On the Squeezed Number States and their Phase Space Representations

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Abstract

We compute the photon number distribution, the \(Q(\alpha)\) distribution function and the wave functions in the momentum and position representation for a single mode squeezed number state. We discuss the oscillations which appear in the photon number distribution of squeezed number states for high values of the squeezing parameter. We compare our results with the formalism based on the interference in phase space.
1 Introduction

The study and experimental detection of squeezed states \cite{1,2,3,4} and other non classical states of optical systems, is an interesting issue, from both the fundamental and the technological point of view. On one hand, disentangling the properties of intrinsically quantum states of light enhances our understanding of the behavior and interactions of photons, and improves our knowledge of Quantum Electrodynamics. On the other hand, the use of light states with reduced quantum fluctuations in one of the conjugate quadratures is at the heart of attractive proposals for the detection of very weak signals\cite{5,6}.

In this article we are concerned with the study of the squeezed number states and their phase space representations. These states emerged in the original analysis of Yuen \cite{1} of the Two Photon Coherent States and have received since then a moderate amount of attention in the literature \cite{7,8,9}. As we will show, they are particularly suitable for discussing the departure from the semiclassical regime in the context of quantum optics and for this reason were chosen for this study, but they also may have a practical interest of their own \cite{10}.

The one mode squeezed number states are obtained by applying the squeeze operator to the Fock states. As we discuss below, their wave functions in the position and momentum like quadratures are obtained by literally compressing or stretching the corresponding wave functions of the Fock states. An important aspect which holds our attention through this work is that they present oscillations in the photon number distribution which have been reported in Refs. \cite{7,8}. These oscillations were first observed in the photon number distributions of squeezed states \cite{11} which are equivalent to the Two-Photon Coherent States, and present several interesting properties.

In a series of papers, \cite{11,12,13,14,15}, Wheeler et al. explain these oscillations as the result of the interference of different contributions to probability amplitudes. These contributions were interpreted as the overlapping of the regions assigned to each quantum state on the phase space. For the squeezed number states oscillations, a closely related discussion was sketched in \cite{7,8}. This treatment is physically very appealing, but is based on the semiclassical representation of states as bounded areas in phase space, a picture which is justified only when the Wigner distributions of the states \cite{16} are well localized and structureless.

To establish the area of overlapping interference hypothesis as a general rule it would be necessary to develop a case independent method to assign a specific shape to the representation of the state in phase space. In Ref. \cite{14} for the case of the squeezed states this area was chosen by weighting the Bohr-Sommerfeld representation with the Wigner-Cohen \cite{17} distribution function. The same prescription was followed in Ref. \cite{7} for the squeezed number states. To work a more systematic approach to the area of overlapping interference hypothesis it seems natural to study other phase space distributions. For this reason, in this paper we study the squeezed number states using the $Q(\alpha)$ distribution to characterize the phase space properties of the states. The $Q(\alpha)$ distribution presents the advantage of being a true probability density, which
can be interpreted as the probability of a simultaneous measurement of position and momentum within some region of phase space [18]. For a Fock state, the $Q(\alpha)$ distribution is localized on a circular ring, a fact which allows one to make contact with the semiclassical representation.

To compare with the area of overlapping approach we compute explicitly the probability amplitudes and distributions of the squeezed number states in the Fock, position and momentum representations and then we argue that a complete description of the photon number oscillations should go beyond the interference of areas of overlapping hypothesis of Wheeler et. al.

Our computations are done using a generating function technique which can be applied quite generally to compute expectation values of arbitrary observables in different representations. In particular we compute the $Q(\alpha)$ function for a squeezed number state and show that for high squeezing, it presents intrinsic oscillations in the space of the complex labels $\alpha$ which can be correlated with those in the momentum and Fock representations. This result challenges the simple view of the oscillations as the phase space interference and prompt us to discuss its validity.

