Cosmological and black hole brane-world Universes in higher
derivative gravity

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Abstract

General model of multidimensional $R^2$-gravity including Riemann tensor square term (non-zero $c$ case) is considered. The number of brane-worlds in such model is constructed (mainly in five dimensions) and their properties are discussed. Thermodynamics of S-AdS BH (with boundary) is presented when perturbation on $c$ is used. The entropy, free energy and energy are calculated. For non-zero $c$ the entropy (energy) is not proportional to the area (mass). The equation of motion of brane in BH background is presented as FRW equation. Using dual CFT description it is shown that dual field theory is not conformal one when $c$ is not zero. In this case the holographic entropy does not coincide with BH entropy (they coincide for Einstein gravity or $c = 0$ HD gravity where AdS/CFT description is well applied).

Asymmetrically warped background (analog of charged AdS BH) where Lorentz invariance violation occurs is found. The cosmological 4d dS brane connecting two dS bulk spaces is formulated in terms of parameters of $R^2$-gravity. Within proposed dS/CFT correspondence the holographic conformal

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anomaly from five-dimensional higher derivative gravity in de Sitter background is evaluated.

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I. INTRODUCTION

It is quite an old idea going back to Kaluza and Klein that theory of fundamental interactions is some manifestation of the presence of extra dimensions. The investigation of higher dimensional theories has appeared recently in the form of brane-world physics [1,2]. According to this (simplified) picture the fundamental dynamics occurs in $d+1$ dimensional bulk manifold with $d$ dimensional boundary (brane). The evolution of observable universe corresponds to brane.

Brane-world description [1] looks really attractive due to the following reasons. First of all, in most models the gravity on the brane is localized [1]. In other words, despite the fact that gravity is multidimensional one, the Newton law is recovered on the brane. It happens even with large (infinite) extra dimensions. Moreover, brane matter may be localized as well. Second, this approach provides very natural explanation of hierarchy between the gravitational and electroweak mass scales. It opens the window for construction of new classes of unified models from higher dimensional gravity. Third, standard cosmology is modified by the presence of extra dimensions. Moreover, standard Friedmann-Robertson-Walker (FRW) cosmology is recovered at low energies. On the same time the role of brane-world effects to very early universe may be quite significant. In particular, higher dimensions effects may give the origin of dark matter. There is much activity in the study of brane-world cosmology, see refs. [3–5]. Fourth, brane-world physics very naturally appears in frames of string and M-theory. For example, some brane-world cosmological scenarios (like Brane New World [6–8]) are very much connected with AdS/CFT correspondence [9]. Moreover, such approach is really useful to test the holography predictions. In a sense, the very much related elements of new physics (holographic principle, AdS/CFT correspondence, brane-worlds) appeared almost simultaneously.

The priority area is currently the search of realistic brane universes within different theories. The most attention is paid so far to Einstein gravity (with matter). Nevertheless, there is interest in the study of brane universes in higher derivative (HD) gravity [10]. It is not strange as usually at low energies HD gravity approaches to Einstein one, i.e. the difference between these theories is typical only at high energies where brane universe occurs. It is known that four dimensional HD gravity has better ultraviolet behaviour than Einstein one. HD gravitational terms appear in low-energy effective action of string theory as well as in holographic renormalization group [11]. Moreover, some variants of HD gravity may possess the interesting additional symmetry (Weyl gravity). Gauss-Bonnet (GB) gravity which is another variant of HD gravity is typically associated with superstring [12]. It has very attractive ultraviolet behaviour of propagator for number of backgrounds in $d$ dimensions.

The theory we are going to investigate in this work in connection with brane-world approach is multidimensional HD (or $R^2$-) gravity. The construction of different warped spacetimes (brane-worlds) in HD gravity will be given and a number of related questions (thermodynamics of AdS black holes (BH), FRW brane dynamics, de Sitter (dS) brane, etc) will be discussed. The remark may be in order. In four dimensions where a lot of works devoted to HD gravity exists (for review, see [13]) one only needs scalar curvature square and Ricci tensor square terms. Riemann tensor square term may be always discarded using GB topological invariant. In $d$ dimensions, Riemann tensor square term should be kept and as we will see it leads to some new features in the description of AdS BH thermodynamics.
The paper is organized as follows. In the next section the equations of motion for general HD gravity are reviewed in the case of FRW background. Section three is devoted to the construction of boundary terms in $d$ dimensional HD gravity. The role of these terms is to make the variational procedure to be well-defined and to render the classical action to be finite for AdS background. In a sense, such construction is the generalization of Gibbons-Hawking boundary term [14] in Einstein gravity.

In the section four we study thermodynamics of five dimensional AdS BH (with brane) in general model of HD gravity. The entropy, free energy and energy of AdS BH are obtained and compared with the same ones calculated by another method. It is demonstrated that for AdS BH with curved brane the entropy is not proportional to horizon area only if Riemann tensor square term presents in the action. This new, normally non-observed feature of the entropy is presumably caused by above HD correction. In the section five the derivation of brane equation (FRW equation) when brane sits inside AdS BH is done. The dual AdS/CFT description is used. It is demonstrated that presence of Riemann tensor square term breaks the conformal symmetry of dual field theory. It results in the fact that there appears the non-trivial difference between holographic (Hubble) QFT entropy (when brane crosses the horizon) and black hole entropy. In the absence of Riemann tensor square term these entropies coincide for HD as well as for Einstein gravity [15]. In the section six we re-write the results of two previous sections about AdS BH thermodynamics and brane equation for Gauss-Bonnet gravity.

Section seven is devoted to construction of asymmetrically warped spacetime in general HD gravity. Riemann tensor square term produces such background in the way similar to Einstein-Maxwell gravity, however, in our case the Maxwell charge contribution is not necessary. The apparent violation of Lorentz invariance for such background is briefly mentioned.

