The Structure of Screening in QED

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Abstract: The possibility of constructing charged particles in
gauge theories has long been the subject of debate. In the con-
text of QED we have shown how to construct operators which
have a particle description. In this paper we further support
this programme by showing how the screening interactions arise
between these charges. Unexpectedly we see that there are two
different gauge invariant contributions with opposite signs. Their
difference gives the expected result.

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Introduction

The long range nature of the electromagnetic interaction means that the QED coupling cannot be naively switched off. Neglecting this leads to the infra-red problem and the lack of a pole structure in the on-shell Green’s functions and $S$-matrix. This has been taken [1] to mean that one cannot describe charged particles in gauge theories. In a series of papers [2–6] we have shown that this conclusion is overly hasty: it is in fact possible to construct gauge invariant operators whose $S$-matrix elements are free of infra-red divergences. These fields have been shown to asymptotically recover a particle description of charges and to have a rich structure which is physically reflected in the cancellation of both soft and phase divergences. In QCD we have demonstrated [7,8] (in both $2 + 1$ and $3 + 1$ dimensions) that the term responsible for the cancellation of soft divergences generates the anti-screening forces between static quarks. We have then suggested that the factor responsible for cancelling phase divergences must generate the screening interaction. Here we will show that this is indeed the case.

The Structure of Static Charges

Many years ago Dirac [9] suggested that a static charged particle should be described by the locally gauge invariant operator

$$\psi_D(x) \equiv \exp \left(-ie \frac{\partial_i A_i}{\nabla^2}(x)\right) \psi(x) . \quad (1)$$

His argument for this was that, in addition to the essential requirement of gauge invariance, it has the expected equal-time commutator with the electric field operator

$$[E_i(x), \psi_D(y)] = -\frac{e}{4\pi |x-y|} \psi_D(y) , \quad (2)$$

i.e., it recovers the static Coulombic electric field in $3 + 1$ dimensions. This argument also works in $2 + 1$ dimensions.

In [3] we have shown that this electric field requirement is not unique even at lowest order in the coupling. In fact, arguing from a general kinematical point of view (inspired by the heavy quark effective theory), we have shown that the correct description of a static charge is given by the dressed field

$$h^{-1}(x)\psi(x) = e^{-ieK(x)} e^{-ie\chi(x)} \psi(x)$$

$$= \exp \left(-ie \int_{-\infty}^{x_0} \frac{\partial_i E_i(s, x)}{\nabla^2} ds\right) \exp \left(-ie \frac{\partial_i A_i}{\nabla^2}(x)\right) \psi(x) . \quad (3)$$
The new factor is separately gauge invariant and has a vanishing commutator with the electric field in the absence of light charges. We have shown that it (and its generalisation for a moving charge) is essential in the cancellation of the phase divergences associated with pair production processes.

In order to show that this provides the correct dynamical description of physical charges, we have investigated the potential between them. In the non-abelian theory we have demonstrated \[8,7\] that the generalisation to QCD of Dirac’s proposal, \textit{i.e.}, just the minimal part of the dressing, produces the anti-screening interaction at order \(g^4\). We will now show that the new factor in (3) produces the screening effects at the same order of perturbation theory.

**The Potential Between Charges**

As usual we identify \[2,8,7\] the potential with the separation dependent part of the matrix element

\[
\langle 0| h(y')h^{-1}(y) H_0 h(y)h^{-1}(y')|0 \rangle. \tag{4}
\]

For the purposes of this paper we can neglect higher terms in the expansion of the dressing and simply write

\[
h^{-1}(y) = 1 - ie(K(y) + \chi(y)). \tag{5}
\]

Following our discussion above, we will refer to the \(K\) term as the \textit{phase} contribution and \(\chi\) as the \textit{soft} structure.

The relevant part of the free Hamiltonian in \(d\) spatial dimensions is

\[
H_0 = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \tilde{E}^i(p,x_0) \tilde{E}^i(-p,x_0) \tag{6}
\]

\[
+ \int \frac{d^d p}{(2\pi)^d} E_p \left( \tilde{\psi}^+_{-}(\mathbf{p},x_0) \tilde{\psi}^+_+(\mathbf{p},x_0) + \tilde{\psi}^-_-(\mathbf{p},x_0) \tilde{\psi}^+_+(\mathbf{p},x_0) \right), \tag{7}
\]

where we have dropped the irrelevant magnetic part of the Hamiltonian and the terms involving the static charges. In the second term here only light fermions are included and our positive and negative frequency decomposition is defined by