2 Two photon coherent states and squeezed number states

We consider here a single mode of the radiation field described in terms of the creation and annihilation operators $a^\dagger$, $a$, the number operator $\hat{N}_a = a^\dagger a$, and the momentum-like and position-like quadratures $\hat{p} = \frac{1}{\sqrt{2}}(a - a^\dagger)$ and $\hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger)$. Number (Fock) states $|n\rangle$ and coherent states $|\beta\rangle$ are the eigenstates of $\hat{N}_a$ and $a$ with eigenvalues $n$ and $\beta$ respectively. Coherent states may also be characterized as displaced vacuum states and are minimal uncertainty states satisfying $\Delta \hat{q}^2 = \Delta \hat{p}^2 = \frac{1}{2}$. We are working in units such that $\hbar = 2\pi$. In the number state representation they are written as:

$$|\beta\rangle = e^{-|\beta|^2/2} \sum_n \frac{\beta^n}{\sqrt{n!}} |n\rangle. \quad (1)$$

Squeezed states and squeezed number states may be obtained directly from $|\beta\rangle$ and $|n\rangle$ by the application of the squeeze operator. This operator depends on a complex parameter $\xi \equiv re^{-i\phi}$. For simplicity we restrict our computation to the case of real $\xi$. We define $S(r)$ as the squeeze operator, and therefore

$$|\beta, r\rangle = S(r)|\beta\rangle \quad (2)$$
$$|m, r\rangle = S(r)|m\rangle \quad (3)$$

where $S(r)$ is given by

$$S(r) = \exp \left( \frac{1}{2} r (a^\dagger)^2 - \frac{1}{2} r a^2 \right). \quad (4)$$

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It is also useful to define a transformed annihilation operator $b$ as:

$$b = S a S^\dagger = \cosh(r)a + \sinh(r)a^\dagger$$  \hspace{1cm} (5)$$

and then, one has

$$b|\beta, r\rangle = \beta|\beta, r\rangle$$  \hspace{1cm} (6)$$

and

$$\hat{N}_b|m, r\rangle = b^\dagger b|m, r\rangle = m|m, r\rangle.$$  \hspace{1cm} (7)$$

Given the linear nature of the transformation on $a$, (5) one readily computes the coherent states and Fock states amplitudes for the squeezed states [1],

$$\langle \alpha|\beta, r\rangle = \frac{1}{\cosh^{1/2}(r)} \exp \left( -\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 - \frac{\tanh(r)}{2} \alpha^*\beta \right)$$  \hspace{1cm} (8)$$

$$\langle n|\beta, r\rangle = \frac{\tanh^{n/2}(r)}{(2^n n! \cosh(r))^{1/2}} H_n \left( \frac{\beta}{(2 \sinh(r) \cosh(r))^{1/2}} \right) \times \exp \left( -\frac{1}{2} |\beta|^2 + \frac{1}{2} \tanh(r)\beta^2 \right)$$  \hspace{1cm} (9)$$

where $H_n$ is the $n$th order Hermite polynomial.

For $\beta \in \mathbb{R}$, the wave functions in $q$ and $p$ are then given by:

$$\langle q|\beta, \xi\rangle = (2\pi \Delta^2 q)^{-1/4} \exp \left( -\frac{(q - q_0)^2}{4\Delta^2 q} \right)$$  \hspace{1cm} (10)$$

$$\langle p|\beta, \xi\rangle = \left( \frac{2\Delta^2 q}{\pi} \right)^{1/4} \exp \left( -\frac{p^2 \Delta^2 q - ipq_0}{2} \right),$$  \hspace{1cm} (11)$$

with $q_0 = \sqrt{2} e^r \beta$, $\Delta^2 q = e^{-2r}/2$ and $\Delta^2 p = e^{2r}/2$.