In the section eight we discuss dS brane solutions in bulk de Sitter space. The explicit construction of such brane-worlds motivated by dS/CFT correspondence is presented. In section nine, assuming dS/CFT correspondence the calculation of central charge (holographic conformal anomaly) from five dimensional HD gravity on dS space is made. In the absence of string framework for dS/CFT correspondence, (somehow speculative) example of specific five dimensional de Sitter HD gravity model which has dual $Sp(N)\mathcal{N}=2$ super Yang-Mills theory description up to the next-to-leading terms in $1/N$ expansion is presented. Summary of results and outlook are given in the last section. Appendix is devoted to description of AdS BH thermodynamics for HD gravity without Riemann tensor square term.

**II. FRW DYNAMICS FROM MULTIDIMENSIONAL $R^2$-GRAVITY**

HD gravity attracts the attention of researchers due to different reasons. Clearly, this is very natural generalization of Einstein gravity. Moreover, HD terms appear in low-energy effective action of superstring theory. On the quantum level, HD gravity seems to be renormalizable in four dimensions (for a review, see [13]) forgetting the unitarity problem (for recent discussion of life with ghosts in HD theories, see [16]).

HD gravity appears also very naturally in brane-world physics and AdS/CFT set-up. Indeed, imagine one substitutes the classical solution into the action of the supergravity on AdS background. The action diverges due to the infinite volume of AdS. In order that
AdS\textsubscript{d+1}/CFT\textsubscript{d} correspondence [9] was well-defined, we need to add the surface term to cancel the divergence. The leading divergence has the form of the cosmological term in \(d\) dimensional spacetime and the next-to-leading term is Einstein-Hilbert action (scalar curvature term). The next-to-next-to-leading terms contain the square of the curvatures, which correspond to the trace anomaly when \(d = 4\). In the context of the Brane New World [6,7], the gravity localized on the brane in the Randall-Sundrum model [1] corresponds to the remnant after cancelling the leading divergent cosmological-like term. Therefore the gravity on the brane always contains the \(R^2\)-gravity as the correction to the Einstein gravity.

Let us review FRW-dynamics in general multidimensional HD gravity. The general action of \(d\) dimensional \(R^2\)-gravity with cosmological constant and matter is given by:

\[
S = \int d^d x \sqrt{-g} \left\{ a R^2 + b R_{\mu\nu} R^{\mu\nu} + c R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} + \frac{1}{\kappa^2} R - \Lambda + L_{\text{matter}} \right\} .
\] (1)

Here \(L_{\text{matter}}\) is the Lagrangian density for the matter fields. By the variation over the metric tensor \(g_{\mu\nu}\), we obtain the following equation

\[
0 = \frac{1}{2} g^{\mu\nu} \left( a R^2 + b R_{\mu\rho} R^{\mu\rho} + c R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} + \frac{1}{\kappa^2} R - \Lambda \right) + a \left( -2 R R^{\mu\nu} + \nabla^\mu \nabla^\nu R + \nabla^\nu \nabla^\mu R - 2 g^{\mu\nu} \nabla^\rho \nabla^\rho R \right) + b \left( -2 R_{\mu}^\rho R^\rho_{\nu} + \nabla^\rho \nabla^\nu R^\rho_{\mu} + \nabla^\nu \nabla^\mu R^\rho_{\nu} - \Box R^\rho_{\mu} - g^{\mu\nu} \nabla^\rho \nabla^\sigma R^\rho_{\sigma} \right) + c \left( -4 R^{\mu\rho\sigma\tau} R^\rho_{\sigma\tau} - 2 \nabla^\rho \nabla^\sigma R^\mu_{\rho\sigma\tau} - 2 \nabla^\rho \nabla^\sigma R^\mu_{\rho\sigma} \right) - \frac{1}{\kappa^2} R^{\mu\nu} - T^{\mu\nu}_{\text{matter}} .
\] (2)

Here \(T^{\mu\nu}_{\text{matter}}\) is the energy-momentum tensor of the matter fields. As we are interested in the cosmological problem, we choose the spacetime metric in the following form:

\[
ds^2 = -dt^2 + l^2 e^{2 A(t)} \tilde{g}_{ij}(x^k) dx^i dx^j .
\] (3)

Here \(\tilde{g}_{ij}\) is the metric of the \(d - 1\) dimensional Euclidean Einstein manifold defined by

\[
\tilde{R}_{ij} = k \tilde{g}_{ij} .
\] (4)

Here \(k\) is a constant and \(\tilde{R}_{ij}\) is the Ricci tensor defined by \(\tilde{g}_{ij}\). In the following, we denote the quantities given by \(\tilde{g}_{ij}\) using \(\tilde{\cdot}\). The natural assumption is

\[
\tilde{R}_{ijkl} = \frac{k}{d-1} (\tilde{g}_{ik} \tilde{g}_{jl} - \tilde{g}_{il} \tilde{g}_{jk}) .
\] (5)

Taking the metric as in (3), one gets the following expressions for the connection and the curvatures:

\[
\Gamma^t_{it} = \Gamma^t_{ti} = \Gamma^t_{it} = \Gamma^i_{tt} = 0 , \quad \Gamma^t_{ij} = l^2 e^{2A} A_t \tilde{g}_{ij} ,
\]
\[
\Gamma^i_{jt} = \Gamma^i_{tj} = \delta^i_j A_t , \quad \Gamma^i_{jk} = \Gamma^i_{jk} ,
\]
\[
R_{itlj} = -R_{ljit} = -R_{litj} = R_{itlj} = -e^{2A} \left( A_{tt} + A_{t}^2 \right) \tilde{g}_{ij} ,
\] (6)
\[ R_{ijkl} = \left( \frac{k}{d-2} l^2 e^{2A} + l^4 e^{4A} A_{,t}^2 \right) \left( \tilde{g}_{ik} \tilde{g}_{jl} - \tilde{g}_{il} \tilde{g}_{jk} \right) , \]
other components of Riemann tensor = 0 ,
\( R_{tt} = -(d-1) \left( A_{,tt} + A_{,t}^2 \right) , \quad R_{tt} = R_{tt} = 0 , \)
\[ R_{ij} = \left( k + l^2 e^{2A} \left( A_{,tt} + (d-1) A_{,t}^2 \right) \right) \tilde{g}_{ij} , \]
\[ R = (d-1)k l^{-2} e^{-2A} + 2(d-1)A_{,tt} + d(d-1)A_{,t}^2 . \]