\[
\tilde{\psi}(\mathbf{p},x_0) = \frac{1}{\sqrt{2E_p}} \left( b^{\alpha}(\mathbf{p}) u^{\alpha}(\mathbf{p}) e^{-iE_p x_0} + d^{\alpha\dagger}(\mathbf{p}) \psi\alpha(-\mathbf{p}) e^{iE_p x_0} \right)
\]

\[
= \tilde{\psi}^+_+(\mathbf{p},x_0) + \tilde{\psi}^-_-(\mathbf{p},x_0). \tag{8}
\]
We recall that Gauss’ law in momentum space reads:

\[ p_i \tilde{E}_i(p, x_0) = ie \int \frac{d^d q}{(2\pi)^d} \tilde{\psi}^\dagger(q, x_0) \tilde{\psi}(p - q, x_0), \]  

(9)

where again, we neglect the heavy, static charges.

**Lowest Order**

It is easy to see that at lowest order the momentum space contribution comes from the commutator of the Hamiltonian with the soft terms in the dressings:

\[ \tilde{V}(q, k) = e^2 \int \frac{d^d p}{(2\pi)^d} \langle 0 | [\tilde{E}_i(p, x_0), \tilde{\chi}(q, x_0)] [\tilde{E}_i(-p, x_0), \tilde{\chi}(k, x_0)] | 0 \rangle \]

\[ = -(2\pi)^d e^2 \delta(q + k) \frac{1}{q^2}. \]

(10)

Performing the \( k \) integral recovers the usual result

\[ \tilde{V}(q) = -e^2 \frac{1}{q^2} = -\frac{4\pi \alpha}{q^2}. \]

(11)

Note that this gives the correct \( d \)-dimensional configuration space Coulombic potential between heavy charges at a separation \( r \)

\[ V(r) = -e^2 \frac{\Gamma\left(\frac{d}{2} - 1\right)}{4\pi \frac{d}{2}} \frac{1}{|r|^{d-2}}. \]

(12)

The extension of this soft-soft contribution to the non-abelian theory gives anti-screening.

In the absence of light charges (12) is the full result in QED and it is easy to see that there is no contribution from the phase dressing. The presence of light fermions, however, modifies the potential and we expect a screening effect. We now want to show that our full dressing generates this screening force. There are two contributions to this and we will analyse them in turn.

**Phase-Phase Contribution**

The first term we want to calculate comes from the phase-phase analogue of the soft-soft structure in (10)

\[ \tilde{V}_{pp}(q, k) = 2e^2 \int \frac{d^d p}{(2\pi)^d} E_p \left\{ \text{tr} \langle 0 | [\tilde{\psi}^\dagger(-p, x_0), \tilde{K}(q, x_0)][\tilde{\psi}(p, x_0), \tilde{K}(k, x_0)] | 0 \rangle \right. 

\[ + \text{tr} \langle 0 | [\tilde{\psi}(-p, x_0), \tilde{K}(q, x_0)][\tilde{\psi}^\dagger(-p, x_0), \tilde{K}(k, x_0)] | 0 \rangle \left\}. \]  

(13)
After using Gauss’ law (9) to rewrite the phase dressing as

$$\tilde{K}(p, x_0) = -e \int \frac{d^d q}{(2\pi)^d} \int_{x_0}^{x_0} ds \frac{1}{p^2} \tilde{\psi}^\dagger(q, s) \tilde{\psi}(p - q, s),$$

we get

$$\tilde{V}_{pp}(q, k) = -2(2\pi)^d e^4 \frac{1}{q^2} \delta(q + k) \int \frac{d^d p}{(2\pi)^d} \frac{E_p}{(E_p + E_{q-p})^2}$$

\[\text{tr} \left( \mathcal{P}_-(q - p) \mathcal{P}_+(p) + \mathcal{P}_+(q - p) \mathcal{P}_-(p) \right), \tag{15}\]

where $\mathcal{P}_\pm(p) = (\phi \pm m)\gamma^0/2E_p$ are the projectors onto positive/negative frequencies.