## 3 Representations of the squeezed number states

To discuss the properties of the squeezed number states, we observe that for any operator $\hat{R}$ and any state $|\psi\rangle$, the completeness of the Fock states and Eq. (1) allow one to write,

$$\langle \psi|\hat{R}|\beta, r\rangle = \sum_m \langle \psi|\hat{R}S(r)|m\rangle \langle m|\beta\rangle = e^{-|\beta|^2/2} \sum_m \frac{\beta^n \langle \psi|\hat{R}|m, r\rangle}{(m!)^{1/2}}.$$  \hspace{1cm} (12)$$

This provides a generating function for the matrix element $\langle \psi|\hat{R}|m, r\rangle$ which can then be obtained as,
\[ \langle \psi | \hat{R} | m, r \rangle = \frac{1}{(m!)^{1/2}} \left[ \frac{\partial^m}{\partial \beta^m} (e^{[\beta^2/2] \langle \psi | \hat{R} | \beta, r \rangle}) \right]_{\beta=0}. \] (13)

For example, if \( \hat{R} = U(t) \) is the evolution operator, and \( |\psi\rangle = |k\rangle \) with \( \{|k\rangle\} \) being any complete basis of the radiation states, one can compute in this form the time dependent probability amplitudes in the \( \{|k\rangle\} \) representation. For \( t = 0 \) we get the probability amplitudes of stationary squeezed number states.

Generalizing the same idea, we also note that
\[ \langle \alpha, r | \hat{R} | \beta, r \rangle = e^{-1/2 |\alpha|^2} e^{-1/2 |\beta|^2} \sum_n \sum_m \alpha^* n \frac{\beta^m}{n! (m!)^{1/2}} \langle n, r | \hat{R} | m, r \rangle \] (14)
so that we can write,
\[ \langle n, r | \hat{R} | m, r \rangle = \frac{1}{(n!)^{1/2} (m!)^{1/2}} \left[ \frac{\partial^n}{\partial \alpha^n} \frac{\partial^m}{\partial \beta^m} \left( e^{[1/2 |\alpha|^2} e^{[1/2 |\beta|^2] \langle \alpha, r | \hat{R} | \beta, r \rangle} \right) \right]_{\alpha=0, \beta=0}. \] (15)

To continue, let us generate with this observation the probability distributions for the squeezed number states in the Fock, position, momentum, and coherent states representations. We begin computing the amplitude of the squeezed number states in the Fock states basis. Taking \( |\psi\rangle = |n\rangle \) and \( \hat{R} = 1 \), the amplitude for the squeezed number states is,
\[ \langle n | m, r \rangle = \frac{1}{(n!)^{1/2} (m!)^{1/2}} \left[ \frac{\partial^m}{\partial \beta^m} \left( e^{[1/2 |\beta|^2] \langle n | \beta, r \rangle} \right) \right]_{\beta=0} = \frac{(\tanh(r/2))^{n/2}}{\sqrt{n! m! \cosh(r)}} \left[ \frac{\partial^m}{\partial \beta^m} H_n \left( \frac{\beta}{\sqrt{2 \sinh(r) \cosh(r)}} \right) e^{\beta^2 \tanh(r)/2} \right]_{\beta=0}. \] (16)

Next we note that,
\[ \left[ H_n \left( \frac{\beta}{(2 \sinh(r) \cosh(r))^{1/2}} \right) \right]_{\beta=0} = \frac{2^l (-1)^{n-l} n! (2 \sinh(r) \cosh(r))^{-1/2}}{(n-l)/2)!} \] (17)
and
\[ \left( e^{\beta^2 \tanh(r)/2} \right)_{\beta=0} = \left\{ \begin{array}{ll} 0 & \text{if } l \text{ odd} \\ \left( \frac{\tanh(r)}{2} \right)^{l/2} \frac{n!}{(n-l)/2)!} & \text{if } l \text{ even} \end{array} \right. \] (18)

Using the Cauchy formula
\[ \{ f(x) g(x) \}^{(m)} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} f^{(m-k)}(x) g^{(k)}(x) \] (19)
we get,

$$\langle n|m, r \rangle = \frac{(m!n!)^{1/2}}{\cosh(r)\left(\frac{m+n+1}{2}\right)} \sum_k \frac{\sinh(r)}{2} \frac{(-1)^{\frac{n-k}{2}}}{k!(\frac{m-k}{2})!(\frac{n-k}{2})!}$$

(20)

$$k = \begin{cases} 
0, 2, 4, 6, \ldots & n, m \text{ even} \\
1, 3, 5, 7, \ldots & n, m \text{ odd} 
\end{cases}$$

This result is in agreement with the one reported in Ref. [7], which was obtained by the application of a normally ordered squeeze operator to a Fock state. The most salient features of this photon distribution are that it oscillates and that only the photon number states of the same parity of \( m \) are represented in the expansion for \( |m, r \rangle \). The latter reflects of course the quadratic dependence of \( S(r) \) in \( a \) and \( a^\dagger \). The oscillations of the distribution \( P_{n,m} \equiv |\langle n|m, r \rangle|^2 \) for \( m > 1 \) and \( r \) sufficiently large have a fixed number of maxima; \( m/2 + 1 \) when \( m \) is even and \( (m+1)/2 \) when \( m \) is odd. These characteristics can be related to the structure of the \( Q(\alpha) \) function, as we discuss below. However it should be noted that the parity related characteristics of the distributions are difficult to explain within the phase space interference hypothesis,[7]. However.

For the squeezed number states with \( m = 0 \) (the squeezed vacuum) and \( m = 1 \) the sum in equation (20) is just a single term:

$$|\langle n|0, r \rangle|^2 = \frac{n!}{((\frac{n}{2})!)^{2^n}} \frac{\tanh^n(r)}{\cosh(r)}$$

(21)

$$n = 0, 2, 4, 6, \ldots$$

$$|\langle n|1, r \rangle|^2 = \frac{n!}{((\frac{n-1}{2})!)^{2^{n-1}} - \frac{1}{\cosh(r)}}$$

(22)

$$n = 1, 3, 5, 7, \ldots$$

As an example, we show in Figure (1) the photon number distribution for a squeezed number state with \( m = 7 \) and \( r = 1.4 \). The distribution \( |\langle n|7, 1.40 \rangle|^2 \) has 4 maxima in \( n \) located at 1, 11, 37 and 89 photons respectively.

The support of the distribution, and the oscillations, become wider for greater \( r \). For the squeezed number state with \( m = 7 \), this is illustrated by Figure (2). In this figure, we plot the positions of \( n \) for the last three maxima as functions of \( r \). These values shift to the right as \( r \) grows. It is possible to observe a tendency for each of the maxima, which can be approximated by exponential functions.

Let us study the other representations of squeezed number states. Using (10) and (13) we have

$$\langle q|m, r \rangle = m!^{-1/2}(2\pi)^{-1/4}(2^{-1/2}e^{-r})^{-1/2} \frac{\partial^m}{\partial \beta^m} \left\{ e^{-\left(\frac{-i(\beta + \frac{\pi}{2})}{2\alpha} \right)^2} \right\}_{\beta = 0}.$$
And using again the Cauchy formula we have
\[
\frac{\partial^m}{\partial \beta^m} \left\{ e^{\left( \frac{-i(q-e^{-2r}r)\beta^2}{2} \right)} \right\} = \sum_{k=0}^{m} \frac{m!}{(m-k)!k!} \frac{\partial^k}{\partial \beta^k} \left\{ e^{\left( \frac{-i(q-e^{-2r}r)\beta^2}{2} \right)} \right\}.
\]

Evaluating at \( \beta = 0 \) we obtain
\[
\langle q|m,r \rangle = \pi^{-1/4} e^{r/2} e^{\left( \frac{-2r^2}{2} \right)} 2^{-m/2} m! 1/2 \sum_{k=0}^{m} \frac{2^{k/2}}{k!((m-k)/2)!} H_k \left( \frac{e^r q}{2^{1/2}} \right)
\]
\[k = \begin{cases} 0, 2, 4, 6... & m \text{ even} \\ 1, 3, 5, 7... & m \text{ odd} \end{cases}
\]

In order to simplify the above expression we use the following identity of the Hermite polynomials:
\[
\frac{1}{m!} H_m(x) = \sum_{k=0}^{m} \frac{2^{k/2}}{k!((m-k)/2)!} H_k \left( \frac{x}{2^{1/2}} \right)
\]

Finally, the amplitude (25) becomes:
\[
\langle q|m,r \rangle = \pi^{-1/4} e^{r/2} e^{\left( \frac{-2r^2}{2} \right)} 2^{-m/2} m!^{-1/2} H_m \left( \frac{q}{e^{-r}} \right)
\]

Thus, the amplitude is that of a Fock state, depending on a squeezed quadrature variable \( e^r q \). Moreover, the amplitude of a Fock state is given by the limit of 27 when \( r \to 0 \).

The representation of the squeezed number states in terms of the other quadrature follows along the same lines. We get
\[
\langle p|m,r \rangle = \frac{e^{-r/2}}{m!^{1/2} \pi^{1/4}} e^{\left( \frac{-2r^2}{2} \right)} 2^{-m/2} \sum_{k=0}^{m} \frac{m!(-2i e^{-r} p)^k}{k!((m-k)/2)!} H_m(e^{-r} p)
\]

and
\[
\langle p|m,r \rangle = \frac{e^{-r/2}}{m!^{1/2} \pi^{1/4}} e^{\left( \frac{-2r^2}{2} \right)} 2^{-m/2} (-i)^m H_m(e^{-r} p)
\]

The amplitude depends on \( e^{-r} p \) and, for \( r > 0 \), \(|\langle p|m,r \rangle|^2\) has a support that is \( e^r \) wider than the support of a Fock state. The distribution has \( m+1 \) maxima and \( m \) minima (which are also zeroes). In Figure (3) we show the momentum probability distribution for the \( m = 7 \) squeezed number state with \( r = 1.4 \). The oscillations in this distribution come from the Hermite polynomial. For large squeezing, these oscillations reappear in the \( Q(\alpha) \) distribution function.
Finally, let us turn to the $Q(\alpha)$ function, which is given by the diagonal elements of the density matrix $\rho$ in the coherent states basis. For a squeezed number state one has,

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle = \frac{1}{\pi} |\langle \alpha | m, \xi \rangle|^2.$$  

(30)

The function $Q(\alpha)$ gives the probability of being in a minimum dispersion state around an average position and momentum proportional to $Re(\alpha)$ and $Im(\alpha)$ [18]. Following the previous procedure and using (8), we have

$$\langle \alpha | m, r \rangle = \left( \frac{1}{m! \cosh(r)} \right)^{1/2} e^{-1/2|\alpha|^2 \text{tanh}(r) |\alpha|^2/2} \frac{\partial^m}{\partial \beta^m} \left. e^{\text{tanh}(r) \beta^2 + \frac{\text{tanh}(r)}{\beta}} \right|_{\beta=0}$$

(31)

which can be reduced to

$$\langle \alpha | m, r \rangle = m^{1/2} \cosh(r)^{-1/2} e^{-1/2|\alpha|^2 \text{tanh}(r) |\alpha|^2/2}$$

$$\times \sum_{p=0}^{[m/2]} \frac{2^{-p} \sinh^p(r) \cosh^{m-p}(r) (\alpha^*)^2}{(m-2p)!p!} (\alpha^{*})^{m-2p}.$$  

(32)

The $Q(\alpha)$ function of the squeezed number states is, therefore:

$$Q(\alpha) = \frac{1}{\pi} m! \cosh(r)^{-1} e^{-|\alpha|^2 \text{tanh}(r) (|\alpha|^2 + |\alpha|^2)/2}$$

$$\times \sum_{p=0}^{[m/2]} \frac{2^{-p} \sinh^p(r) \cosh^{m-p}(r) (\alpha^*)^2}{(m-2p)!p!} (\alpha^{*})^{m-2p} |\beta=0|^2.$$  

(33)

The shape of $Q(\alpha)$ depends strongly on the squeezing parameter $r$, as illustrated by Figures (4), (5) and (6), (7); which show the Eq. (33) for $m = 7$, $r = 0.50$ and $r = 1.40$.

