Topological String Defect Formation During the Chiral Phase Transition

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Abstract

We argue for the existence of topological string defects in chiral sigma models. We discuss the formation of these strings during the chiral phase transitions in the early Universe and in the high energy heavy-ion collisions. Chiral symmetry as well as deconfinement are restored in the core of these defects. Formation of a dense network of string defects is likely to play an important role in the dynamics following the chiral phase transition. We speculate that such a network can give rise to non-azimuthal distribution of transverse energy in heavy-ion collisions.

I. INTRODUCTION

QCD with massless quarks possesses $U(N_f)_R \times U(N_f)_L$ symmetry. However, the low energy spectrum of QCD does not exhibit this full symmetry and suggests that part of this symmetry is spontaneously broken. It is the axial vector part of this symmetry which is spontaneously broken at low energy, $U(N_f)_R \times U(N_f)_L \rightarrow U(N_f)_{V=R+L} \equiv U(N_f)_{V}$. Pseudoscalar mesons are the Goldstone bosons of this spontaneous symmetry breaking (SSB). The vacuum at low energy/temperature, i.e in the broken phase, is characterized by a non-zero value of the chiral condensate.

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In the real world, the $U(N_f)_A$ transformations, apart from spontaneous breaking, are also explicitly broken because of nonzero quark masses. Due to this the Goldstone bosons become massive. Even in the chiral limit, the $U(1)_A$ part of $U(N_f)_A$ is spontaneously broken to $Z(N_f)$ due to anomaly [1]. This makes the corresponding Goldstone boson, the $\eta'$ meson, heavier compared to other pseudoscalar mesons. Finite temperature field theory calculations on lattice show that $SU(N_f)_A$ is restored above a critical temperature $T_\chi$ in the chiral limit and approximately restored for the case of small quark masses. Also lattice results [2] show that topological susceptibility sharply drops above $T_\chi$ effectively restoring the $U(1)_A$ symmetry. Such chiral symmetry restoration could have occurred in the early Universe, when the temperature of the universe was above $T_\chi$. Also, one of the main objectives of relativistic heavy-ion collisions is to detect and study such chiral symmetry restoration, or spontaneous breaking of the chiral symmetry, in QCD.

One of the interesting aspects of spontaneous symmetry breaking phase transitions is that topological defects may form during these transitions, depending on the structure of the vacuum manifold. Topological defects are extended, non-zero energy solutions of the equations of motion. Their stability typically originates from a non-zero topological charge. There are numerous examples of topological defects in condensed matter systems, such as flux tubes in super-conductors, vortices in super-fluids, strings and monopoles in nematic liquid crystals etc.. Topological defects like cosmic strings, monopoles and domain walls etc. arise in various particle physics models of the early Universe. Even though topological defects arise in the symmetry broken phase, in the core of these defects symmetry is restored, as if the phase with higher symmetry is trapped in the background of broken phase.

For the case of small explicit symmetry breaking, these topological field configurations are no more static solutions of the theory. Nevertheless they will always form during phase transitions, though their interaction dynamics may be dominated by explicit symmetry breaking. These defects have structures which are stable against small fluctuations of the order parameter (even though the defect may not be a static solution) and can only decay by annihilation with defects of opposite topological charge. This makes them long lived and can lead to important effects on the dynamics of the system.

We will argue below that in QCD, in the chiral limit, spontaneous breaking of $U(N_f)_R \times U(N_f)_L \longrightarrow U_V(N_f)$ leads to static topological string solutions. Even with explicit symmetry breaking of the $U(1)_A$ of the $U(N_f)_A$ due to anomaly, these topological strings remain static, although the field configuration is different from the case of zero quark mass case. In the framework of linear sigma model we give the field configurations of these topological string defects in the chiral limit including the effects of anomaly. Also we will argue that chiral symmetry as well as deconfinement are restored in the core of the string which could have interesting implications. These topological string solutions become non-static for non-zero quark masses.

In the next section we briefly discuss the connection between topological defects and spontaneous symmetry breaking. In section III we will discuss the essential features of the linear sigma model Lagrangian which admits static topological string defects in the chiral limit [3]. Following this in section IV we will give the static string solutions in the chiral limit taking parameters from reference [3]. In section V we will discuss how these defects will form during the chiral transition in the chiral limit with some remarks for the case of non-zero quark masses. In section VI we will discuss the implications of formation of such
II. TOPOLOGICAL DEFECTS AND SSB

The type of defects formed in a phase transition and the symmetry breaking pattern are intimately related. Complete SSB of $U(1)$ symmetry gives rise to vortices and strings in 2-D and 3-D physical space respectively. SSB of $SU(2)$ to $U(1)$ gives rise to monopoles in 3-D physical space. For the chiral transition one can argue that SSB of chiral symmetry $U(N_f) \times U(N_f)$ to $U(N_f)$ leads to strings. Energetics of the defects is primarily decided by whether the SSB is that of a local or global symmetry. In the case of local symmetry one usually has finite energy topological defect solutions, on the other hand for global symmetry, like in the chiral transition, energy of the solutions diverges logarithmically with the volume.

