Pseudo-Hermiticity of Hamiltonians under imaginary shift of the co-ordinate: real spectrum of complex potentials

Zafar Ahmed
Nuclear Physics Division, Bhabha Atomic Research Centre
Trombay, Bombay 400 085, India
zahmed@apsara.barc.ernet.in
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Abstract

We propose that the real spectrum and the orthogonality of the states for several known complex potentials of both types, \( \mathcal{PT} \)-symmetric and non-\( \mathcal{PT} \)-symmetric can be understood in terms of currently proposed \( \eta \)-pseudo-Hermiticity (Mostafazadeh, quant-ph/0107001) of a Hamiltonian, provided the Hermitian linear automorphism, \( \eta \), is introduced as \( e^{-\theta p} \) which affects an imaginary shift of the co-ordinate:

\[
e^{-\theta p} x e^{\theta p} = x + i \theta.
\]

Until year 1998 [1], Hermiticity of the Hamiltonian was supposed to be the necessary condition for having real spectrum. A conjecture due to Bender and Boettcher [1], has relaxed this condition in a very inspiring way by introducing the concept of \( \mathcal{PT} \)-symmetric Hamiltonians. Here, \( \mathcal{P} \) denotes the parity operation (space reflection) : \( x \rightarrow -x \) and \( \mathcal{T} \) mimics the time-reversal : \( i \rightarrow -i \). Let \( \chi \) denote \( \mathcal{PT} \) then if (i)- \( \chi H \chi^{-1} = H \) and if (ii) - \( \chi \Psi(x) = \mp 1 \Psi(x) \) the eigenvalues will be real and complex conjugate pairs if the latter condition is not satisfied. Several \( \mathcal{PT} \)-symmetric potentials have been witnessed to hold real discrete spectrum [1-10]. A fully exactly solvable \( \mathcal{PT} \)-symmetric potential model containing both the scenarios (real and complex conjugate pairs of eigenvalues) is available [7]. Orthogonality conditions for the eigenstates of such potentials have been proposed [7,11,12]. Recent conceptual developments can be seen in Refs. [12-14].

However, a non-\( \mathcal{PT} \)-symmetric complex Morse potential,

\[
V(x) = (A + iB)^2 \exp(-2x) - (2C + 1)(A + iB) \exp(-x),
\]

(1)
which holds real discrete spectrum [10] remains a well known exception in this regard. This complex potential (1) being exactly solvable, it becomes even more interesting to find the underlying structure responsible for the real spectrum, for a possible new direction. The potentials discussed so far [1-10] can mainly be categorized as of two types: \( V_I(x) = \alpha V_e(x) + i\beta V_o(x) \) (e: even, o: odd function) and \( V_{II}(x) = \alpha V_e(x - i\gamma) + i\beta V_o(x - i\gamma) \) [8-10]. Let us write a new category of potentials as \( V_{III}(x) = \alpha V(x - \beta - i\gamma) \), where \( V(x) \) need not have a definite parity. Such potentials are non-\( \mathcal{PT} \)-symmetric, a few potentials of this category have been generated [10] by group theoretic techniques and found to have real spectrum. The complex Morse potential (1) is one such example. In these expressions the parameters \( \alpha, \beta \) and \( \gamma \) are assumed to be real.

Currently, in a very interesting work [15], Mostafazadeh points out that the potentials of the type (I,II) are actually \( \mathcal{P} \)-pseudo-Hermitian and claims that the \( \eta \)-pseudo-Hermiticity

\[
\eta H \eta^{-1} = H^\dagger, \tag{2}
\]

is the necessary condition for having real spectrum, where \( \eta \) has been referred to as Hermitian linear automorphism. Let \( \mathcal{V} \) be an inner product space, for two arbitrary elements \( u \) and \( v \), \( \eta \) satisfies

\[
<\eta u|v> = <u|\eta v>. \tag{3}
\]

In this Letter, we propose the imaginary shift of the co-ordinate

\[
e^{-\theta p}xe^{\theta p} = x + i\theta, \tag{4}
\]

as a Hermitian linear automorphism, where \( \theta \) is real and \( p = -i\frac{d}{dx}, (\hbar = 1) \).

