{\alpha\beta} = T_{\alpha\beta}$. The procedure to go from 5D to 4D is by now well known (see Liu and Mashhoon 1995 or Wesson 1999), and is guaranteed by Campbell’s theorem. There are many classes of solutions of the 5D field equations known, including the much-discussed Ponce de Leon (1988) cosmologies.

A new class, which extends the FRW solutions and may be verified by computer, is given by

$$
\begin{align*}
    dS^2 &= B^2 dt^2 - A^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) - dy^2 \\
    A^2 &= (\mu^2 + k) y^2 + 2\nu y + \frac{\nu^2 + K}{\mu^2 + k} \\
    B &= \frac{1}{\mu} \frac{\partial A}{\partial t} \equiv \dot{A} / \mu
\end{align*}
$$

(1)

Here $\mu = \mu(t)$ and $\nu = \nu(t)$ are arbitrary functions, $k$ is the 3D curvature index ($k = \pm 1, 0$) and $K$ is a constant. After a lengthy calculation we find that the 5D Kretschmann invariant takes the form

$$
I = R_{ABCD} R^{ABCD} = \frac{72K^2}{A^8},
$$

(2)

which shows that $K$ determines the curvature of the 5D manifold (1). Note that the result (2) agrees, after rescaling parameters, with that obtained in the so-called canonical coordinates (Liu and Mashhoon 1995). From (1) we can see that the form of $Bdt$ is invariant under an arbitrary transformation $t = t(\tilde{t})$. This gives us a freedom to fix one of the two arbitrary functions $\mu(t)$ and $\nu(t)$ without changing the form of solutions (1). The other can be seen to relate to the 4D properties of matter, to which we now proceed.
The 4D line element is

$$ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta = B^2 dt^2 - A^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (3)$$

This has the Robertson-Walker form which underlies the standard FRW models, and allows us to calculate the non-vanishing components of the 4D Ricci tensor:

$$R_{00}^4 = -\frac{3}{B^2} \left( \frac{A}{A} - \frac{\dot{A} B}{A B} \right)$$

$$R_{11}^4 = R_{22}^4 = R_{33}^4 = \frac{1}{B^2} \left[ \frac{\ddot{A}}{A} + \frac{\dot{A}}{A} \left( \frac{2A}{A} - \frac{\dot{B}}{B} \right) + 2k \frac{B^2}{A^2} \right]. \quad (4)$$

Now from (1) we have

$$B = \frac{\dot{A}}{\mu}, \quad \dot{B} = \frac{\ddot{A}}{\mu} - \frac{\dot{A} \mu}{\mu \mu}.$$  

Using these in (4), we can eliminate $B$ and $\dot{B}$ from them to give

$$R_{00}^4 = -\frac{3\mu \ddot{\mu}}{A \ddot{A}}$$

$$R_{11}^4 = R_{22}^4 = R_{33}^4 = -\left[ \frac{\mu \ddot{\mu}}{A A} + \frac{2(\mu^2 + k)}{A^2} \right]. \quad (5)$$

These yield the 4D Ricci scalar

$$R^4 = -6 \left( \frac{\mu \ddot{\mu}}{A A} + \frac{\mu^2 + k}{A^2} \right). \quad (6)$$

This together with (5) enables us to form the 4D Einstein tensor $(4)G_{\alpha \beta}^4 \equiv (4)R_{\alpha \beta}^4 - \delta^\alpha_\beta (4)R/2$. Its non-vanishing components are

$$G_{00}^4 = \frac{3(\mu^2 + k)}{A^2}$$

$$G_{11}^4 = G_{22}^4 = G_{33}^4 = \frac{2\mu \ddot{\mu}}{A A} + \frac{\mu^2 + k}{A^2} \quad . \quad (7)$$

These give the components of the induced energy-momentum tensor since Einstein’s equations $G_{\beta}^\alpha = T_{\beta}^\alpha$ hold.
Let us suppose that the induced matter is a perfect fluid with density \( \rho \) and pressure \( p \) moving with a 4-velocity \( u^\alpha \equiv dx^\alpha/ds \), plus a cosmological term whose nature is to be determined:

\[
^{(4)}G_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + (\Lambda - p) g_{\alpha\beta} \quad . 
\]

