Evolutionarily Stable Strategies in Quantum Hawk- Dove Game

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Abstract
Hawk-Dove is an interesting and important game of evolutionary biology. We consider the game from point of view of evolutionarily stable strategies (ESSs). In the classical version of the game only a mixed ESS exists. We find a quantum version of this game where both pure and mixed ESSs can exist.

1 Introduction

The idea of ESS was originally given by Maynard Smith and Price [1] with the central idea of uninvadability against mutants i.e. a population playing an ESS can withstand a small invading group. If a strategy $A$ is played by almost all members of a population and rest of the population forms a small group of mutants playing strategy $B$ constituting a fraction $\epsilon$ of total population. The strategy $A$ is said to be ESS against $B$ if

$$\langle A, (1-\epsilon)A + \epsilon B \rangle > \langle B, (1-\epsilon)A + \epsilon B \rangle$$

where $\langle A, B \rangle$ is payoff to player playing $A$ against player playing $B$. For all sufficiently small positive $\epsilon$ there exists an $\epsilon_0$ such that for all $\epsilon \in [0, \epsilon_0]$ the above inequality is satisfied [2] If for the given $A$ and $B$ the $\epsilon_0$ is as large as possible the $\epsilon_0$ is called invasion barrier. If $B$ comes at a frequency larger than $\epsilon_0$ it will lead to an invasion.

For symmetric bi-matrix game $A$ is an ESS with respect to $B$ [3, 2] if

$$\langle A, A \rangle > \langle B, A \rangle$$

and if $\langle A, A \rangle = \langle B, A \rangle$ then $\langle A, B \rangle > \langle B, B \rangle$. For asymmetric case NE with strict inequality must hold in case it is an ESS. For example a strategy pair $(A^*, B^*)$ is an ESS if $\langle A^*, B^* \rangle > \langle A, B^* \rangle$ for all $A^* \neq A$ and $\langle B^*, B^* \rangle > \langle A^*, B \rangle$ for all $B \neq B^*$.

Iqbal and Toor [4, 5] considered ESSs in quantum versions of Prisoner’s Dilemma and Battle of Sexes games. They showed that evolutionary stability of Nash equilibria in symmetric as well as asymmetric games can be controlled by changing parameters of an initial quantum state like

$$|\psi_0\rangle = a |S_1 S_1\rangle + b |S_2 S_2\rangle$$
where $S_1, S_2$ represent pure strategies. Because a transition between a classical and a quantum form of these games also occurs by changing those parameters, therefore, it becomes possible that a Nash equilibrium not ESS in classical game can be an ESS in certain quantum form of the same game.

We take Hawk-Dove game in which neither Hawk nor the Dove is a pure ESS, i.e. there is no pure ESS, however, there exists a mixed ESS. When the game is quantized we find that the game remains symmetric if an additional restriction is imposed on initial state parameters. It leads to two cases of the game: symmetric and asymmetric. We discuss both these cases and show that there exist pure ESSs in both but mixed ESS can exist in symmetric case only.

1.1 Classical Hawk-Dove Game

Hawk-Dove \cite{7} is a simple game where two behavioral strategies employ very different means to obtain the resources. Hawks are very aggressive and always fight for some resource. These fights are very brutal affairs with the loser being the one who first sustains the injury. The winner takes the sole possession of the resource. The Hawks who lose are only injured, mathematics of the game requires that they do not die and are fully mended before their next expected contest. In a Hawk-Hawk contest each has a chance of 50% winning.

Dove never fights for a resource. It displays and if attacked it immediately withdraws before getting injured. Thus it will always lose a conflict with a Hawk, but it is not hurt when confronts a Hawk, hence these interactions are neutral with respect to Doves fitness. Dove do not display for a long against a Hawk. In a contest when it knows that the opponent is the Hawk it withdraws without paying a meaningful cost. On the other hand if a Dove meets a Dove there will be period of displaying with some cost (time, energy for display) but no injury. It is assumed that doves are equally good in displaying and they adopt strategy of waiting for random time thus when a Dove meets another Dove each has 50% chance of winning. The winner is the individual willing to pay more \cite{7}.