Here the following conventions of curvatures are used

\[ R = g^{\mu \nu} R_{\mu \nu} , \]
\[ R_{\mu \nu} = R^{\lambda}_{\mu \lambda \nu} , \]
\[ R^{\lambda}_{\mu \nu \rho} = -\Gamma^{\lambda}_{\mu \rho , \nu} + \Gamma^{\lambda}_{\nu \rho , \mu} - \Gamma^{\eta}_{\mu \nu} \Gamma^{\lambda}_{\rho \eta} + \Gamma^{\eta}_{\nu \rho} \Gamma^{\lambda}_{\mu \eta} , \]
\[ \Gamma^{\eta}_{\mu \rho} = \frac{1}{2} g^{\mu \nu} (g_{\nu \lambda, \rho} + g_{\lambda \nu, \rho} - g_{\lambda \nu, \rho} \gamma^{\lambda}_{\mu} \rho) . \]

When \( a = b = c = 0 \) the \((\mu, \nu) = (t, t)\) component in (2) corresponds to the first FRW equation and \((i, j)\) component to the second one.

The \((t, t)\) component in (2) gives

\[ 0 = -\frac{(d-1)k}{2 \kappa^2 r^2} - \frac{(d-1)(d-2)}{2 \kappa^2} H^2 - \rho \]
\[ + \left( 2(d-1)^2 a + \frac{d(d-1)}{2} b + 2(d-1)c \right) H_t^2 
+ \left( -4(d-1)^2 a - d(d-1)b - 4(d-1) \right) HH_{,tt} 
+ \left( -4(d-1)^3 a - d(d-1)^2 b - 4(d-1)^2 \right) H^2 H_{,t} 
+ \left( \frac{d(d-1)^2(d-4)}{2} a - \frac{(d-1)^2(d-4)}{2} b - (d-1)(d-4)c \right) H^4 
+ \frac{k}{r^2} \left( -(d-6)(d-1)^2 a + 2(d-1)b + 2(d-1)c \right) H^2 
+ \left( -\frac{(d-1)^2}{2} a - \frac{d-1}{2} b - \frac{d-1}{2} c \right) \frac{k^2}{r^4} , \]

which gives a generalization of the first FRW equation for the presence of HD terms. Here

\[ r = le^{A} , \quad H = A_{,t} . \]

and \( \rho \) is the energy density. On the other hand, \((\mu, \nu) = (i, j)\) components give

\[ 0 = \frac{1}{\kappa^2} \left\{ \frac{(d-3)k}{2 r^4} + \frac{(d-2)H_{,t}}{r^2} + \frac{(d-1)(d-2)}{2} H^2 \right\} - \rho 
+ (4(d-1)a + db + 4c) H_{,,tt} 
+ \left( 8(d-1)^2 a + 2d(d-1)b + 8(d-1)c \right) HH_{,,tt} \]
\[ + \left( 6(d-1)^2 a + \frac{3d(d-1)}{2} b + 6(d-1) c \right) H^2_+ \]
\[ + \left( (6d^2 - 16d + 4)(d-1) a + (d^2 + d - 8)(d-1) b \right) \]
\[ + (4d^2 - 4d - 12)c H^2 H_t \]
\[ + \left( \frac{d(d-1)^2(d-4)}{2} a + \frac{(d-4)(d-1)^2}{2} b + (d-1)(d-4)c \right) H^4 \]
\[ + \frac{k}{r^2} \left\{ 2(d-1)(d-6) a - 4b - 4c \right\} H_t \]
\[ + \left\{ (d-1)(d^2 - 9d + 18) a - 2(d-3)b - 2(d-3)c \right\} H^2 \}
\[ + \frac{k^2}{r^4} \left( \frac{(d-1)^2}{2} a + \frac{d-1}{2} b + \frac{d-1}{d-2} c \right) . \]

(13)

Here \( p \) is the pressure. When \( a = b = c = 0 \), Eqs.(11) and (13) reduce into the usual FRW equations

\[ 0 = -\frac{(d-1)k}{2\kappa^2 r^2} - \frac{(d-1)(d-2)}{2\kappa^2} H^2 - \rho \]

\[ 0 = \frac{1}{\kappa^2} \left\{ -\frac{k}{r^4} + \frac{(d-2)H_t}{r^2} \right\} - (\rho + p) . \]

(14)

(15)

When \( d = 4 \), Eqs.(11) and (13) have the following forms:

\[ 0 = -\frac{3k}{2\kappa^2 r^2} - \frac{3}{\kappa^2} H^2 - \rho \]
\[ + (3a + b + c) \left( 6H^2_+ - 12H H_{tt} - 36H^2 H_t + 6H^4 \right) \]
\[ + \frac{6kH^2}{r^2} - \frac{3k^2}{2r^4} \] , \quad (16)
\[ 0 = \frac{1}{\kappa^2} \left\{ \frac{k}{2r^4} + \frac{2H_t}{r^2} + 3H^2 \right\} - p \]
\[ + (3a + b + c) \left\{ 4H_{tt} + 24H H_{tt} + 18H^2 + 36H^2 H_t \right\} \]
\[ + \frac{k}{r^2} \left( -4H_t - 2H^2 \right) + \frac{3k^2}{2r^4} \] . \quad (17)

Especially if we choose the combination of \( R^2 \) terms in (1) to be proportional to the Gauss-Bonnet invariant \( G \) or to the square of the Weyl tensor \( F \)

\[ G = R^2 - 4R_{\mu\nu}R_{\mu\nu} + R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} , \]
\[ F = \frac{1}{3} R^2 - 2R_{\mu\nu}R_{\mu\nu} + R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \] , \quad (18)

that is, \( a = c, b = -4a \) or \( a = \frac{2}{3}, b = -2a \), the coefficient \( 3a + b + c \) vanishes, therefore \( R^2 \) terms do not give any contribution. As we mentioned first, in the Brane New World scenario,
$R^2$ terms are produced by the trace anomaly, therefore $R^2$ terms are given by a combination of the Gauss-Bonnet invariant and the square of the Weyl tensor, therefore, they do not give any contribution. The situation is true if we start with the bulk (5d) $R^2$-gravity, the holographic trace anomaly is the combination of the Gauss-Bonnet invariant and the square of the Weyl tensor [17]. Therefore it would be natural that the form of the FRW equation itself does not change even if we start with the bulk $R^2$-gravity [18].