**Proof** :

\[
x e^{\theta p} - e^{\theta p}x = [x, e^{\theta p}]
= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} [x, p^n]
= \sum_{n=0}^{\infty} \frac{i\theta^n p^{n-1}}{n!}
= i\theta e^{\theta p}. \tag{5}
\]

Multiplying from left on both sides by \( e^{-\theta p} \), we prove the proposition (4). The linear operator \( e^{-\theta p} \) being Hermitian the condition (3) gets automatically satisfied as \( (e^{-\theta p} \psi(x))^\dagger = \psi(x) \).
$\Psi^*(x) e^{-\theta p}$. Here, it is worthwhile to recall that the usual linear operator, $e^{-ip}$, affecting a real shift in $x$ is unitary and not Hermitian. The operator $e^{-\theta p}$ is endowed with the following important properties:

A constant scalar $c$ (real or complex) commutes with $p$, so we have

\[ e^{-\theta p} c e^{\theta p} = c. \tag{6a} \]

The linear momentum $p$ commutes with $e^{-\theta p}$ to remain unaffected under imaginary shift of the co-ordinate. So we have

\[ e^{-\theta p} p e^{\theta p} = p. \tag{6b} \]

By nothing that $e^{-\theta p} x^2 e^{\theta p} = e^{-\theta p} x e^{\theta p} e^{-\theta p} x e^{\theta p} = (x + i\theta)^2$, the method of induction leads to

\[ e^{-\theta p} x^n e^{\theta p} = (x + i\theta)^n. \tag{6c} \]

A further generalization is by noting that the potential operator $V(x)$ can be expanded in the powers of $x$, to write

\[ e^{-\theta p} V(x) e^{\theta p} = V(x + i\theta). \tag{6d} \]

Reminding us of the Taylor series expansion, the operator $e^{-\theta p}$ would act on a wavefunction as

\[ e^{-\theta p} \Psi(x) = \Psi(x + i\theta). \tag{6e} \]

The abovementioned $\mathcal{PT}$-symmetric potentials of the types $V_I(x)$ [1-7] and $V_{II}(x)$ [8-10] have been discussed as being $\mathcal{P}$-pseudo-symmetric [15]. Nevertheless, the complex Morse potential (1) and the potentials of the types $V_{III}(x)$ [10], despite entailing real spectrum require a Hermitian linear automorphism for pseudo-Hermiticity to be invoked. Having introduced the properties of $e^{-\theta p}$, we claim it to be the required Hermitian linear automorphism for this exceptional class of the potentials. Notice that the potential in Eq. (1) satisfies

\[ e^{-\theta p} V(x) e^{\theta p} = V(x + i\theta) = V^*(x), \quad \theta = [2 \tan^{-1}(B/A)], \tag{7} \]

rendering the Hamiltonian, $H = p^2 + V(x)$, as pseudo-Hermitian (see Eq. [2]) under the imaginary shift of the co-ordinate. The real spectrum of the potential (1) has been predicted by a group theoretic technique [10].
In the following, we propose to investigate the complex Morse potential (1) in the light of pseudo-Hermiticity under imaginary shift of the co-ordinate as an illustrative example. To this end, it is worthwhile to solve the Schrödinger equation for a general Morse potential

\[ V(x) = V_1 \exp(-2x) - V_2 \exp(-x) \]  

using the first principles. By assuming \( \hbar = 1 = 2m \), we write the Schrödinger equation for (8)

\[ \frac{d^2 \Psi(x)}{dx^2} + [E - V_1 \exp(-2x) + V_2 \exp(-x)] \Psi(x) = 0. \]  

Next, a change of variable, namely \( z = 2\sqrt{V_1} \exp(-x) \) leads to

\[ z^2 \frac{d^2 \Psi(x)}{dz^2} + z \frac{d \Psi(z)}{dz} + [E - \frac{z^2}{4} + \frac{V_2}{2\sqrt{V_1}}] \Psi(x) = 0 \]

an important form which clearly indicates that the energy eigenvalues for a general Morse potential will be a function of the effective parameter \( \frac{V_2}{2\sqrt{V_1}} \). Further, the fact that for the complex Morse potential (1) this effective parameter is real \( (= C + 1/2) \) explains the real spectrum to be found in the sequel. Next by using a transformation, \( \Psi(z) = z^\alpha \exp(-z/2) F(z) \), in Eq. (10) we get the confluent hypergeometric equation [16] as

\[ z F''(z) + (2\alpha + 1 - z) F'(z) - (\alpha - C) F(z) = 0, \quad \alpha = \sqrt{-E}. \]