As in the FRW models, we can take the matter to be comoving in 3D, so \( u^\alpha = (u^0, 0, 0, 0) \) and \( u^0 u_0 = 1 \). Then (8) and (7) yield

\[
\rho + \Lambda = \frac{3 (\mu^2 + k)}{A^2} \\
p - \Lambda = -\frac{2 \mu \frac{\ddot{\varnothing}}{A}}{A^2} - \frac{\mu^2 + k}{A^2} \quad .
\]

These are the analogs for our solution (1) of the Friedmann equations for the FRW solutions. As there, we are free to choose an equation of state, which we take to be the isothermal one

\[
p = \gamma \rho \quad .
\]

Here \( \gamma \) is a constant, which for ordinary matter lies in the range (dust) \( 0 \leq \gamma \leq 1/3 \) (radiation or ultrarelativistic particles). Using (10) in (9) we can isolate the density of matter and the cosmological term:

\[
\rho = \frac{2}{1 + \gamma} \left( \frac{\mu^2 + k}{A^2} - \frac{\mu \ddot{\varnothing}}{A \ddot{A}} \right) \\
\Lambda = \frac{2}{1 + \gamma} \left[ \left( 1 + 3 \gamma \right) \left( \frac{\mu^2 + k}{A^2} \right) + \frac{\mu \ddot{\varnothing}}{A \ddot{A}} \right] \quad .
\]

In these relations, \( \mu = \mu(t) \) is still arbitrary and \( A = A(t, y) \) is given by (1). The matter density therefore has a wide range of forms and the cosmological “constant” is variable.

Let us now consider singularities of the 5D manifold (1). Since metric (1) is 5D Ricci flat, we have \( R = 0 \) and \( R^{AB} R_{AB} = 0 \). The third 5D invariant is given in (2), from which we see that \( A = 0 \) (with \( K \neq 0 \)) corresponds to a 5D singularity. This is a physical singularity and, as is in general relativity, can be naturally explained as a “big bang” singularity. From (6) and (11) we also see that if \( \ddot{A} = 0 \) then all \( ^{(4)}R, \rho \) and \( \Lambda \) become infinity. This means that \( \ddot{A} = 0 \) (with \( A, \mu, \ddot{\varnothing} \neq 0 \)) represents a second kind of singularities where the 4D scalar curvature \( ^{(4)}R \), the 4D induced mass density \( \rho \) and \( \Lambda \) diverge. Clearly, this is a 4D singularity at which the 5D invariant (2) may keep normal. We can naturally explain this one as a “big bounce” singularity at which the scale factor \( A \) reaches its minimum with \( \ddot{A} = 0 \). Note that \( \ddot{A} = 0 \) (with \( \mu \neq 0 \)) implies \( B = 0 \) by (1). So we conclude that our 5D models (1) may have
two kinds of singularities, “big bang” and “big bounce”, characterized by $A = 0$ and $B = 0$, respectively. Here $A$ is the scale factor of the 3D space and we can call $B$ the scale factor of the time. In the following sections we will see a realization of the 5D “big bounce” models.

3. THE SPATIALLY-FLAT MODEL

Astrophysical data, including the age of the universe, are compatible with an FRW model with flat space sections (Leonard and Lake 1995; Overduin 1999). We therefore put $k = 0$ in (1), which gives

$$dS^2 = B^2 dt^2 - A^2 \left( dr^2 + r^2 d\Omega^2 \right) - dy^2$$

$$A^2 = \left( \mu y + \frac{\nu}{\mu} \right)^2 + \frac{K}{\mu^2}$$

$$B = \frac{\dot{A}}{\mu} .$$

(12)