Hence, we reviewed $d$ dimensional FRW dynamics from HD gravity. The corresponding equations will be used in the next sections.

III. BOUNDARY TERMS IN BULK $R^2$-GRAVITY

In section 2, $R^2$-gravity has been considered as an effective theory on the $d$ dimensional brane in the $d + 1$ dimensional bulk space. In this section, we consider $d + 1$ dimensional $R^2$-gravity as the gravity in the bulk AdS spacetime. Bulk $R^2$-gravity can appear as the next-to-leading $1/N$ correction in AdS/CFT correspondence.

The problem of surface (boundary) counterterms in multidimensional HD gravity already has been discussed in [19]. In this section, we reconsider the counterterms corresponding to the Gibbons-Hawking one in the Einstein gravity, in $R^2$-gravity and make the arguments more rigorous. We start from the general $R^2$ part of the total action of $d + 1$ dimensional $R^2$-gravity:

$$ S_{R^2} = \int \sqrt{-G} \left\{ a \hat{\mathcal{R}}^2 + b \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + c \hat{R}_{\mu\nu\xi\sigma} \hat{R}^{\mu\nu\xi\sigma} \right\} . $$

(19)

By introducing the auxiliary fields $\hat{A}$, $\hat{B}_{\mu\nu}$, and $\hat{C}_{\mu\nu\rho\sigma}$, one can rewrite the action (19) in the following form:

$$ \tilde{S}_{R^2} = \int \sqrt{-\hat{G}} \left\{ a \left( 2 \hat{A} \hat{R} - \hat{A}^2 \right) + b \left( 2 \hat{B}_{\mu\nu} \hat{R}^{\mu\nu} - \hat{B}_{\mu\nu} \hat{B}^{\mu\nu} \right) 
+ c \left( 2 \hat{C}_{\mu\nu\xi\sigma} \hat{R}^{\mu\nu\xi\sigma} - \hat{C}_{\mu\nu\xi\sigma} \hat{C}^{\mu\nu\xi\sigma} \right) \right\} . $$

(20)

Using the equation of the motion

$$ \hat{A} = \hat{R} , \quad \hat{B}_{\mu\nu} = \hat{R}_{\mu\nu} , \quad \hat{C}_{\mu\nu\rho\sigma} = \hat{R}_{\mu\nu\rho\sigma} , $$

(21)

we find the action (20) is equivalent to (19). Let us impose a Dirichlet type boundary condition, which is consistent with (21), $\hat{A} = \hat{R} \big|_{\text{at the boundary}}$, $\hat{B}_{\mu\nu} = \hat{R}_{\mu\nu} \big|_{\text{at the boundary}}$, and $\hat{C}_{\mu\nu\rho\sigma} = \hat{R}_{\mu\nu\rho\sigma} \big|_{\text{at the boundary}}$ and $\delta \hat{A} = \delta \hat{B}_{\mu\nu} = \delta \hat{C}_{\mu\nu\rho\sigma} = 0$ on the boundary. However, the conditions for $\hat{B}_{\mu\nu}$ and $\hat{C}_{\mu\nu\rho\sigma}$ are, in general, inconsistent. For example, even if $\delta B_{\mu\nu} = 0$, we have $\delta B_{\mu} = \delta G^{\nu\rho} B_{\mu\rho} \neq 0$. Then one can impose boundary conditions on the scalar quantities:

$$ \hat{A} = \hat{B}_{\mu} = \hat{R} , \quad n^{\mu} n^{\nu} \hat{B}_{\mu\nu} = n^{\mu} n^{\nu} \hat{C}_{\mu\nu\rho} = n^{\mu} n^{\nu} \hat{R}_{\mu\nu} . $$

(22)

and
\[ \delta \hat{A} = \delta \left( \hat{B}_\mu \right) = \delta \left( n^\mu n^\nu \hat{B}_{\mu \nu} \right) = \delta \left( n^\mu n^\nu \hat{C}_{\mu \nu \rho} \right) = 0. \quad (23) \]

Here \( n^\mu \) is a unit vector perpendicular to the boundary. The equations given by the variations over other components could be automatically satisfied by the coordinate or gauge choice, as we will see later.

Using the conventions of curvatures in (10), one can further rewrite the action (20) in the following form:

\[
\tilde{S}_{R^2} = 2 \int_{\text{surface}} d^d x \sqrt{-\hat{g}} \left( -\hat{\Gamma}^\lambda_{\mu \rho} n_\nu + \hat{\Gamma}^\lambda_{\mu \nu} n_\rho \right) \\
\times \left( a \delta^\rho_{\lambda} \hat{G}^{\mu \nu} \hat{A} + b \delta^\rho_{\lambda} \hat{B}^{\mu \nu} + c \hat{C}_{\lambda \mu \nu} \right) \\
+ \int d^{d+1} \left[ \cdots \right]. \quad (24)
\]