For small squeezing, the $Q(\alpha)$ function is appreciably nonzero over an elliptical ring, with an eccentricity $e^r$. When $r = 0$ this ring is circular, the $Q(\alpha)$ function being that of a Fock state. For greater squeezing, the $Q(\alpha)$ function shows prominent oscillations with a number of maxima which for the $m$ squeezed number state stabilize at $m + 1$. As can be shown by direct numerical computation, the transition occurs at $r \simeq \frac{1}{2} \ln(m)$. This corresponds to $\sqrt{m} e^{-r} = 1$ and is interpreted as the point where the inner boundary of the deformed ring touch itself. The oscillations for $r > \frac{1}{2} \ln(m)$ occur alone g the axis of $Im(\alpha)$ and are correlated to the oscillations of the momentum wave function. For large squeezing, the $Q(\alpha)$ function evaluated on the $Im(\alpha)$ axis is proportional to the square of the momentum wave function:

$$Q(\alpha = i \sqrt{2p}) \sim | \langle p | m, r \rangle |^2$$  

(34)
As an example, consider the $m = 7$ squeezed number state with $r = 1.4$. The maxima in its momentum distribution (see Figure (3)) correspond to those of the $Q(\alpha)$ function (figure 6) in the $\text{Im}(\alpha)$ axis.

In the same way, for large squeezing, the $Q(\alpha)$ oscillations are also correlated to the oscillations of the photon number distribution $P_{n,m}$. Each maximum $i\alpha_{\text{max}}$ ($\alpha_{\text{max}} \geq 1$) of $Q$ is associated to a maximum $n_{\text{max}}$ of $P_{n,m}$, such that,

$$n_{\text{max}} \simeq |\alpha_{\text{max}}|^2.$$ 

If we use the $Q(\alpha)$ function to assign an area in phase space which represent the states, this result does not contradict the area overlapping hypothesis of Wheeler et al. Nevertheless the oscillations in this case does not arise from the interference of different contributions of the overlapping areas but directly from the phase space representation of the state. The highly squeezed number states cannot be represented by simple elliptical rings in phase space but one should use a disconnected area with zones arranged along the quadrature that is not squeezed (see Figure (7)).

4 Conclusion

We have computed the photon number distribution, the momentum like and position like wave functions, and the $Q(\alpha)$ function for squeezed number states and we have shown that each of them has characteristic oscillations which depend on the squeeze parameter $r$.

For highly squeezed number states we observed that the oscillations in the different probability distributions are fixed in number and have a correspondence between them.

As a consequence we argue that a semiclassical analysis of the distributions based on the overlapping areas assigned to the states in phase space should distinguish between the low ($r < \frac{1}{2}\ln(m)$) and the high ($r > \frac{1}{2}\ln(m)$) squeezing cases. For the former the accepted explanation of the oscillations in terms of the interference of different sectors of the overlapping area is well sustained. For the latter one finds that a major effect should be ascribed to the particular phase space structure of the state which we explore with the aid of $Q(\alpha)$.

From this analysis we conclude that a precise description of quantum states in terms of bounded areas in the phase space should allow in some cases to assign a non-connected area to the state. In these cases when one computes the amplitude of the state in some basis, the effect of the interference of the overlapping areas may be screened by the more direct effect of the geometry of the state in phase space.

It should be also mentioned that the generating function formalism used throughout this work has been shown to be a straightforward and very valuable procedure for calculating probability amplitudes and matrix elements for any base obtained transforming number states by an unitary operator, chosen in an arbitrary manner.
References

Figure 1: Photon number distribution for the squeezed number states with $m = 7$ and $r = 1.40$, $|\langle n | 7, 1.40 \rangle|^2$. 
Figure 2: Position $n$ of the maxima for the photon number distribution $|\langle n|7, r \rangle|^2$ of a squeezed number states, versus $r$
Figure 3: Momentum probability distribution $|\langle p|m, r \rangle|^2$ for the $m = 7$ squeezed number state with $r = 1.4$. 
Figure 4: $Q(\alpha)$ function of the $m = 7$ squeezed number state with $r = 0.50$
Figure 5: Contour graphic of the $Q(\alpha)$ function of the $m = 7$ squeezed number state with $r = 0.50$
Figure 6: $Q(\alpha)$ function of the $m = 7$ squeezed number state with $r = 1.40$
Figure 7: Contour graphic of the $Q(\alpha)$ function of the $m = 7$ squeezed number state with $r = 1.40$