The corresponding relations (11) become

$$\rho = \frac{2}{1+\gamma} \frac{\mu}{A} \left( \frac{\mu}{A} - \frac{\dot{\mu}}{\dot{A}} \right)$$

$$\Lambda = \frac{2}{1+\gamma} \frac{\mu}{A} \left[ \frac{1+3\gamma}{2} \frac{\mu}{A} + \frac{\dot{\mu}}{\dot{A}} \right] .$$

(13)

But as mentioned above, one of the functions $\mu(t)$ and $\nu(t)$ which appear in (12) and (13) is free to choose without loss of generality. A convenient choice and the corresponding form of $A(t, y)$ are given by:

$$\nu = -C \mu^2 t$$

$$A^2 = \mu^2 (y - Ct)^2 + \frac{K}{\mu^2} .$$

(14)

Here $C$ is a constant and $\mu = \mu(t)$ is the only function left undetermined.

It is well known in general relativity that the isolation of a cosmological term from densities $\rho$ and $p$ is not unique: one can always get $\Lambda$ absorbed into $\rho$ and $p$. The only thing which is of physical importance is the equation of state satisfied by $\rho$ and $p$. This has things to do with the vacuum energy. It is expected that the vacuum energy could be contributed by either $\Lambda$ or dark matter, or by both. Without a better model for the dark matter, we do not know how to separate them. So we assume that all constituents of $\Lambda$ which evolve like an
ordinary matter through \( p = \gamma \rho \) have already been absorbed into \( \rho \) and \( p \). Furthermore, we also assume that the remaining \( \Lambda \) is small compared with \( \rho \) in the present universe. Then, to fix \( \mu(t) \), we will impose a condition on the cosmological term \( \Lambda \) in equations (13) that as the model expands to infinity, \( \Lambda \) tends to zero more rapidly than \( \rho \). [The relative sizes of \( \Lambda \) and \( \rho \) will then be given by (13) and depend on the epoch.] To implement this condition, we recall that in the \( k = 0 \) FRW model the scale factor is of a power law. This reminds us to try to use a power law for \( \mu(t) \) in (14), which give.

\[
\mu(t) = t^n
\]

\[
A^2 = t^{2n} (y - Ct)^2 + Kt^{-2n}.
\]  

The last of these shows that the power \( n \) determines the behavior of the scale factor \( A(t,y) \) on the hypersurface \( y = \text{constant} \) which is spacetime. Specifically, we find that \( A \propto t^{n+1} \) for \( n \geq -1/2 \) as \( t \to \infty \). This into (13) gives

\[
\rho \propto \left( 1 - \frac{n}{n+1} \right) \frac{1}{t^2}, \quad t \to \infty
\]

\[
\Lambda \propto \left( \frac{1 + 3\gamma}{2} + \frac{n}{n+1} \right) \frac{1}{t^2}, \quad t \to \infty.
\]  

These are observationally acceptable behaviors for ordinary matter (Leonard and Lake 1995) and the cosmological term (Overduin 1999). However, the condition that \( \Lambda \) decays faster than \( \rho \) means that we must have \( n = -(1 + 3\gamma)/3(1 + \gamma) \). This with \( A \propto t^{n+1} \) and \( \mu = t^n \) in (12) gives \( B^2 \to \text{constant} \), for \( t \to \infty \). This means that the usual proper time is recovered if we let \( B \to 1 \) for \( t \to \infty \). Then the constant \( C \) in (14) is

\[
C = \begin{cases} 
\frac{3}{2}(1 + \gamma) & \text{for } 0 \leq \gamma < \frac{1}{3} \\
\sqrt{4 - K} & \text{for } \gamma = \frac{1}{3}
\end{cases}
\]  

(17)

We see that the value of this parameter depends on whether the matter is cold or hot.