We can write the solution of Eq. (10) in terms confluent hypergeometric function : \( \Psi(x) = z^\alpha e^{-z/2} \, _1F_1[\alpha - C; 2\alpha + 1; z] \). The condition that the series \( _1F_1 \) be truncated and become a polynomial gives the quantization condition, \( \alpha - C = -n, n = 0, 1, 2.. < C \), which in turn gives energy eigenvalues as

\[ E_n = -(n - C)^2, \quad n = 0, 1, 2,... < C = \frac{V_2}{2\sqrt{V_1}} - 1. \]

It can also be checked that \( \Psi_n(\pm \infty) = 0 \), as long as \( n < C \). Finally, the eigenfunctions can be re-written in terms of associated Laguerre polynomials as

\[ \Psi_n(x) = z^{C-n} e^{-z/2} \, L_n^{2C-2n}(z), \quad z = 2\sqrt{V_1} \exp(-x). \]

The Schrödinger equation (10) also being a Strum-Liouville equation the orthogonality condition, \( \int_{-\infty}^{\infty} \Psi_m(x) \, \Psi_n(x) dx = N_n^2 \delta_{m,n} \), ought to be satisfied. However, surprisingly, in the mathematical handbooks [16,17], we do not find the interesting ensuing integral, namely
\[ \int_{0}^{\infty} z^{2c-(m+n+1)} e^{-z} L_{m}^{2c-2m}(z) L_{n}^{2c-2n}(z) \, dz = N_{n}^{2} \delta_{m,n}. \]  

(14)

We have, therefore, checked the result (14) using \text{NIntegrate} of Mathematica for several cases. We think it remains desirable to prove the result in Eq. (14) analytically.

Having confirmed the real spectrum of the pseudo-Hermitian potential (1) under the imaginary shift of the co-ordinate (4), we now show that it conforms to the proposed definition of \( \eta \)-orthogonality [15], namely

\[ (E_{1}^{*} - E_{2}) \int_{-\infty}^{\infty} \Psi_{1}^{*}(x) \eta \Psi_{2}(x) \, dx = 0. \]  

(15)

The \( \mathcal{PT} \)-orthogonality has earlier [7,12] been proposed as

\[ (E_{1}^{*} - E_{2}) \int_{-\infty}^{\infty} \Psi_{1}^{PT}(x) \Psi_{2}(x) \, dx = 0, \]  

(16)

which is not different from (15) in case the Hamiltonian is \( \mathcal{P} \)-pseudo-Hermitian. Let us notice the operation of imaginary shift of the co-ordinate on \( z(x) = 2(A + iB) e^{-x} \) using Eq. (6d) for the special case of complex Morse potential (1) as

\[ e^{-\theta p} z(x) e^{\theta p} = z(x + i\theta) = 2(A + iB) e^{-(x+i\theta)} = 2(A + iB) \frac{A - iB}{A + iB} e^{-x} = z^{*}(x). \]  

(17)

Similarly, using the property in Eq. (6e) for an arbitrary complex eigenstate from Eq. (10), we find that \( e^{-\theta p} \Psi_{n}(z) = \Psi_{n}(z^{*}) = \Psi_{n}^{*}(x) \) to display the orthogonality effectively as

\[ \int_{-\infty}^{\infty} \Psi_{m}^{*}(x) \Psi_{n}^{*}(x) \, dx = N_{n}^{2} \delta_{m,n}, \]  

in conformity with the condition (15) and the mathematical result (14).