Topological defects arise when the ground state of the theory is degenerate and the order parameter space (OPS) consisting of all possible values of the order parameter (OP) is topologically non-trivial. For example when $U(1)$ symmetry is spontaneously broken the OP is given by $\psi = \eta e^{i\theta}$ with non-zero fixed $\eta$. The OPS in this case consists of all possible values of $\theta$ between 0 and $2\pi$, hence the OPS is a circle $S^1$. An OPS has non-trivial $n$-th homotopy group $\pi_n$ if there exist non-trivial mappings from $n$-sphere $S^n$ to the OPS, or in other words if one can have non-contractible loops, closed surfaces etc. in the OPS. In general topological defects are characterized by the homotopy group $\pi_n(OPS)$ of the OPS. Group $\pi_n(OPS)$ consists of all non-trivial mappings from $S^n$ to OPS. Vortices or strings are characterized by $\pi_1(OPS)$, monopoles by $\pi_2(OPS)$ etc.. For the case of OPS=$S^1$ only $\pi_1(S^1)$ is non-trivial and each element of this group relates to total variation of phase of the OP around a vortex or string. The total phase variation around a string or vortex is always an integer multiple $2\pi n$ of $2\pi$. Because of this, the group $\pi_1(S^1)$ is isomorphic to the set of integers $\mathbb{Z}$. $n$ here is called the winding number which is the topological charge of the string or vortex configuration.

For the SSB of chiral symmetry $U(N_f)_R \times U(N_f)_L \rightarrow U(N_f)$ the OPS topologically is equivalent to $U(N_f) = \frac{SU(N_f) \times U(1)}{\mathbb{Z}_{N_f}}$. One can show that there exist non-trivial closed loops in $U(N_f)$ [5]. Closed loops in the OPS can be generated by the $U_A(1)$ generator. These correspond to topological defects known as the abelian strings. Also there are non-trivial loops in OPS generated by linear combination of the $U_A(1)$ and $SU(N_f)$ generators, which give rise to non-abelian strings. We consider here the abelian topological strings and will discuss non-abelian strings in a separate work. In the next section we will consider the linear sigma model Lagrangian [3] and will give the configuration of the winding one string in the chiral limit.

III. THE MODEL LAGRANGIAN

The $U(N_f)_R \times U(N_f)_L$ linear sigma model for $N_f$ quark flavors is given by [3,4],

$$L(\Phi) = \text{Tr} \left( \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi \right) - \lambda_1 \left[ \text{Tr} \left( \Phi^\dagger \Phi \right) \right]^2 - \lambda_2 \text{Tr} \left( \Phi^\dagger \Phi \right)^2 + c \left[ \text{Det} (\Phi) + \text{Det} (\Phi^\dagger) \right] + \text{Tr} \left[ H (\Phi + \Phi^\dagger) \right].$$

(1)
Φ is a complex $N_f \times N_f$ matrix field parameterizing the scalar and pseudoscalar mesons,  
\begin{equation}
    \Phi = T_a \phi_a = T_a (\sigma_a + i \pi_a),
\end{equation}
\(a = 0, ..., N_f^2 - 1\) and $T_a$, $a \neq 0$, are a basis of generators for the $SU(N_f)$ Lie algebra, the analogues of $\lambda_a$ for $N_f = 3$. $T_0 = 1$ is the generator for the $U_A(1)$ Lie algebra.

The field $\Phi$ transforms under the chiral transformation $U(N_f)_R \times U(N_f)_L$ as  
\begin{equation}
    \Phi \longrightarrow U_r \Phi U^\dagger_\ell, \quad U_{r,\ell} \equiv \exp \left( i \omega^a_{r,\ell} T_a \right).
\end{equation}

One can rewrite the left-right transformations in terms of vector-axial vector transformations with parameters $w_{r,A}^a = (w_r^a \pm w_l^a)/2$. The infinitesimal form of the $U(N_f)_R \times U(N_f)_L$ symmetry transformation (3) is  
\begin{equation}
    T_a \phi_a \longrightarrow T_a \phi_a - i \omega^a_{r,\ell} [T_a, T_b] \phi_b + i \omega^a_{A} \{ T_a, T_b \} \phi_b. \quad (4)
\end{equation}

$\Phi$ is a singlet under the $U(1)_V$ transformations ($\exp(iw^0_V T^0)$). In QCD, it gives rise to a conserved charge identified with baryon number.

In the above model the determinant term takes into account the instanton effect which explicitly breaks the $U(1)_A$ symmetry [4]. It is not clear whether such a term can be justified for high temperatures, because vanishing $\Phi$ makes the instanton effects to disappear [6], though lattice results show similar behavior for the chiral condensate and topological susceptibility [2]. Both drop sharply above the critical temperature $T_\chi$ suggesting that they become vanishingly small in a narrow range of temperatures. We expect that for such a narrow range of temperatures the basic picture of string defect formation during the chiral transition will not be affected. The last term is due to non-zero quark masses, where $H = T_a h_a$.

When $c = H = \lambda_2 = 0$, $U(N_f) \times U(N_f)$ is spontaneously broken for $m^2 < 0$ and $\lambda_1 > 0$. For $\lambda_2 \neq 0$ $U(N_f)_A$ is spontaneously broken to identity. This results in $N_f^2$ Goldstone bosons, for $N_f = 3$ these Goldstone bosons are the $\pi$’s, $K$’s, $\eta$ and $\eta'$. However when just $c \neq 0$, the $U(1)_A$ is further broken to $Z(N_f)$ by the axial anomaly, making the $\eta'$ massive compared to other pseudoscalar mesons. $SU(N_f)_V$ is still the symmetry of the Lagrangian. All these symmetries are in addition explicitly broken by non-zero quark masses making all the Goldstone bosons massive.

In the absence of anomaly and $H = 0$, the non-trivial loops in the OPS can be generated by exponentiating $T_0$ or a linear combination of $T_0$ and $T_a$, $a = 1, ..., N_f^2 - 1$ [5]. For $c \neq 0$ and $H = 0$, only the $N_f$ equally spaced points $\Phi_0 e^{i2\pi n/N_f}$, $n = 1, 2, ..., N_f$ on the $U_A(1)$ circle $\langle \Phi_0 e^{i\theta} : 0 \leq \theta \leq 2\pi, \Phi_0$ a suitable fixed point in OPS $\rangle$ are degenerate and all other points have higher effective potential or free energy regardless of the values of $\lambda_i$’s. However one can still have static string configurations with non-zero topological charge. For $H \neq 0$ as well, only one point on OPS is the absolute minimum of the free energy. There exists no static topological configuration in this case, however for small $H \neq 0$ one can expect topological string field configurations to form during the chiral transition. In the following we find the topological string solutions for both $c = 0$ and $c \neq 0$ in the chiral limit for $N_f = 3$. 

4
IV. THE STRING SOLUTION

For the abelian topological string solution, it is enough to consider the variation of $\Phi$ generated by $T_0$. Topological aspects of the string will not change by considering variation of other components of $\Phi$. However in the core of the string all components of $\Phi$ do vanish in the linear sigma model we are considering here. We assume $\Phi = T_0(\phi_1 + i\phi_2)$, $\phi_2$ is the field for $\eta'$. With this restriction on $\Phi$, the effective Lagrangian reduces to that of a complex scalar field given by the following equation.

$$L(\Phi) = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2)$$  \hspace{1cm} (5)

where

$$V(\phi_1, \phi_2) = \frac{m^2}{2} (\phi_1^2 + \phi_2^2) + \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2 - \frac{c}{3} (\phi_1^3 - 3\phi_1\phi_2^2) - 6h_0\phi_1,$$  \hspace{1cm} (6)

$$h_0 = -\frac{1}{3} Tr(HT_0).$$  \hspace{1cm} (7)

We consider here $h_0 = 0$ for which one can solve the field equations for static string configurations. The static string configuration is straight and has a certain symmetry around the length of the string. For $c = 0$ the solution of the string has cylindrical symmetry. However for $c \neq 0$ the string does not have cylindrical symmetry, but possesses a symmetry of rotation by $2\pi/3$ about the axis of the string. In this case the string is the junction of 3 domain walls which interpolate between the different $Z(3)$ ground states. In the following we consider the string solution first for $c = 0$ and then for $c \neq 0$.