To complete this section, it is useful to recap the analysis and state the result explicitly: The class of exact solutions (1) is very rich algebraically; so we appeal to astrophysics to argue that \( k = 0 \) which gives (12), and to argue that \( \Lambda \) decays faster than \( \rho \) which fixes \( \mu(t) \) and gives (17). The result is a spatially-flat cosmological solution with good physical properties given by

\[
dS^2 = B^2 dt^2 - A^2 (dr^2 + r^2 d\Omega^2) - dy^2
\]

\[
A^2 = \mu^2 (y - Ct)^2 + \frac{K}{\mu^2}
\]

\[
B = \frac{A}{\mu}
\]
\[
\mu(t) = t^n, \quad n = -\frac{1 + 3\gamma}{3(1 + \gamma)}
\]

\[
C = \begin{cases} 
\frac{3}{2}(1 + \gamma) & \text{for } 0 \leq \gamma < \frac{1}{3} \\
\sqrt{4 - K} & \text{for } \gamma = \frac{1}{3}
\end{cases} 
\] (18)

This solution has some features in common with others in the literature and some which are
different. A catalog of exact solutions of the 5D field equations \((R_{AB} = 0)\) has been given by
Wesson (1999), who has also discussed the standard technique wherein these equations are
reduced to Einstein’s \((G_{\alpha\beta} = T_{\alpha\beta})\) to obtain the properties of matter. In this regard, it is
important to note that matter is related to the 4D space and its curvature. Thus the much-
discussed 5D cosmologies due to Ponce de Leon (1988) are flat in 5D but curved in 4D. That
is why they contain matter. However, the Ponce de Leon cosmologies are separable in \(t\) and
\(y\), reducing to FRW ones on the hypersurfaces \(y = \text{constants}\). In contrast, the cosmologies
(1) and their 3D-flat subset (18) are not separable. To distinguish between models of the
Ponce de Leon kind and those discussed here will require new methods which probe the
dynamics associated with the \(y\)-dependence of the solutions. This could involve new tests
of gravity in the solar system, such as the planned Space Test of the Equivalence Principle
(Wesson 1996; Overduin 2000). But tests of 5D models could also be made astrophysically,
and to this end we will examine in the following two sections the implications of (18) for
cold and hot matter.

4. THE COLD 3D-FLAT MODEL

For \(\gamma = 0\) in (10) the equation of state is \(p = 0\) for dust, which is believed to describe
the matter content of the late universe. In this case by (18) we have \(n = -1/3, C = 3/2\) for
the constants and \(\mu(t) = t^{-1/3}\) for the function which describes the scale factor. The latter
is given explicitly by

\[
A^2 = \frac{9}{4}t^{4/3} + Kt^{2/3} - 3yt^{1/3} + y^2t^{-2/3} 
\] (19)

We have examined this numerically for various values of the hypersurfaces \(y = \text{constants}\)
and the three possible values of the normalized 5D curvature parameter \(K\) which appears in
(2), namely \(K = \pm 1, 0\). Note that \(y = 0\) is too special to represent a typical \(y = \text{constant}\)
hypersurface, because in that case the last two terms in (19) do not contribute to the scale
factor \(A\). We find that in most other cases the results are similar. Here we present only
one for the purpose of illustration. Thus Figure 1 is a plot of \(A\) versus \(t\) for \(K = 1\) and
\(y = -1, -2/3, -1/3\), respectively. We see that there is a finite minimum of \(A\) at \(t = t_m\),
before which the universe contracts and after which it expands. This kind of behavior can
happen in 4D FRW models which have a large value of a constant $\Lambda$ (Robertson and Noonan 1968). But most FRW models have a singularity in the geometry and divergent properties of matter because $A(t) \to 0$ for $t \to 0$, defining the big bang. Here the situation is different. By (2), the 5D geometry would be singular if $A(t) \to 0$, but by (19) this would only happen on the special hypersurfaces defined by the roots of that equation. In general, on $y = \text{constant}$ hypersurfaces the 5D curvature invariant (2) is finite. However, at $t = t_m$ we have $\dot{A} = 0$ and $\mu(t) = t^{-1/3} \neq 0$. So by (18) we have $B = 0$ at $t = t_m$. Thus, as discussed in section 2, we get a “big bounce” model in which all $^{(4)}R$, $\rho$ and $\Lambda$ diverge at $t = t_m$ except the 5D invariant (2), which keeps normal.