Interestingly, the potentials, \( V_{I}(x) \) [1-7] and \( V_{II}(x) \) [8-10] are both \( \mathcal{PT} \)-symmetric and \( \mathcal{P} \)-Pseudo-Hermitian as well. However, we assert that the \( III \) category of potentials, \( V_{III}(x) = \alpha V(x - \beta - i\gamma) \) which are non-\( \mathcal{PT} \)-symmetric are only pseudo-Hermitian under the imaginary shift of the co-ordinate \( (x \rightarrow x + 2i\gamma) \):

\[ e^{-2\gamma p} \alpha V(x - \beta - i\gamma) e^{2\gamma p} = \alpha V(x + 2i\gamma - \beta - i\gamma) = \alpha V^{*}(x - \beta - i\gamma). \]  

(18)

The wavefunctions will be such that \( e^{-2\gamma p} \Psi_{n}(x - \beta - i\gamma) = \Psi(x - \beta + i\gamma) \) to satisfy the orthogonality condition (15), given the fact that the states, \( \Psi_{1}(x) \) and \( \Psi_{2}(x) \), of the real potential, \( V(x) \), are orthogonal in \([-\infty, \infty]\). We point out that in Ref. [6], the potential, \( V(x) = [z \cosh(2x) - iM]^{2} \) is actually pseudo-Hermitian under \( x \rightarrow x + i\pi/2 \). If, \( \hbar = 1 = m \), three Harmonic-oscillator potentials : \( V_{1}^{HO}(x) = \frac{1}{2} x^{2}, V_{2}^{HO}(x) = \frac{1}{2}(x - i\gamma)^{2} \) and
\[ V_3^{HO}(x) = \frac{1}{2} (x - \beta - i\gamma)^2 \] will have the same real spectrum, \( E_n = n + 1/2 \), with different sets of eigenfunctions. The potential \( V_1^{HO} \) is Hermitian, \( V_2^{HO} \) is both \( \mathcal{PT} \)-symmetric and pseudo-Hermitian as well, and \( V_3^{HO} \) is only pseudo-Hermitian. Both the times the pseudo-Hermiticity is under the imaginary shift of the co-ordinate (4).

When \( \hbar = 1 = 2m \), three Eckart potentials \( V_1^E(x) = -\alpha \text{sech}^2(x), V_2^E(x) = -\alpha \text{sech}^2(x - i\gamma) \) and \( V_3^E(x) = -\alpha \text{sech}^2(x - \beta - i\gamma) \), will all have a common real spectrum, \( E_n = -[n + 1/2 - \sqrt{\alpha + 1/4}]^2, n = 0, 1, 2... < \sqrt{\alpha + 1/4} \) with different sets of eigenstates. Once again, \( V_1^E \) is Hermitian, \( V_2^E \) is both \( \mathcal{PT} \)-symmetric and pseudo-Hermitian and \( V_3^E \) is only pseudo-Hermitian. Once again, both the times the pseudo-Hermiticity is under imaginary shift of the co-ordinate (4). Apparently similar looking potentials \( V_2^E \) and \( V_3^E \) can be made to look entirely different from \( V_1^E(x) \), in order to visualize and analyze them like their partner \( V_1^E(x) \), if their real and imaginary parts are separated out.

Further, we can cite the exactly solvable well known potentials Pöschal-Teller-II and generalized Pöschal-Teller [9,10] which can contribute the pseudo-Hermitian potentials in the proposed sense (4). They are \( V^{PT-II}(x - \beta - i\gamma) \) and \( V^{GPT}(x - \beta - i\gamma) \); they will have a common real spectrum as that of \( V^{PT-II}(x) \) and \( V^{GPT}(x) \), respectively.

The formalism [15] does not claim the \( \eta \)-pseudo-Hermiticity to be sufficient, however, it has been claimed to be the necessary condition on the complex potential for having the real spectrum. We remark that the latter issue is not without a practical difficulty. The main problem in this regard is to actually identify the Hermitian linear automorphism, \( \eta \). In some instances it might be apparent, in others it might even be elusive for the pseudo-Hermiticity to be talked about. We feel that the identification of the imaginary shift of the co-ordinate, \( e^{-\theta \rho} \) to act as \( \eta \) is the main contribution of the present Letter. In this light, several complex potentials of both the types, \( \mathcal{PT} \)-symmetric and non-\( \mathcal{PT} \)-symmetric, entailing real spectrum have been argued to be pseudo-Hermitian. The complex Morse potential turns out to be a bit more interesting instance of pseudo-Hermiticity under imaginary shift of the co-ordinate. Further, introduction or identification of a new Hermitian linear automorphism, \( \eta \), will help in generating a new class of non-Hermitian Hamiltonians having real spectrum.

REFERENCES:


