We provide a new canonical approach for studying dissipation by considering the quantum mechanical damped harmonic oscillator. Explicit expressions for lagrangians characterising forward and backward time propagations are given. A hamiltonian analysis, showing the equivalence with the lagrangian approach, is also done.

The study of dissipation phenomena is of great interest, both from the theoretical and application points of view. Dissipation effects are relevant in high energy physics, in the study of early universe, in many body theories, in phase transitions, in quantum optics and in numerous practical applications of finite temperature quantum field theory. The paradigm of a dissipative system is the quantum mechanical damped harmonic oscillator (dho). Since this is not time reversal invariant, a canonical formulation beginning from a lagrangian is problematic. The canonical quantisation is formulated by doubling the degrees of freedom such that the dho is complemented by its time reversed image [1 - 3]. This leads to a closed system for which a lagrangian formulation is viable. A natural question that arises in this context is to seek some lagrangian description that simulates the forward and backward time propagations so that the composite system indeed resembles the above mentioned closed system.

In the present paper we provide explicit expressions for lagrangians characterising forward and backward time propagations. The composite lagrangian obtained by combining the two constituent pieces correctly reproduces the lagrangian for the closed system used in the study of the quantum mechanical dho. A hamiltonian formulation is also provided which shows the equivalence with the lagrangian approach.

We have shown that the two cases of overdamped and (oscillatory) underdamped motions correspond to distinct regimes characterised by real and complex parameters, respectively, of the constituent lagrangians. The hamiltonian corresponding to the complex lagrangian is found to be pseudo-hermitian [4]. Such hamiltonians have recently occurred in the study of PT symmetric theories [5].

We begin with a review of the problem of the damped harmonic oscillator (dho). The equation of motion of the one-dimensional damped harmonic oscillator is

$$m \ddot{x} + \gamma \dot{x} + kx = 0 \quad (1)$$

The parameters m, γ and k are independent of time. If the ratio

$$R = \frac{k}{4m} \quad (2)$$

is greater than one, the motion is oscillatory with exponentially decaying amplitude. Otherwise, the motion is nonoscillatory i.e. overdamped. Since the system (1) is dissipative a straightforward lagrangian description leading to a consistent canonical quantization is not available. To develop a canonical formalism we require to consider (1) along with its time reversed image [1]

$$m \ddot{y} - \gamma \dot{y} + ky = 0 \quad (3)$$

which accounts for the bath where the dissipated energy flows. The system (1) and (3) can be derived from the lagrangian

$$L = m \dot{x} \dot{y} + \frac{\gamma}{2} (x \dot{y} - \dot{x} y) - kxy \quad (4)$$

Introducing the hyperbolic coordinates $x_1$ and $x_2$ [3] where,

$$x = \frac{1}{\sqrt{2}} (x_1 + x_2); y = \frac{1}{\sqrt{2}} (x_1 - x_2) \quad (5)$$

the above lagrangian can be written in a compact notation as

$$L = \frac{m}{2} g_{ij} \dot{x}_i \dot{x}_j - \frac{\gamma}{2} \epsilon_{ij} \dot{x}_i \dot{x}_j - \frac{k}{2} g_{ij} x_i x_j \quad (6)$$
where the pseudo - Euclidean metric $g_{ij}$ is given by $g_{11} = -g_{22} = 1$ and $g_{12} = 0$.

The lagrangian (6) is invariant under the SU(1,1) transformation

$$x_i \rightarrow x_i + \theta \sigma_{ij} x_j$$

(7)

where

$$\sigma = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$$

(8)

and $\theta$ is an infinitesimal parameter.

Since a straightforward lagrangian formalism for the dissipative systems is unavailable, one wonders whether individual lagrangians can be constructed for the time ordered modes separately so that the lagrangian (6) can be considered as a synthesis of the separate lagrangians. This is motivated by the fact that the composite lagrangian (6) is analogous to the general bidimensional oscillator lagrangian

$$L = \frac{m}{2} \ddot{x}_i^2 + \frac{B}{2} \epsilon_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} k x_i^2$$

(9)

studied recently [6] in connection with the Landau problem. The analogy becomes evident, after a suitable mapping of the corresponding parameters, by converting the pseudo - Euclidean metric to the Euclidean one by $g_{ij} \rightarrow \delta_{ij}$. In the Landau problem (9) one exploits dual aspects of the rotation symmetry in analysing it in terms of opposite chiralities [7]. Symmetry of (6) under (7) thus offers a possibility of analysing the composite theory in its elementary modes.

Accordingly we introduce the lagrangian

$$L_+(x) = \frac{\Gamma}{2} \epsilon_{ij} x_i \dot{x}_j - \frac{k_+}{2} g_{ij} x_i x_j$$

(10)

and its counterpart

$$L_-(x) = -\frac{\Gamma}{2} \epsilon_{ij} x_i \dot{x}_j - \frac{k_2}{2} g_{ij} x_i x_j$$

(11)

which are separately invariant under (7). Since the lagrangians (10) and (11) are already first order in time derivative we can read the algebras

$$\{ x_i, x_j \} = \pm \frac{1}{\Gamma} \epsilon_{ij}$$

(12)

and the hamiltonians

$$H_\pm = \frac{k_\pm}{2} g_{ij} x_i \dot{x}_j$$

(13)

The Noether charges corresponding to the transformations (7) are

$$C_\pm = \pm \frac{\Gamma}{k_\pm} g_{ij} x_i x_j$$

(14)

One can aptly call $C_\pm$ as the pseudochiral charge and (10) and (11) as the pseudochiral oscillators.

The synthesis of $L_+$ and $L_-$ is done by the soldering formalism [8]. We construct

$$L^{(1)}(y, z) = L_+(y) + L_-(z)$$

(15)

and consider its change under a gauge transformation

$$\delta y_i = \delta z_i = \Lambda_i(t)$$

(16)

where $\Lambda_i$ are some arbitrary functions of time. Then

$$\delta L^{(1)}(y, z) = \Lambda_i (J^{(+)i}(y) + J^{(-)i}(z))$$

(17)

where the currents $J^{(\pm) i}$ are given by

$$J^{(\pm) i}(x) = \pm \Gamma \sigma_{ij} \dot{x}_j - k_\pm x_i$$

(18)

The synthesis of $L_+$ and $L_-$ is realized by modifying $L^{(1)}(y, z)$ so that it becomes invariant under the gauge transformation (16). Writing,

$$L(y, z) = L^{(1)}(y, z) - B_i (J^{(+) i}(y) + J^{(-) i}(z))$$

(19)

where $B_i$ is an intermediate auxiliary field which also transforms as (16), it is easy to verify that $\delta L(y, z) = 0$ under the complete set of transformations. Eliminating the $B_i$ field by using its equation of motion, the residual lagrangian, which is found to depend only on $x_i = y_i - z_i$, exactly reproduces (6) provided the following identification is made

$$m = -\frac{\Gamma^2}{(k_+ k_-)}, \ \gamma = \frac{\Gamma (k_+ - k_-)}{k_+ k_-}, \ k = \frac{k_+ k_-}{(k_+ + k_-)}$$

(20)

The identification (20) has an immediate consequence. The ratio (2) is now given by

$$R = 1 - \frac{(k_+ + k_-)^2}{(k_+ - k_-)^2}$$

(21)
For real $k_+, k_-, R \leq 1$. Then the parameters identified by (20) correspond to an overdamped motion of the dho. Also note that to get the coefficients $m$ and $k$ to be positive we require $k_+$ and $k_-$ to be of opposite sign, with a suitable choice of their absolute values. Finally, for positive $\gamma$, $\Gamma > 0$ is required.

Now the physically more important situation is the underdamped motion of the dho where the motion is oscillatory with decaying amplitude. Here the parameters of (6) are such that the ratio $R > 1$. As already observed this condition cannot be simulated by the identification (20) for real values of $k_{\pm}$. However, if $k_+$ and $k_-$ are continued to complex values so that

$$k_+ = \kappa \quad k_- = \kappa^*$$

(22)

$$R = 1 + \left(\frac{\text{Re} \kappa}{\text{Im} \kappa}\right)^2$$

(23)

then $R > 1$, which is the required condition for oscillatory motion. Now equation (20) gives

$$m = -\frac{\Gamma^2}{2 \text{Re} \kappa}, \quad \gamma = i \Gamma \text{Im} \kappa / \text{Re} \kappa, \quad k = |\kappa|^2 / 2 \text{Re} \kappa$$

(24)

Taking $\kappa$ of the form $\kappa_1 - i \kappa_2$ with $\kappa_{1,2}$ positive we find that $\Gamma$ must be purely imaginary

$$\Gamma = ig, \quad g > 0$$

(25)

so that the parameters in (24) are positive. Substituting (22) and (25) in (10) and (11) we get

$$L_+(x) = \frac{g}{2} \epsilon_{ij} x_i \dot{x}_j - \frac{\kappa}{2} \eta_{ij} x_i x_j$$

$$L_-(x) = -\frac{i g}{2} \epsilon_{ij} x_i \dot{x}_j - \frac{\kappa^*}{2} \eta_{ij} x_i x_j$$

(26)

which are the elementary modes of (6) pertaining to the oscillatory limit. Remarkably, the lagrangians $L_{\pm}$ are complex conjugates of each other.

The emergence of complex lagrangians is not unexpected in dissipation problems. The equations of motion following from $L_+(x)$ in (26) are

$$\ddot{x}_i + \left(\frac{\kappa_1 - i \kappa_2}{g}\right) x_i = 0$$

(27)

which are simple harmonic oscillator equations with complex frequency $\pm ig \kappa_{1,2}$. The solutions of (27) are

$$x_i = \exp \pm \frac{ig \kappa_1 t}{g} \exp \pm \frac{ig \kappa_2 t}{g}$$

(28)

which are oscillatory solutions with exponentially growing or decaying amplitude. The lagrangians (26) thus really catch the essence of the independent modes of (6).