To analyze the game quantitatively let $v$ be the value of resource, a positive number; injury to self is $i$, a negative number; losing a resource costs 0 and cost of display is $d$, a negative number. The payoff matrix takes the form

\begin{equation}
\begin{array}{ccc}
 & Hawk(H) & Dove(D) \\
Hawk(H) & (v^2 + i^2, v^2 + i^2) & (v, 0) \\
Dove(D) & (0, v) & (v^2 + d, v^2 + d)
\end{array}
\end{equation}

From the payoff matrix (1.1) it is clear that Hawk is a pure ESS if either $(H, H) > (D, H)$ or $(H, H) = (D, H)$ and $(H, D) > (D, D)$. In these conditions we see from payoff matrix (1.1) that if $(H, H) > (D, H)$ then $(v + i) >
0. The second condition implies $v + i = 0$ and $v^2 > d$. As $v^2 > d$ always holds therefore Hawk is pure ESS whenever $v + i \geq 0$. Since $v^2 > d$ so $(H, D) > (D, D)$ and Dove can never be pure ESS [7]. Selecting the values as [7]

<table>
<thead>
<tr>
<th>Value of resource $v$</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Injury to self $i$</td>
<td>-100</td>
</tr>
<tr>
<td>Cost of display $d$</td>
<td>-10</td>
</tr>
<tr>
<td>Resource cost $c$</td>
<td>0</td>
</tr>
</tbody>
</table>

and the payoff matrix (1.1) takes the form

\[
\begin{array}{c|cc}
 & H & D \\
\hline
H & (-25, -25) & (50, 0) \\
D & (0, 50) & (15, 15) \\
\end{array}
\]

It is clear that there is no pure ESS in this game [7].

1.1.1 Mixed ESS

For pure strategies $H$ and $D$ the corresponding fitnesses $W(H)$, $W(D)$ are defined as [7]

\[
\begin{align*}
W(H) &= $(H, H)h + $(H, D)(1 - h) \\
W(D) &= $(D, H)h + $(D, D)(1 - h)
\end{align*}
\]

where $h$ is the classical frequency of the pure strategy $H$. For a mixed ESS both strategies must have same fitness i.e. $W(H) = W(D)$. At such a mixed equilibrium we have [7]

\[
\begin{align*}
h1 - h &= $(D, D) - $(H, D)$(H, H) - $(D, H) \\
and h &= 2d - v2d + i
\end{align*}
\]

Using values from the payoff matrix (1.1) the frequencies of Hawks and Doves are $h = .583$ and $1 - h = .417$ respectively.

2 Quantum Hawk-Dove Game

We use Marinatto and Weber’s scheme [9] to quantize this game assuming that two players, call them Alice and Bob, can play pure strategies $H$ and $D$. The players have the following entangled state at their disposal
\[ |\psi_{in}\rangle = a |HH\rangle + b |DD\rangle + c |HD\rangle + d |DH\rangle \]

where \( |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \) \hspace{1cm} (6)

where first position is reserved for Alice’s strategy and the second for Bob’s. The associated density matrix takes the form

\[
\rho_{in} = |a|^2 |HH\rangle \langle HH| + ab^* |HH\rangle \langle DD| + ac^* |HH\rangle \langle HD| + ad^* |HH\rangle \langle DH|
+ ba^* |DD\rangle \langle HH| + |b|^2 |DD\rangle \langle DD| + bc^* |DD\rangle \langle HD| + bd^* |DD\rangle \langle DH|
+ ca^* |HD\rangle \langle HH| + cb^* |HD\rangle \langle DD| + |c|^2 |HD\rangle \langle HD| + cd^* |HD\rangle \langle DH|
+ da^* |DH\rangle \langle HH| + db^* |DH\rangle \langle DD| + dc^* |DH\rangle \langle HD| + |d|^2 |DH\rangle \langle DH|
\]

Let \( C \) be a unitary and Hermitian operator such that

\[
C |H\rangle = |D\rangle, \ C |D\rangle = |H\rangle, \ C = C^\dagger = C^{-1}
\]

If Alice uses \( I \), the identity operator, with probability \( p \) and \( C \) with probability \( (1-p) \) and Bob uses these operators with probability \( q \) and \( (1-q) \) respectively then final density matrix takes the form \([9]\),

\[
\rho_f = pq I_A \otimes I_B \rho_{in} I_A^\dagger \otimes I_B^\dagger + p(1-q) I_A \otimes C_B \rho_{in} I_A^\dagger \otimes C_B^\dagger
+ q(1-p) C_A \otimes I_B \rho_{in} C_A^\dagger \otimes I_B^\dagger + (1-p)(1-q) C_A \otimes C_B \rho_{in} C_A^\dagger \otimes C_B^\dagger
\]