From the result that

$$\text{tr} \left( \mathcal{P}_-(p) \mathcal{P}_+(q) \right) = \frac{(d + 1)n_f}{2E_p 2E_q} (E_p E_q + p \cdot q - m^2),$$

where $n_f$ is the number of light fermion species, we can trivially integrate out $k$ to obtain the phase-phase contribution to the potential at $e^4$

$$\tilde{V}_{pp}(q) = -e^4(d + 1)n_f \frac{1}{q^2} \int \frac{d^d p}{(2\pi)^d} \frac{E_p E_{q-p} + p \cdot (q - p) - m^2}{E_{q-p}(E_p + E_{q-p})^2}. \tag{17}\]

Expanding around large $p$ here gives the following divergent correction in $d = 3 - 2\epsilon$ dimensions

$$\tilde{V}_{pp}(q) = -\frac{4\pi\alpha_0 \alpha_0 n_f}{q^2} \frac{1}{3} \left[ \frac{1}{\epsilon} - \ln \left( \frac{q^2}{\mu^2} \right) \right]. \tag{18}\]

The sign here, however, corresponds to anti-screening!

**Screening**

In addition to this phase-phase contribution, there are also two identical soft-phase cross-terms. These structures yield

$$\tilde{V}_{sp}(q, k) = 2e^2 \int \frac{d^d p}{(2\pi)^d} \text{tr} \left\langle 0 | [\tilde{E}_i(p, x_0), \chi(q, x_0)] [\tilde{E}_i(-p, x_0), \tilde{K}(k, x_0)] | 0 \right\rangle. \tag{19}\]

Using Gauss’ law (9) this becomes

$$\tilde{V}_{sp}(q, k) = 2ie^4 \frac{1}{q^2} \frac{1}{k^2} \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d p'}{(2\pi)^d} \int_{-\infty}^{x_0} ds \text{tr} \left\langle 0 | [\tilde{\psi}^\dagger(p, x_0) \tilde{\psi}(p - q, x_0), \tilde{\psi}^\dagger(p', s) \tilde{\psi}(k - p', s)] | 0 \right\rangle. \tag{20}\]
After a little algebra, we obtain
\[ \tilde{V}_{sp}(q, k) = 2(2\pi)^d e^4 \frac{1}{q^4} \delta(q + k) \int \frac{d^d p}{(2\pi)^d} \frac{1}{E_p + E_{q-p}} \times \text{tr} \left( P_-(p) P_+(q - p) + P_-(p) P_+(p - q) \right). \] (21)

This is then
\[ \tilde{V}_{sp}(q, k) = e^4(2\pi)^d (d + 1) n_t \frac{1}{q^4} \delta(q + k) \int \frac{d^d p}{(2\pi)^d} \frac{E_p E_{q-p} + p \cdot (q - p) - m^2}{E_p E_{q-p} (E_p + E_{q-p})}. \] (22)

Note that this is almost identical to the phase-phase term. The only difference being the overall sign and the denominator term in the momentum integral. In \( d = 3 - 2\epsilon \) dimensions, we find
\[ \tilde{V}_{sp}(q) = 4\pi\alpha_0 \frac{2\alpha_0 n_t}{q^2} \frac{1}{\pi} \left[ \frac{1}{\epsilon} - \ln \left( \frac{q^2}{\mu^2} \right) \right]. \] (23)

Adding this to (18) we obtain the total (divergent) contribution
\[ \tilde{V}(q) = -4\pi\alpha_0 \frac{n_t}{q^2} \left\{ 1 - \frac{\alpha_0 n_t}{\pi} \left[ \frac{1}{\epsilon} - \ln \left( \frac{q^2}{\mu^2} \right) \right] \right\}. \] (24)

up to order \( \alpha^2 \). Charge renormalisation in QED corresponds to \( \alpha_0 = Z\alpha \), where \( Z\alpha = 1 + \frac{\alpha \pi}{\pi - \frac{1}{\epsilon}} \). We thus see that the divergences cancel as expected and we obtain the usual screening result. We have thus shown at next to leading order that the structures of the physical dressing generate the interaction between charges.

**Conclusion**

We have seen that the physical description of a charge which we propose indeed contains the effects which screen static charges. In a concrete calculation we have seen that the overall screening forces between such charges arise from two distinct, gauge invariant contributions. One has an anti-screening effect, but it is only half the size of the dominant screening term. This separation is not apparent in other methods (such as Wilson loops [10,11] and non-relativistic perturbation theory [12,13]) and it is intriguing to speculate on similar structures in the gluonic screening of QCD.

This result is a further vindication of our approach to the fundamental question of how to describe charged particles in gauge theories. We have seen that, from general principles, the dressing around a charge has a rich...
structure which is reflected in the infra-red properties of the fields and in the forces between charges. This shows a previously unobserved intimate connection between the soft structures ($\chi$) of gauge theories and anti-screening and also between the phase structures ($K$) and the overall screening effect.

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