It will be instructive to look at the problem from the hamiltonian approach. The hamiltonian following from (6) is

$$H = \frac{1}{2m} \left(p_1 - \frac{\gamma}{2} x_2\right)^2$$

$$+ \frac{k}{2} x_1^2 - \frac{1}{2m} \left(p_2 + \frac{\gamma}{2} x_1\right)^2 - \frac{k^2}{2} x_2^2$$

(29)

where $p_1 = m \dot{x}_1 + \frac{\gamma}{2} \dot{x}_2$, $p_2 = -m \dot{x}_2 - \frac{\gamma}{2} x_1$ are the canonical momentum conjugate to $q_1$ and $q_2$. Introduce a canonical transformation from $(q_1, q_2, p_1, p_2)$ to $(q_+ ,q_-; p_+, p_-)$ where

$$p_\pm = \left(\frac{\omega_\pm}{2m \Omega}\right)^{\frac{1}{2}} p_1 \pm i \left(\frac{m \Omega \omega_\pm}{2}\right)^{\frac{1}{2}} x_2$$

$$x_\pm = \left(\frac{m \Omega}{2 \omega_\pm}\right)^{\frac{1}{2}} x_1 \pm i \left(\frac{1}{2m \Omega \omega_\pm}\right)^{\frac{1}{2}} p_2$$

(30)

with

$$\Omega = \left(\frac{1}{m} \left(k - \frac{\gamma^2}{4m}\right)\right)^{\frac{1}{2}}$$

(31)

and

$$\omega_\pm = \Omega \pm \frac{i \gamma}{2m}$$

(32)

Such transformations, though involving only real parameters, were used in [9], [10]. Now the composite hamiltonian (29) diagonalises as

$$H = H_+ + H_-$$

(33)

where,

$$H_\pm = \frac{p_\pm^2}{2} + \frac{\omega_\pm x_\pm^2}{2}$$

(34)

Using the definition of $\Omega$ and the identifications (24) we find that the frequencies occurring in (33) are identical to those obtained from the equations of motion following from (26). The composite hamiltonian is then really equivalent to the hamiltonian of two oscillators with complex frequencies $\mp \frac{\kappa_1}{g}$ and $\mp \frac{\kappa_2}{g}$.
\[ L(q_+) = \frac{1}{2} \dot{q}_+^2 - \frac{1}{2} \left( \frac{\kappa}{g} \right)^2 q_+^2 \]  

(35)

where \( q_+ = \sqrt{-\frac{\kappa}{g^2}}x_2 \). The hamiltonian following from (35) is \( H_+ \) given by (34). Similarly, \( H_- \) follows from \( L_- \). The hamiltonian decomposition (33) thus conclusively demonstrates that the elementary modes (26) constitute the system (6).

A question may arise regarding the interpretation of the complex hamiltonians \( H_{\pm} \) found in the constituent pieces. The first point to note is that they satisfy

\[ H_{\pm}^\dagger = H_{\mp} \]  

(36)

This hermitian conjugation property corresponds to the time reversal operation that connects the doubled degrees of freedom (1) and (3) of the closed theory. Also, this property manifestly ensures the hermiticity of the complete hamiltonian (33).

Although \( H_{\pm} \) are not hermitian, they satisfy a crucial property,

\[ H_{\pm}^\dagger = \eta H_{\pm} \eta^{-1} \]  

(37)

where \( \eta \) is the PT operator. To see this note that,

\[ \eta x_i \eta^{-1} = g_{ij} x_j, \quad \eta p_i \eta^{-1} = -g_{ij} p_j \]  

(38)

from which (37) follows using the basic expressions (30).

The property (37) shows that \( H_{\pm} \) are pseudo-hermitian operators. The concept of pseudo-hermiticity, which generalises the usual notions of hermiticity, has been introduced recently [4]. Indeed (37) is the defining relation for a pseudo-hermitian operator for some choice of \( \eta \). Of course for \( \eta = 1 \), pseudo-hermiticity simply reduces to hermiticity. Pseudo-hermitian hamiltonians have occurred in the study of PT-symmetric quantum mechanics [5], in minisuperspace quantum cosmology and other constructions [4].

To summarise, we have provided a new canonical formulation for studying dissipation by considering the example of the quantum mechanical damped harmonic oscillator. Explicit structures for lagrangians characterising the dissipative and the bath degrees of freedom were given. These structures naturally involved the pseudo-Euclidean metric which is an essential ingredient in the study of dissipative problems [1 - 3]. Furthermore, the hamiltonians obtained from the basic lagrangians were found to be complex in the damped oscillatory sector. Obviously these hamiltonians were not hermitian. But we showed that these were pseudo-hermitian with respect to the PT-operation. The eigenvalues for such operators are either real or occur in complex conjugated pairs [4]. It is quite noteworthy that pseudo hermitian operators, which occur in the study of a class of PT-symmetric quantum mechanics [5], have appeared naturally in our canonical formulation. This suggests a possible connection between our proposed approach to dissipation and the broadened formulation of quantum mechanics discussing PT-symmetry [5]. The validity of our approach was shown by reproducing familiar expressions for the composite lagrangian and the hamiltonian which simultaneously contain both the dissipative and bath variables [1 - 3]. Our approach can naturally be used to explore other disciplines like thermofield dynamics [11], which are based on the doubling of the degrees of freedom.