The payoff operators for Alice and Bob are defined as \([9]\)

\[
P_A = (v2 + i2) |HH\rangle \langle HH| + v |HD\rangle \langle HD| + (v2 + d) |DD\rangle \langle DD| \hspace{1cm} (8)
\]
\[
P_B = (v2 + i2) |HH\rangle \langle HH| + v |DH\rangle \langle DH| + (v2 + d) |DD\rangle \langle DD| \hspace{1cm} (9)
\]

The payoff functions are obtained as mean values of these operators i.e.

\[
SA(p, q) = Tr(P_A \rho_f) \hspace{1cm} SB(p, q) = Tr(P_B \rho_f) \hspace{1cm} (10)
\]

From relations (2) and (2) we obtain expression for the final density matrix which depends on parameters \( p, q, a, b, c, d \). The expected payoff functions for both the players are obtained from eqs. \((10)\)

\[
SA(p, q) = (v2 + i2) [pq |a|^2 + p(1-q) |c|^2 + q(1-p) |d|^2 + (1-p)(1-q) |b|^2]
+ v[pq |c|^2 + p(1-q) |a|^2 + q(1-p) |b|^2 + (1-p)(1-q) |d|^2]
+ (v2 + d)[pq |b|^2 + p(1-q) |d|^2 + q(1-p) |c|^2 + (1-p)(1-q) |a|^2]
\]
the subscripts \( A \) and \( B \) are not necessary so \( \$A(p, q) = \$B(p, q) = \$B(p, q) \). The NE inequality becomes

\[
\$B(p, q) = (v^2 + i2)[pq |a|^2 + p(1 - q) |c|^2 + q(1 - p)(1 - q) |b|^2] \\
+ v[pq |d|^2 + p(1 - q) |b|^2 + q(1 - p) |a|^2 + (1 - p)(1 - q) |c|^2] \\
+ (v^2 + d)[pq |b|^2 + p(1 - q) |d|^2 + q(1 - p) |c|^2 + (1 - p)(1 - q) |a|^2]
\]

with values from matrix (1.1) eq. (2) becomes

\[
\$A(p, q) = p[q \{-60 |a|^2 - 60 |b|^2 + 60 |c|^2 + 60 |d|^2\} \\
- 25 |c|^2 + 25 |b|^2 + 35 |a|^2 - 35 |d|^2] \\
+ q[75 |b|^2 - 75 |d|^2 - 15 |a|^2 + 15 |c|^2] - 25 |b|^2 + 50 |d|^2 + 15 |a|^2 \\
\$B(p, q) = q[p \{-60 |a|^2 - 60 |b|^2 + 60 |c|^2 + 60 |d|^2\} \\
- 25 |d|^2 + 25 |b|^2 + 35 |a|^2 - 35 |c|^2] \\
+ p[75 |b|^2 - 75 |c|^2 + 15 |d|^2 - 15 |a|^2] - 25 |b|^2 + 50 |c|^2 + 15 |a|^2
\]

In a symmetric game an interchange of \( p \) and \( q \) changes \( \$A(p, q) \) into \( \$B(p, q) \). For the general quantum state \( |\psi_{in}\rangle \) (??) it is clear from eq. (2) that an additional restriction of \( c = d \) on the initial state parameters is required to get a symmetric quantum game. We will discuss both these cases.

### 2.1 Symmetric case (c=d)

In a symmetric game \( G = (M, M^T) \), where \( M \) is a square matrix and \( T \) is for transpose, an ESS is a symmetric NE with an additional “stability property” [7]. From eq. (2) the game becomes symmetric when \( c = d \), and the payoff functions take the form

\[
\$A(p, q) = p[q \{-60 |a|^2 - 60 |b|^2 + 120 |c|^2\} + 35 |a|^2 + 25 |b|^2 - 60 |c|^2] \\
+ q[-15 |a|^2 + 75 |b|^2 - 60 |c|^2] + 15 |a|^2 - 25 |b|^2 + 50 |c|^2 \\
\$B(p, q) = q[p \{-60 |a|^2 - 60 |b|^2 + 120 |c|^2\} + 35 |a|^2 + 25 |b|^2 - 60 |c|^2] \\
+ p[-15 |a|^2 + 75 |b|^2 - 60 |c|^2] + 15 |a|^2 - 25 |b|^2 + 50 |c|^2 \tag{11}
\]

Since the game is symmetric, a game where players are anonymous, therefore the subscripts \( A \) and \( B \) are not necessary so \( \$A(p, q) = \$B(p, q) = \$B(p, q) \). The NE inequality becomes

\[
\$B(p^*, q^*) - \$B(p, q) \geq 0 \\
\Rightarrow (p^* - p)[q^* \{-60 |a|^2 - 60 |b|^2 + 120 |c|^2\} + 35 |a|^2 + 25 |b|^2 - 60 |c|^2] \geq 0
\]

Three Nash equilibria arise from this inequality.
2.1.1 Case 1 \((p^* = q^* = 0)\)

The inequality (2.1) requires \(35 |a|^2 + 25 |b|^2 - 60 |c|^2 < 0\). This holds, for example, for \(|a|^2 = 116, |b|^2 = 14, |c|^2 = 1116\). Therefore from eqs. (11)

\[
\begin{align*}
\$(0, 0) &= 15 |a|^2 - 25 |b|^2 + 50 |c|^2 = 46516 \\
\$(p, 0) &= (35 |a|^2 + 25 |b|^2 - 60 |c|^2)p + 15 |a|^2 - 25 |b|^2 + 50 |c|^2 \\
&= 46516 - 52516p.
\end{align*}
\]

So that \(\$(0, 0) > \$(p, 0) \forall 0 < p < 1\). Hence \(p^* = q^* = 0\) is an ESS.

2.1.2 Case 2 \((p^* = q^* = 1)\)

In this case eq. (2.1) demands \(-25 |a|^2 - 35 |b|^2 + 60 |c|^2 > 0\). This inequality holds, for example, for \(|a|^2 = 116, |b|^2 = 18, |c|^2 = 1316\). From eqs. (11)

\[
\begin{align*}
\$(1, 1) &= -25 |a|^2 + 15 |b|^2 + 50 |c|^2 = 65516 \\
\$(p, 1) &= (-25 |a|^2 - 35 |b|^2 + 60 |c|^2)p + 50 |b|^2 - 10 |c|^2 \\
&= -158 + 68516p.
\end{align*}
\]

Since \(\$(1, 1) - \$(p, 1) = 68516(1 - p) > 0 \forall 0 < p < 1\), therefore, \(p^* = q^* = 1\) is an ESS.

2.1.3 Case 3 (mixed ESS)

From inequality (2.1) the mixed NE is \(p^* = q^* = -7 |a|^2 - 5 |b|^2 + 12 |c|^2 12(− |a|^2 - |b|^2 + 2 |c|^2)\).

For classical game, \(|a|^2 = 1\), we get \(p^* = q^* = 712\). In quantum version of the game we can obtain this value, for example, for \(|a|^2 = 12, |b|^2 = |c|^2 = 16\). The initial state, then, takes the form

\[
|\psi_{in}\rangle = 1\sqrt{\sigma}|HH\rangle + 1\sqrt{\sigma}|DD\rangle + 1\sqrt{\sigma}|HD\rangle + 1\sqrt{\sigma}|DH\rangle
\]

Now from eq. (11) \(\$(p^*, q^*) = \$(p, q^*) = 8.75\), \(\$(q, q) = -60 q^2 + 20q + 353\), \(\$(p^*, q) = -600q + 66536\) and \(\$(p^*, q) - \$(q, q) = 720q^2 - 820q + 24536 > 0\) \(\forall 0 < q < 1\). It implies that \(\$(p^*, q) > \$(q, q)\). Therefore \(p^* = q^* = -7 |a|^2 - 5 |b|^2 + 12 |c|^2 12(− |a|^2 - |b|^2 + 2 |c|^2)\) is a mixed ESS for the above values of initial quantum state parameters.

2.2 Asymmetric case \((c \neq d)\)

For asymmetric bi-matrix game \(G = (M, N)\), where \(M\) and \(N\) are square matrices and \(N \neq M^T\) an ESS is defined with strict Nash inequality [10]. In this case a strategy pair \((A^*, B^*)\) is an ESS if NE conditions with strict inequalities.
hold i.e. $\$ _A(A^*, B^*) > \$ _A(A, B^*)$ for all $A \neq A^*$ and $\$ _B(A^*, B^*) > \$ _B(A^*, B)$ for all $B \neq B^*$. Nash inequalities from (2) are then:

$$\$ _A(p^*, q^*) - \$ _A(p, q^*) \geq 0$$

$$\Rightarrow (p^* - p)[q^* \{-60 |a|^2 - 60 |b|^2 + 60 |c|^2 + 60 |d|^2\} + 35 |a|^2 + 25 |b|^2 - 25 |c|^2 - 35 |d|^2] \geq 0$$

$$\Rightarrow (q^* - q)[p^* \{-60 |a|^2 - 60 |b|^2 + 60 |c|^2 + 60 |d|^2\} + 35 |a|^2 + 25 |b|^2 - 35 |c|^2 - 25 |d|^2] \geq 0$$

From these inequalities three Nash equilibria arise.

### 2.2.1 Case 1 ($p^* = q^* = 0$)

From inequalities (2.2), (2.2) and for $p^* = q^* = 0$ we get:

$$35 |a|^2 + 25 |b|^2 - 25 |c|^2 - 35 |d|^2 < 0 \quad (15)$$

$$35 |a|^2 + 25 |b|^2 - 35 |c|^2 - 25 |d|^2 < 0 \quad (16)$$

respectively. Both these inequalities (15) and (16) are satisfied, for example, for $|a|^2 = 116, |b|^2 = 14, |c|^2 = 916, |d|^2 = 18$. Therefore from eq. (2)

$$\$ _A(0,0) = 15 |a|^2 - 25 |b|^2 + 50 |d|^2 = 1516$$

$$\$ _A(p,0) = 1516 + p(35 |a|^2 + 25 |b|^2 - 25 |c|^2 - 35 |d|^2)$$

$$= 1516 - 10p \quad (17)$$

$$\$ _B(0,0) = -25 |b|^2 + 50 |c|^2 + 15 |a|^2 = 36516$$

$$\$ _B(0,q) = 36516 + q(35 |a|^2 + 25 |b|^2 - 35 |c|^2 - 25 |d|^2)$$

$$= 36516 - 23016q \quad (18)$$

Since $\$ _A(0,0) > $\$ _A(p,0), \forall 0 < p < 1$, and $\$ _B(0,0) > $\$ _B(0,q), \forall 0 < q < 1, therefore strict inequality holds and $p^* = q^* = 0$ is an ESS.

### 2.2.2 Case 2 ($p^* = q^* = 1$)

Similarly from inequalities (2.2), (2.2) and for $p^* = q^* = 1$ we get:

$$-25 |a|^2 - 35 |b|^2 + 35 |c|^2 + 25 |d|^2 > 0 \quad (19)$$

$$-25 |a|^2 - 35 |b|^2 + 25 |c|^2 + 35 |d|^2 > 0 \quad (20)$$

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respectively. These inequalities are satisfied, for example, for $|a|^2 = 116$, $|b|^2 = 18$, $|c|^2 = 916$, $|d|^2 = 14$. Hence from eq. (2)

\[
\begin{align*}
\$A(1, 1) &= -25 |a|^2 + 15 |b|^2 + 50 |c|^2 = 45516 \\
\$A(p, 1) &= (-25 |a|^2 + 25 |d|^2 + 35 |c|^2) - 35 |b|^2)p + \\
&\quad (-25 |d|^2 + 50 |b|^2 + 15 |c|^2) = 13516 + 20p
\end{align*}
\]

\[
\begin{align*}
\$B(1, 1) &= -25 |a|^2 + 25 |d|^2 + 15 |b|^2 = 20516 \\
\$B(1, q) &= (-25 |a|^2 + 35 |d|^2 - 35 |b|^2 + 25 |c|^2)q + \\
&\quad 15 |d|^2 + 50 |b|^2 - 25 |c|^2 = 27016q - 6516
\end{align*}
\]

From these equations it is clear that $\$A(1, 1) > $\$A(p, 1) $\forall 0 < p < 1$ and $\$B(1, 1) > $\$B(1, q) $\forall 0 < q < 1$. As strict inequality holds in this case, therefore, $p^* = q^* = 1$ is an ESS.

### 2.2.3 Case 3 (mixed ESS)

For asymmetric case and from inequalities (2.2), (2.2) we have

\[
p^* = -7 |a|^2 - 5 |b|^2 + 7 |c|^2 + 5 |d|^2 \frac{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}, q^* = -7 |a|^2 - 5 |b|^2 + 5 |c|^2 + 7 |d|^2 \frac{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}
\]

Strict inequality does not hold for these values, therefore

\[
p^* = -7 |a|^2 - 5 |b|^2 + 7 |c|^2 + 5 |d|^2 \frac{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}, q^* = -7 |a|^2 - 5 |b|^2 + 5 |c|^2 + 7 |d|^2 \frac{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}
\]

is not an ESS.

### 3 Conclusion

We showed that there exists pure ESS in quantum version of the Hawk-Dove game whereas no such pure ESS can exist in classical version of this game. We found that in quantum two players symmetric Hawk-Dove game an additional restriction on parameters of initial quantum state is needed. We considered the game in both symmetric and asymmetric situations and showed that pure ESS can exist in both whereas mixed ESS only in symmetric situation.

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