To investigate this and other aspects of the model, we note from Figure 1 that $t_m$ is of order unity so we can study the cases $t \gg t_m$ and $t \ll t_m$.

For $t \gg t_m$ (19) gives

$$A = \frac{3}{2} t^{2/3} \left[ 1 + \frac{2}{9} K t^{-2/3} - \frac{2}{3} y t^{-1} + O(t^{-4/3}) \right], \quad (20)$$

whose derivative in (18) gives

$$B = \frac{\dot{A}}{\mu} = 1 + \frac{y}{3 t} + O(t^{-4/3}). \quad (21)$$

The 5D line element of (18) now reads

$$dS^2 = [1 + O(t^{-1})] dt^2 - \left[ \frac{3}{2} t^{2/3} + \frac{1}{3} K + O(t^{-1/3}) \right]^2 (dr^2 + r^2 d\Omega^2) - dy^2. \quad (22)$$

This is similar to the 4D Einstein-de Sitter metric, insofar as $t$ tends to the proper time and the scale factor then varies as $t^{2/3}$. Using the constants noted at the beginning of this section and $A$ of (20), we can obtain the density of matter and the cosmological term from (13). They are

$$\rho = \frac{4}{3 t^2} \left[ 1 - \frac{10}{27} K t^{-2/3} + O(t^{-1}) \right]$$

$$\Lambda = -\frac{8 K}{81 t^{8/3}} [1 + O(t^{-1/3})]. \quad (23)$$

From this and (20) we find that

$$\rho A^3 = \frac{9}{2} \left[ 1 + \frac{8}{27} K t^{-2/3} + O(t^{-1}) \right]$$

$$\frac{\partial}{\partial t} (\rho A^3) = \frac{8}{9} K t^{-5/3} [1 + O(t^{-1/3})]. \quad (24)$$
This means that the mass of the (uniform) cosmological fluid within a spherical volume defined by the scale factor changes with time. By comparison with the corresponding situation in general relativity, this may be thought of as being due to the effective pressure associated with the cosmological term (Henriksen, Emslie and Wesson 1983). By (23) and (24), for $K > 0$ we have $\Lambda < 0$ and $(\rho A^3) < 0$, while for $K < 0$ we have $\Lambda > 0$ and $(\rho A^3) > 0$. [If $K = 0$, (23) has to be calculated to higher order.] The generation of mass in quantum field theory has been studied before, particularly in the context of inflationary cosmology (Linde 1990). Here we have a classical analog of that process due to a variable cosmological “constant”, that operates even at late times.

For $t \ll t_m$ (19) gives

$$A = y t^{-1/3} \left[ 1 - \frac{3}{2y} t^2 - \frac{K}{2y^2} t^{4/3} + O \left( t^2 \right) \right], \quad (25)$$

whose derivative in (18) gives

$$B = \frac{\dot{A}}{\mu} = -\frac{y}{3t} \left[ 1 + \frac{3}{y} t - \frac{3K}{2y^2} t^{4/3} + O \left( t^2 \right) \right]. \quad (26)$$

The 5D line element of (18), correct to the leading order in (25), now reads

$$dS^2 = \left( \frac{y}{3t} \right)^2 dt^2 - (yt^{-1/3})^2 \left( dr^2 + r^2 d\Omega^2 \right) - dy^2. \quad (27)$$

If we carry out the coordinate transformation $t = L^3 e^{3\tau/L}$ on this (where $L$ is a constant), it becomes

$$dS^2 = \frac{y^2}{L^2} \left[ d\tau^2 - e^{-2\tau/L} \left( dr^2 + r^2 d\Omega^2 \right) \right] - dy^2. \quad (28)$$

This is a metric of the canonical form (Liu and Mashhoon 1995; Wesson 1999). And the part inside the square brackets is in fact the de Sitter metric, which in 4D would be interpreted as having $\rho = 0$, $\Lambda = 3/L^3$. Here, however, the density and cosmological term are obtained as before by substituting (25) and the other appropriate quantities into (13). They are

$$\rho = \frac{9}{y^3} \left[ 1 - \frac{4K}{9y} t^{1/3} + O \left( t^2 \right) \right],$$

$$\Lambda = \frac{3}{y^2} \left[ 1 + \frac{K}{3y^2} t^{4/3} + O \left( t^2 \right) \right]. \quad (29)$$

These show that $\rho \to 0$ and $\Lambda \to 3/y^2$ for $t \to 0$, so the universe was empty of matter and $\Lambda$-dominated at its start in $t$-time. However, (28) shows that the proper time in 4D is not $t$ but $\tau$, where by our coordinate transformation $t = 0$ corresponds to $\tau = -\infty$. So in $\tau$-time
the universe was empty and existed forever, before its matter was produced around the “big bounce”. The generation of matter by quantum tunneling has been studied before, as a means of creating universes from nothing (Vilenkin 1982). Here we have a classical analog of that process.

5. THE HOT 3D-FLAT MODEL

For $\gamma = 1/3$ in (10) the equation of state is $p = \rho/3$ for radiation or ultrarelativistic particles, which is believed to describe the matter content of the early universe. In this case by (18) we have $n = -1/2$, $C = (4 - K)^{1/2}$ for the constants and $\mu(t) = t^{-1/2}$ for the function that describes the scale factor. The latter is given explicitly by

$$A^2 = t^{-1}(y - Ct)^2 + Kt = 4t - 2Cy + \frac{y^2}{t}. \quad (30)$$

Figure 2 is a plot of $A$ versus $t$ for $K = 1$ and $y = -1, -2/3, -1/3$, respectively. We see that the behavior is similar to that for the $\gamma = 0$ model studied in the preceding section, with a “big bounce” at $t = t_m$.

For $t \gg t_m$ (30) gives

$$A = 2t^{1/2} \left[ 1 - \frac{Cy}{4t} + O(t^{-2}) \right], \quad (31)$$

whose derivative in (18) gives

$$B = \frac{\dot{A}}{\mu} = 1 + \frac{Cy}{4t} + O(t^{-2}). \quad (32)$$

The 5D line element of (18) now reads

$$dS^2 = \left[ 1 + \frac{Cy}{2t} + O(t^{-2}) \right] dt^2 - 4t \left[ 1 - \frac{Cy}{2t} + O(t^{-2}) \right] \left( dr^2 + r^2d\Omega^2 \right) - dy^2. \quad (33)$$

This is similar to the 4D radiation metric, insofar as $t$ tends to the proper time and the scale factor then varies as $t^{1/2}$. We also note that the combination $y/t$ in (33) is self-similar, which is a characteristic of 4D astrophysical systems (Henriksen, Emslie and Wesson 1983). Using the constants noted at the beginning of this section and $A$ of (31), we can obtain the density of matter and the cosmological term from (13). They are

$$\rho = \frac{3}{4t^2} \left[ 1 + \frac{Cy}{4t} + O(t^{-2}) \right]$$

$$\Lambda = \frac{3Cy}{16t^3} \left[ 1 + O(t^{-1}) \right]. \quad (34)$$
¿From this and (31) we find that

$$\rho A^4 = 12 \left[ 1 - \frac{3Cy}{4t} + O(t^{-2}) \right] .$$

(35)

This is similar to the situation in general relativity, where $\rho A^4 = \text{constant}$ because the number density of photons decreases as $A^3$ due to the expansion and their energy decreases as $A$ due to the redshift.

For $t \ll t_m$ (30) gives

$$A = yt^{-1/2} \left[ 1 - \frac{Cy}{y} t + O(t^2) \right] ,$$

(36)

whose derivative in (18) gives

$$B = \frac{\dot{A}}{\mu} = -\frac{y}{2t} \left[ 1 + \frac{Cy}{y} t + O(t^2) \right] .$$

(37)

The 5D line element of (18), correct to the leading order in (36), now reads

$$dS^2 = \left( \frac{y}{2t} \right)^2 dt^2 - y^2 t^{-1} (dr^2 + r^2 d\Omega^2) - dy^2 .$$