Considering the string along the $z$-direction, $\phi_1(x, y)$ and $\phi_2(x, y)$ for the static string satisfy the following field equations:

$$\nabla^2 \phi_1 = \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_1},$$

$$\nabla^2 \phi_2 = \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_2}.$$  \hspace{1cm} (8)

For winding one cylindrically symmetric string solution one considers the following ansatz for the string:

$$\phi(\vec{r}) = \phi(r) \frac{\vec{r}}{r}.$$  \hspace{1cm} (9)

Here $\phi = \sqrt{\phi_1^2 + \phi_2^2}$ and $r = \sqrt{x^2 + y^2}$. Now $\phi(r)$ satisfies the field equation

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d \phi}{dr} - \frac{\phi}{r^2} = m^2 \phi + \left( \lambda_1 + \lambda_2 \frac{3}{r^2} \right) \phi^3.$$  \hspace{1cm} (10)
We solve the above equation using the parameters \([3]\) corresponding to \(m_\sigma = 1. GeV\) and vanishing masses for all pseudoscalar mesons. In the left of FIG.1, we plot \(-\phi\) and in the right plot the vector field of \(\phi(\vec{r})\). Magnitude \(\phi\) is proportional to the length of the vectors and the angle these vectors make with the positive \(x\)-axis is the \(U(1)_A\) phase. Configuration of \(U(1)_A\) phase is symmetric around the string in this case. In the core of the string \(\phi(\vec{r})\) vanishes restoring the chiral symmetry.

\[
\frac{d^2 \phi_1}{dt^2} + \alpha \frac{d\phi_1}{dt} - \nabla^2 \phi_1 = - \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_1}
\]

FIG. 1. Configuration of the string for \(c = 0\) and \(h_0 = 0\). Plotted is \(-\phi\) of the string in the left figure and figure on the right gives the vector field plot of the full field \(\phi_1(\vec{r}) + i\phi_2(\vec{r})\).

For the case of \(c \neq 0\) it is difficult to find the static solution by solving the field equations. We have found an approximate solution by evolving a cylindrically symmetric string configuration by putting an additional dissipative term in the equations derived from (6),
FIG. 2. Configuration of the string for \( c = 2.729 \) and \( h_0 = 0 \). Figure on the left gives \(-\sqrt{\phi_1^2 + \phi_2^2}\) and figure on the right gives the vector field plot of the full field \( \phi_1(\vec{r}) + i\phi_2(\vec{r}) \).

\[
\frac{d^2 \phi_2}{dt^2} + \alpha \frac{d\phi_2}{dt} - \nabla^2 \phi_2 = -\frac{\partial V(\phi_1, \phi_2)}{\partial \phi_2},
\]

where \( \alpha \) is the dissipation coefficient. The dissipation term has been used to converge the initial configuration (the circular configuration, which is not the correct configuration in the presence of the anomaly) to the approximate configuration of string connected with domain walls. Once the evolution is done for sufficiently long time the configuration now does not evolve with time even after the dissipation is switched off, which suggests that the configuration is more or less static and so an approximate solution. On evolving the initial cylindrically symmetric solution with the above equations, domain walls connected to the string develop. In this case we considered the \( \eta' \) mass to be the experimental value \( m_{\eta'} \sim 950\text{MeV} \) and set other pseudoscalar meson masses to zero. In the FIG.2 we show
the configuration of the string with the domain walls joined to it. In the left figure we plot $-\sqrt{\phi_1^2 + \phi_2^2}$, and in the right plot the vector field of $\Phi(\vec{r}) = (\phi_1(\vec{r}), \phi_2(\vec{r}))$. Again the magnitude of the field is proportional to the length of the vectors and the angle these vectors make with +ve x-axis is the $U(1)_A$ phase. Clearly there is no symmetric distribution of the $U(1)_A$ phase around the string. Here as well the magnitude of the field vanishes in the core of the string restoring the chiral symmetry.

V. FORMATION OF STRING NETWORK DURING THE CHIRAL TRANSITION

In the above we have shown the existence of static topological string configurations in the chiral limit in the linear sigma model. These topological configurations become non-static for non-zero quark masses. In this case the phase variation around the string will not even have the 3-fold symmetry as in Fig.2. There will be a domain wall, with phase variation of $2\pi n$ across, connecting a string with topological charge $n$. In the chiral limit one can use the picture of conventional mechanism of defect production, known as the Kibble mechanism [7] to make an estimate of the density of strings. (We mention that in situations where the magnitude of the order parameter undergoes huge fluctuations, a new mechanism of defect formation becomes effective, as discussed in [8]. We do not consider such possibilities.) For simplicity we consider here the string configurations which are specified by the $U_A(1)$ phase only. However consideration of non-abelian phases will increase the probability of formation of defects.

The basic picture of this mechanism is the following. For a second order phase transition, after the transition, the physical space develops a domain like structure, with the typical size of the domain being of the order of a relevant correlation length $\xi$ (which depends on the dynamics of the transition). Inside a given domain, the $U(1)$ phase is roughly uniform, but varies randomly from one domain to the other. The phase in between any two adjacent domains is assumed to vary such that the variation of the phase is minimum. This leads to the formation of string defects in the junction of three or more domains. Numerical simulations [9] show that on an average the expected number of strings per domain is 0.88. However for the case of explicit symmetry breaking other mechanisms can also contribute to the formation of these defects. In generic situations formation of topological defects is enhanced in systems with explicit symmetry breaking [10]. However, this enhancement crucially depends on the nature of the dynamics of phase transition. In contrast, Kibble mechanism allows for estimates of defect production (per domain) which are reasonably independent of the dynamical details. Therefore, for a rough estimate for the density of defects one can still use the Kibble mechanism. Thus, we will assume a domain-like structure after the phase transition, and assume a random distribution of the $U(1)_A$ phase between $[0, 2\pi]$ in the domains.

So immediately after the chiral transition there will be a network of strings in the broken phase. For an infinite system one would expect both open and closed string to form, for example during the chiral phase transition in the early universe. But for transition taking place in a small volume with outer region in the broken phase, which is the case of heavy-ion collisions, one will expect only string loops to form. Interestingly the number of string loops
close to the boundary will be larger as discussed in ref. [11]. The usual mode of decay of these string loops is via shrinking and breaking of bigger loops to smaller loops.

One of the implications of formation of strings during the chiral transition can be due to the interplay between the chiral transition and deconfinement. Recent lattice results suggest that chiral condensate and the Polyakov loop are strongly correlated [13]. In [14] it was proposed that the chiral condensate acts as an effective field for the Polyakov loop. Such a consideration implies restoration of deconfinement when there is unbroken chiral symmetry. This would lead to the Polyakov loop taking large expectation values inside the core of the string. Baryons are very heavy in the confined phase and very light in the deconfined phase. So baryons in the medium passed by a moving string would like to remain inside the string. This will give rise to increasing baryon number in the whole string loop as it collapses and moves through the medium. In the extreme case, the baryon number inside the string may become large enough to prevent the collapse of the string loop. The only way it can then decay is by quantum fluctuations involving the domain wall.

There can be interesting effects if a defect network forms in the chiral transition in heavy-ion collisions. Since in the core of the string chiral symmetry as well as deconfinement are restored, it will have large energy density of the order of $\sim 500 \text{MeV/fm}^3$. For non-zero quark masses all the phase variation around the string will be confined to a domain wall bounded by the string. Similar domain walls in QCD with large chemical potentials have been discussed before [12]. Across the domain wall the $U(1)_A$ phase varies by $2\pi$. The surface tension of such a wall can be $\sim 500 \text{MeV/fm}^2$, considering that its size is governed by the $\eta'$ mass. So a string loop of radius $3 \text{fm}$ will have a total energy $\sim 10 \text{GeV}$. In case the dynamics of shrinking of the string loop happens after the freeze out, then the shrinking will be less dissipative. Finally the string loops will shrink to a region of $1 \text{fm}$ size depositing $\sim 10 \text{GeV}$ energy making it a hot spot. However an idea of size distribution of the string loops and their number can only be checked by real time lattice simulations of the linear sigma model. Such hot spots can give rise to a non-isotropic/non-azimuthal energy distribution. It will be interesting to see whether the non-isotropy due to this effect can be larger than the usual statistical fluctuations.

In summary, we have argued that the symmetry breaking $U(N_f) \times U(N_f) \rightarrow U(N_f)$ allows for the existence of topological (abelian and non-abelian) string defects. We have studied the static configuration of these defects in the chiral sigma model. These defects may form during the chiral transition in the high energy heavy-ion collisions as well as in the early universe. We speculate that formation and subsequent evolution of the network of these string defects can give rise to inhomogeneous distribution of baryons and also the energy density.

Acknowledgments

This work was supported by DOE and NSF under contract numbers DE - FG02 - 85ERR 40231 and INT - 9908763 and from BMFB (Germany) under grant 06 BI 902. We have benefited greatly from discussions with F. Karsch, E. Laermann, R. Ray, H. Satz and A. M. Srivastava.
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