(38)

If we carry out the coordinate transformation $t = \mathcal{L}^2 e^{2\tau/\mathcal{L}}$ on this (where $\mathcal{L}$ is a constant), it becomes

$$dS^2 = \frac{y^2}{\mathcal{L}^2} \left[ d\tau^2 - e^{-2\tau/\mathcal{L}} \left( dr^2 + r^2 d\Omega^2 \right) \right] - dy^2 .$$

(39)

This is a metric of the canonical form, and in fact the same as (28). The density and cosmological term are again obtained by substituting (36) and the other appropriate quantities into (13). They are

$$\rho = \frac{3C}{y^3} t \left[ 1 + O(t) \right]$$

$$\Lambda = \frac{3}{y^2} \left[ 1 + \frac{C}{y} t + O(t^2) \right] .$$

(40)

These show that $\rho \rightarrow 0$ and $\Lambda \rightarrow 3/y^2$ for $t \rightarrow 0$, which is the same as in the $\gamma = 0$ case considered in the preceding section. [However, (29) and (40) are different for finite $t$..]

Similar comments apply here as there concerning the generation of matter around the “big bounce”.
6. CONCLUSION

We have given a class (1) of exact solutions of the 5D field equations which extends the class of 4D Friedmann-Robertson-Walker solutions. These solutions, unlike others in the literature including the Ponce de Leon cosmologies, are not separable. But as with other 5D solutions, their matter content can be obtained by a standard technique, which gives the density, pressure and cosmological “constant” (9). The last, however, is really a measure of the variable energy density of the vacuum. This generalizes the vacuum of general relativity, and is in conformity with other higher-dimensional extensions of that theory such as strings and membranes. The class of solutions we have discussed is algebraically rich, so we have used input from cosmology to study its properties. The subclass with spatially-flat geometry (12) yields a cold model (19) which is the analog of the 4D dust one, and a hot model (30) which is the analog of the 4D one for radiation or ultrarelativistic particles. Both models have a cosmological “constant” that decays faster than the density decreases, in accordance with astrophysical data including the age of the universe. Note that our cosmological “constant” do not include those parts of the conventional Λ which evolve like an ordinary matter and can be absorbed into ρ and p. Also, both models in general show a big “bounce” as opposed to a big bang, where the 5D geometry is non-singular but where the matter properties and the 4D scalar curvature diverge. We therefore recover the physics of the late and early universe and the fireball, but without a breakdown in the geometry. In view of the major implications of a bounce as opposed to a bang, we have examined the behavior of the models around that event in some detail. The main feature is that matter is created during the bounce, in agreement with models of inflation based on quantum field theory.

It is known that a major difficulty of the 4D big bounce models in standard general relativity (Robertson and Noonan 1968) is that the collapsing phase would typically generate enormous inhomogeneities. Here we wish to emphasize that our 5D big bounce models are different in many aspects from the conventional 4D big bounce models. Firstly, our models are not symmetric before and after the big bounce. Secondly, in our models, the contraction and expansion of the universe around the bounce was accompanied by matter creation. If the process of the contraction of the universe before the bounce was dominated not by collapsing of previously formed matter but by creating of new matter, (this is very possible because in our models the universe was empty at the beginning of the contracting,) then hopefully the inhomogeneity problem could be resolved.

We are aware that the present work is exploratory. While our solutions are quite general, we have restricted their application by assuming that ordinary matter and the vacuum can be described by a perfect fluid, by adopting an isothermal equation of state for the matter, by concentrating on the spatially flat case, and that only for dust and radiation. All of these
restrictions could be removed in future work. However, we have shown that adding only one extra dimension to general relativity makes the big bang a subject of fruitful analysis and not just a no-go singularity.

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REFERENCES


Fig. 1.— Evolution of the scale factor $A$ for the cold 3D-flat model with $K = 1$ and $y = -1, -2/3, -1/3$. 
Fig. 2.— Evolution of the scale factor $A$ for the hot 3D-flat model with $K = 1$ and $y = -1, -2/3, -1/3$. 