Here \( \hat{g}_{mn} \) is the boundary metric induced by \( \hat{G}_{\mu \nu} \). Now the bulk part of the action denoted by \( \left[ \cdots \right] \) does not contain the second order derivative of \( \hat{G}_{\mu \nu} \). Then the variational principle becomes well-defined if we add the following boundary term to the Einstein action:

\[
\tilde{S}_{\text{bdry}} = -2 \int_{\text{surface}} d^d x \sqrt{-\hat{g}} \left( -\hat{\Gamma}^\lambda_{\mu \nu} n_\rho + \hat{\Gamma}^\lambda_{\mu \nu} n_\rho \right) \\
\times \left( a \delta^\rho_{\lambda} \hat{g}^{\mu \nu} \hat{A} + b \delta^\rho_{\lambda} \hat{B}^{\mu \nu} + c \hat{C}_{\lambda \mu \nu} \right). \quad (25)
\]

The action (25) breaks the general covariance. We should note, however, that

\[
\nabla_\mu n_\nu = \partial_\mu n_\nu - \hat{\Gamma}^\lambda_{\mu \nu} n_\lambda, \quad \nabla_\mu n_\nu = \partial_\mu n_\nu + \hat{\Gamma}^\nu_{\mu \lambda} n_\lambda. \quad (26)
\]

Then at least for the following metric

\[
ds^2 = \left( 1 + \mathcal{O}(q^2) \right) dq^2 + \hat{g}_{mn}(q, x^m) dx^m dx^n. \quad (27)
\]

one can write the boundary action (25) as

\[
\tilde{S}_{R^2 \text{ bdry}} = \int d^d x \sqrt{-\hat{G}} \left[ 4a \nabla_\mu n^\mu \hat{A} + 2b \left( n_\mu n_\nu \nabla^{\lambda} n_\lambda + \nabla \nabla_\mu \hat{B}_{\mu \nu} \right) \right. \\
+ \left. 4c n_{\sigma} n_\nu \nabla_\nu n_\rho \hat{C}^{\rho \mu \nu} \right]. \quad (28)
\]

Choosing \( d + 1 = 5 \), we start with the following bulk action:

\[
S = \int d^5 x \sqrt{-\hat{G}} \left\{ a \left( 2\hat{A} \hat{R} - \hat{A}^2 \right) + b \left( 2\hat{B}_{\mu \nu} \hat{R}^{\mu \nu} - \hat{B}_{\mu \nu} \hat{B}^{\mu \nu} \right) \right. \\
+ \left. c \left( 2\hat{C}_{\mu \nu \xi \sigma} \hat{R}^{\mu \nu \xi \sigma} - \hat{C}_{\mu \nu \xi \sigma} \hat{C}^{\mu \nu \xi \sigma} \right) + \frac{1}{\kappa^2} \hat{R} - \Lambda \right\}. \quad (29)
\]

We also add the surface terms \( S^{(1)}_b \) corresponding to Gibbons-Hawking surface term and (28) as well as \( S^{(2)}_b \) which is the leading counterterm corresponding to the vacuum energy on the brane:
We should note that only the components of $\hat{\mathbf{s}}$ space time as follows:

$$S_b = S_b^{(1)} + S_b^{(2)}$$

$$S_b^{(1)} = \int d^4x \sqrt{-\hat{g}} \left[ 4a \nabla_\mu n^\mu \hat{A} + 2b \left( n_\mu n_\nu \nabla_\sigma n^\sigma + \nabla_\mu n_\nu \right) \hat{B}^{\mu\nu} + 8c n_\mu n_\nu \nabla_\tau n_\sigma \hat{C}^{\mu\tau\nu\sigma} + \frac{2}{\kappa^2} \nabla_\mu n^\mu \right]$$

$$S_b^{(2)} = -\eta \int d^4x \sqrt{-\hat{g}} .$$

(30)

Here $\eta$ is a constant, which is determined later. We now take the metric of five dimensional space time as follows:

$$ds^2 = dz^2 + e^{2A(z,\sigma)} \sum_{i,j=1}^4 \tilde{g}_{ij} dx^i dx^j , \quad \tilde{g}_{\mu\nu} dx^\mu dx^\nu \equiv l^2 \left( d\sigma^2 + d\Omega_3^2 \right) .$$

(31)

Under the choice of metric in (31), the curvature components are:

$$\hat{R}_{zz} = e^{2A} \left( -A_{zz} - (A_z)^2 \right) \tilde{g}_{ij}$$

$$\hat{R}_{z\sigma} = -l^2 e^{2A} A_{z\sigma} g_{\sigma\alpha}^{AB}$$

$$\hat{R}_{\sigma\sigma} = \left( -l^2 e^{2A} (A_{z\sigma} - l^2 e^{4A} (A_z)^2) \right) g_{\sigma\alpha}^{AB}$$

$$\hat{R}_{ABCD} = \left( l^2 e^{2A} - l^2 e^{2A} (A_z)^2 - l^4 e^{4A} (A_z)^2 \right) \left( g_{AC}^{AB} g_{BD}^{AB} - g_{AD}^{AB} g_{BC}^{AB} \right)$$

$$\hat{R}_{zz} = 4 \left( -A_{zz} - (A_z)^2 \right)$$

$$\hat{R}_{z\sigma} = -3A_{z\sigma}$$

$$\hat{R}_{\sigma\sigma} = l^2 e^{2A} \left( -A_{zz} - 4 (A_z)^2 \right) - 3A_{z\sigma}$$

$$\hat{R}_{AB} = \left( l^2 e^{2A} - A_{zz} - 4 (A_z)^2 \right) - A_{\sigma\sigma} - 2 (A_\sigma)^2 + 2 \right) g_{\sigma\alpha}^{AB}$$

$$\hat{R} = -8A_{zz} - 20 (A_z)^2 + l^2 e^{-2A} \left( -6A_{z\sigma} - 6(A_\sigma)^2 + 6 \right) .$$

(32)

Here $\gamma_{\mu\nu\ldots} \equiv \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \cdots (\cdot)$. We also write the metric of $S_3$ in the following form:

$$d\Omega_3^2 = \sum_{A,B=1}^3 g_{AB}^{ij} dx^A dx^B .$$

(33)

One gets that $n^\mu$ and the covariant derivative of $n^\mu$ are

$$n^\mu = \delta^\mu_\rho , \quad \nabla_i n^j = \delta^\rho_i A_z (\text{others} = 0) .$$

(34)

Then the action $S_b$ (30) looks like:

$$S_b = l^4 \int d^4x e^{4A} \sqrt{\tilde{g}} \left[ \left( 16a \hat{A} + 2b \left( 4\hat{B}_{zz} + \hat{B}_i^i \right) \right) + 8c \hat{C}_{zz} \hat{C}_i^i \right] \right] .$$

(35)

We should note that only the components of $\hat{A}, \hat{B}_{\mu\nu},$ and $\hat{C}_{\mu\nu\rho\sigma}$ in (22) or (23) appeared in (35) since $\hat{B}_{zz} = n^\mu n^\nu B_{\mu\nu}, \hat{B}_i^i = B_\mu^i - n^\mu n^\nu \hat{B}_{\mu\nu},$ and $\hat{C}_{zz} \hat{C}_i^i = n^\mu n^\nu \hat{C}_{\mu\nu\rho\sigma}$. Therefore we need not to consider the variation of these auxiliary fields on the boundary.
From the variation over $A$, one obtains the following equation on the brane, which lies at $z = z_0$:

$$
\delta S \big|_{z = z_0} = 2V_3 l^4 \int d\sigma \sqrt{g} e^{4A} \times \left\{ -16a \hat{A} - 2b \left( 4 \hat{B}_{zz} + \hat{B}_i^i \right) - 8c \hat{C}_{ziz}^i - \frac{8}{\kappa^2} \right\} \delta A, \\
+ \left\{ 16a \hat{A}, z + 2b \left( 4 \hat{B}_{zz,z} + \hat{B}_i^i, z \right) + 8c \hat{C}_{ziz}, z \right\} \\
+ \left( -16a \hat{A} + b \left( 16 \hat{B}_{zz} - 8 \hat{B}_i^i \right) \\
+ c \left( 16 C_{ziz}^i - 8 C_{ij}^i \right) + \frac{8}{\kappa^2} \right\} \delta A, \right. \\
\delta S_b \big|_{z = z_0} = V_3 l^4 \int d\sigma \sqrt{g} e^{4A} \left[ \left\{ 16a \hat{A} + 2b \left( 4 \hat{B}_{zz} + \hat{B}_i^i \right) + 8c \hat{C}_{ziz}^i + \frac{8}{\kappa^2} \right\} \delta A, \\
+ 4 \left\{ \left( 16a \hat{A} + 2b \left( 4 \hat{B}_{zz} + \hat{B}_i^i \right) + 8c \hat{C}_{ziz}^i - \frac{8}{\kappa^2} \right) A, z + \eta \right\} \delta A \right]. \\
\right. \\
\tag{36}
\delta (S + 2S_b) \big|_{z = z_0} = 2V_3 l^4 \int d\sigma \sqrt{g} e^{4A} \left[ \left\{ 16a \hat{A}, z + 2b \left( 4 \hat{B}_{zz,z} + \hat{B}_i^i, z \right) + 8c \hat{C}_{ziz}, z \right\} \\
+ \left( 48a \hat{A} + 48b \hat{B}_{zz} + c \left( 48 C_{ziz}^i - 8 C_{ij}^i \right) + \frac{24}{\kappa^2} \right) A, z + 4\eta \right\} \delta A \right]. \\
\tag{37}
\right.
\right.
\right.

The factors 2 in front of $S_b$ and $V_3$ come from the fact that we are considering two bulk spaces, which have one common boundary or brane. $V_3$ is the volume of the unit 3 sphere:

$$
V_3 = \int d^3 x \sqrt{g} = 2\pi^2. \\
\tag{39}
$$

Since the terms containing $\delta A, z$ do not appear, the variational principle is well-defined. Then one obtains the following equation on the boundary or brane

$$
0 = 16a \hat{A}, z + 2b \left( 4 \hat{B}_{zz,z} + \hat{B}_i^i, z \right) + 8c \hat{C}_{ziz}, z \\
+ \left( 48a \hat{A} + 48b \hat{B}_{zz} + c \left( 48 C_{ziz}^i - 8 C_{ij}^i \right) + \frac{24}{\kappa^2} \right) A, z + 4\eta \\
= 16a R, z + 2b \left( 4 R_{zz,z} + R_i^i, z \right) + 8c \hat{C}_{ziz}, z \\
+ \left( 48a \hat{A} + 48b \hat{B}_{zz} + c \left( 56 R_{zz} - 8 R_i^i \right) + \frac{24}{\kappa^2} \right) A, z + 4\eta. \\
\tag{40}
$$

In the second line, (21) is used. Especially taking the background as AdS-Schwarzschild black hole, one gets

$$
\hat{R} = \frac{-20}{l^2}, \quad \hat{R}_{\mu \nu} = \frac{-4}{l^2} g_{\mu \nu}. \\
\tag{41}
$$

Here $l$ is the radius of the asymptotic AdS space, given by solving the equation
Then from (40), we have

\[ 0 = -3 \left( \frac{320a}{l^2} + \frac{64b}{l^2} + \frac{32c}{l^2} - \frac{8}{\kappa^2} \right) A_{,z} + 4\eta . \]  

(43)

The parameter \( \eta \) is not the free parameter. It can be determined from the condition that the leading order divergence of the action, which appears when the brane goes to infinity in the asymptotic AdS space, is cancelled. If we consider the asymptotic anti de Sitter space:

\[ ds^2 \sim -e^{2\rho_0} dt^2 + e^{-2\rho_0} dr^2 + r^2 \sum_{i,j} g_{ij} dx^i dx^j , \]

\[ e^{2\rho_0} = \frac{1}{r^2} \left( \frac{kr^2}{2} + \frac{r^4}{l^2} \right) , \]  

(44)

one finds

\[ S = \int d^4x r_0^4 \frac{1}{4} \left\{ \frac{400a}{l^4} + \frac{80b}{l^4} + \frac{40c}{l^4} - \frac{20}{\kappa^2 l^2} - \Lambda \right\} + \mathcal{O} \left( r_0^3 \right) \]

\[ = \int d^4x r_0^4 \left\{ \frac{80a}{l^4} + \frac{16b}{l^4} + \frac{8c}{l^4} - \frac{2}{\kappa^2 l^2} \right\} + \mathcal{O} \left( r_0^3 \right) \]

\[ S_b = - \int d^4x r_0^d \left\{ \frac{320a}{l^2} + \frac{64b}{l^2} + \frac{32c}{l^2} - \frac{8}{l^2 \kappa^2} - \frac{\eta}{l} \right\} + \mathcal{O} \left( r_0^3 \right) . \]  

(45)

Here we assume there is a brane at \( r = r_0 \). In the second line, we have deleted \( \Lambda \) by using (42). Then one gets

\[ \frac{\eta}{l} = \frac{240a}{l^4} + \frac{48b}{l^4} + \frac{24c}{l^4} - \frac{6}{\kappa^2 l^2} \]  

(46)

Then combining Eq.(43) and (46), one obtains a simple equation in the form which appeared in the previous papers [6,18]

\[ 0 = A_{,z} - \frac{1}{l} . \]  

(47)

As a special case, we can consider the case that the \( R^2 \) terms are given by the combination of the Gauss-Bonnet term (18) in four dimensions:

\[ b = -4a , \quad c = a . \]  

(48)

Then Eqs.(42), (43) and (46) have the following forms:

\[ 0 = \frac{24a}{l^4} - \frac{12}{\kappa^2 l^2} - \Lambda \]  

(49)

\[ 0 = -3 \left( \frac{96a}{l^2} - \frac{8}{\kappa^2} \right) A_{,z} + 4\eta \]  

(50)

\[ \frac{\eta}{l} = \frac{72a}{l^4} - \frac{6}{\kappa^2 l^2} . \]  

(51)
Hence, the surface counterterms (which are useful for regularization of thermodynamical quantities) in bulk HD gravity around asymptotically AdS space are derived. These counterterms will be used later on when the discussion of FRW-dynamics induced by brane will be presented.

**IV. THERMODYNAMICS AND ENTROPY OF BULK ADS BLACK HOLE**

In this section we will be interested in thermodynamics of AdS BH with non-trivial boundary (brane) in bulk $R^2$-gravity. The case is considered when HD terms contain the Riemann tensor square term, i.e. $R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$. The calculation of thermodynamical quantities like mass and entropy will be necessary in order to relate them with the corresponding ones in brane FRW Universe. In section 5, by using the brane equation (47) obtained in section 3, we find that the dynamics of the brane is given by the FRW-like equation. From the FRW-like equation, we can find the thermodynamical quantities and we will compare the quantities with those obtained in this section.

The general action of $d + 1$ dimensional $R^2$-gravity is given by (1). When $c = 0$, Schwarzschild-anti de Sitter space is an exact solution:

$$ds^2 = \hat{G}_{\mu\nu}dx^\mu dx^\nu$$

$$= -e^{2\rho_0}dt^2 + e^{-2\rho_0}dr^2 + r^2 \sum_{i,j} g_{ij}dx^i dx^j ,$$

$$e^{2\rho_0} = \frac{1}{r^{d-2}} \left( -\mu + k r^{d-2} + \frac{r^d}{l^2} \right).$$

For non-vanishing $c$, such an S-AdS BH solution may be constructed perturbatively [20]. In this case, it is useful to establish the higher-derivative AdS/CFT correspondence [21] and find the strong coupling limit of super Yang-Mills theory with two supersymmetries in next-to-leading order. In this section, we only consider the case $a = b = 0$ (see also Appendix) for simplicity:

$$S = \int d^{d+1}x \sqrt{-\hat{G}} \left\{ c\hat{R}_{\mu\nu\xi\sigma}\hat{R}^{\mu\nu\xi\sigma} + \frac{1}{\kappa^2} \hat{R} - \Lambda \right\} .$$

When we assume the metric (52) with $\mu = 0$, the scalar, Ricci and Riemann curvatures are given by

$$\hat{R} = -\frac{d(d+1)}{l^2} , \quad \hat{R}_{\mu\nu} = -\frac{d}{l^2} G_{\mu\nu} , \quad \hat{R}_{\mu\nu\xi\sigma} = -\frac{1}{l^2} \left( \hat{G}_{\mu\xi}\hat{G}_{\nu\sigma} - \hat{G}_{\mu\sigma}\hat{G}_{\nu\xi} \right) ,$$

which tell that the curvatures are covariantly constant. The equation of the motion derived from the action (53) is:

$$0 = -\frac{1}{2} \hat{G}_{\zeta\xi} \left\{ c\hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma} + \frac{1}{\kappa^2} \hat{R} - \Lambda \right\}$$

$$+ 2c \hat{R}_{\zeta\mu\nu}\hat{R}^{\mu\nu}_{\xi} + \frac{1}{\kappa^2} \hat{R}_{\zeta\xi} + 4c D_\rho D_\kappa \hat{R}^{\rho\kappa}_{\zeta\xi} .$$
Then substituting Eqs.(54) into (55), one finds the relation between $c$, $\Lambda$ and $l$

$$0 = \frac{2c}{l^4}d(d - 3) - \frac{d(d - 1)}{\kappa^2 l^2} - \Lambda,$$  

(56)

which defines the radius $l$ of the asymptotic AdS space even if $\mu \neq 0$. For $d + 1 = 5$ with $\mu \neq 0$, using Eq.(55), we get the perturbative solution from (52), which looks like [21]:

$$e^{2\rho} = \frac{1}{r^2} \left\{ -\mu + \frac{k}{2} r^2 + \frac{r^4}{l^2} + \frac{2\mu^2 \epsilon}{r^4} \right\}, \quad \epsilon = c\kappa^2.$$  

(57)

Suppose that $g_{ij}$ (52) corresponds to the Einstein manifold, defined by $r_{ij} = kg_{ij}$, where $r_{ij}$ is Ricci tensor defined by $g_{ij}$ and $k$ is the constant. For example, if $k > 0$ the boundary can be three dimensional sphere, if $k < 0$, hyperboloid, or if $k = 0$, flat space. Properly normalizing the coordinates, one can choose $k = 2$, 0, or $-2$. Note that as it will be shown in next section dual QFT corresponding to above background is not CFT. Probably, the fact that there is no exact S-AdS solution of non-zero c HD gravity is manifestation of this property.

The calculation of thermodynamical quantities like free energy $F$, the entropy $S$ and the energy $E$ may be done following [21]. After Wick-rotating the time variable by $t \to i\tau$, the free energy $F$ can be obtained from the action $S$ (53), where the classical solution is substituted:

$$F = -TS.$$

(58)

Multiplying $\hat{G}^{\xi\xi}$ to (55) in case that $D_\rho D_\kappa \hat{R}_\xi^{\rho\kappa} = \mathcal{O}(\epsilon)$ as in the solution (57), we find for $d = 4$

$$\frac{1}{\kappa^2} \hat{R} = -\frac{c}{3} \hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma} + \frac{5}{3} \Lambda + \mathcal{O}(\epsilon^2).$$

(59)

Substituting (59) into the action (53), one arrives to the following expression:

$$S = \int d^5x \sqrt{-\hat{G}} \left\{ \frac{2}{3} c \hat{R}_{\mu\nu\xi\sigma} \hat{R}^{\mu\nu\xi\sigma} + \frac{2}{3} \Lambda \right\}.$$  

(60)

Since

$$\hat{R}_{\mu\nu\xi\sigma} \hat{R}^{\mu\nu\xi\sigma} = \frac{40}{l^2} + \frac{72\mu^2}{r^8} + \mathcal{O}(\epsilon),$$

(61)

and using (56) with $d = 4$, we obtain

$$S = -\int d^5x \sqrt{-\hat{G}} \left( \frac{8}{\kappa^2 l^2} - \frac{32c}{l^4} + \frac{48c\mu^2}{r^8} \right)$$

$$= -\frac{V_3}{T} \int_{r_H}^{\infty} dr r^3 \left( \frac{8}{\kappa^2 l^2} - \frac{32c}{l^4} + \frac{48c\mu^2}{r^8} \right).$$

(62)

Here $V_3$ is the volume of 3d sphere and we assume $\tau$ has a period $\frac{1}{T}$. The expression for $S$ contains the divergence coming from large $r$. In order to subtract the divergence,
we regularize $S$ (62) by cutting off the integral at a large radius $r_{\text{max}}$ and subtracting the solution with $\mu = 0$ in a same way as in [19]:

$$
S_{\text{reg}} = - \frac{V_3}{T} \left\{ \int_{r_{\text{H}}}^{r_{\text{max}}} dr r^3 \left( \frac{8}{k^2 l^2} - \frac{32c}{l^4} + \frac{48c\mu^2}{r^8} \right) - e^{\rho(r=r_{\text{max}})} - \rho(r=r_{\text{max}}; \mu=0) \int_{0}^{r_{\text{max}}} dr r^3 \left( \frac{8}{k^2 l^2} - \frac{32c}{l^4} \right) \right\}. \tag{63}
$$

The factor $e^{\rho(r=r_{\text{max}})} - \rho(r=r_{\text{max}}; \mu=0)$ is chosen so that the proper length of the circle which corresponds to the period $1/T$ in the Euclidean time at $r = r_{\text{max}}$ coincides with each other in the two solutions. Taking $r_{\text{max}} \to \infty$, one finds

$$
F = V_3 \left\{ \left( \frac{l^2 \mu}{8} - \frac{r_H^4}{4} \right) \left( \frac{8}{k^2 l^2} - \frac{32c}{l^4} \right) + \frac{12c\mu^2}{r_H^4} \right\}. \tag{64}
$$

The horizon radius $r_H$ is given by solving the equation $e^{2\rho_0(r_H)} = 0$ in (57). We can solve $r_H$ perturbatively up to first order on $c$ by putting $r_H = r_0 + c\delta r$, where $r_0$ is the horizon radius when $c = 0$. As in [18], $r_0$ is given by

$$
r_0^2 = - \frac{k l^2}{4} + \frac{1}{2} \sqrt{\frac{k^2}{4} l^4 + 4\mu l^2}. \tag{65}
$$

Then the horizon radius $r_H$ is obtained as follows:

$$
r_H = r_0 - \frac{c\mu^2 k^2}{r_0^3 \left( 2\mu - \frac{k r_0^2}{2} \right)}. \tag{66}
$$

We can also rewrite the black hole mass $\mu$ (using $r_H$) up to first order on $\epsilon$, ($\epsilon = c\kappa^2$).

$$
\mu = \frac{k}{2} r_H^2 + \frac{r_H^4}{l^2} + \frac{2 \epsilon}{r_H^4} \left( \frac{k}{2} r_H^2 + \frac{r_H^4}{l^2} \right)^2. \tag{67}
$$

Then $F$ looks like

$$
F = \frac{V_3}{k^2 l^2} \left[ \frac{l^2 k^2}{2} r_H^2 - r_H^4 + \epsilon \left( \frac{l^2 k^2}{2} + \frac{6r_H^4}{l^2} + \frac{12r_H^4}{r_H^4} \left( \frac{k}{2} r_H^2 + \frac{r_H^4}{l^2} \right)^2 \right) \right]. \tag{68}
$$

The Hawking temperature $T_H$ is given by

$$
T_H = \frac{\left( e^{2\rho} \right)'}{4\pi} \bigg|_{r=r_H} \tag{69}
$$

$$
= \frac{1}{4\pi} \left\{ \frac{4r_H}{l^2} + \frac{k}{r_H^4} - \frac{8\epsilon}{r_H^4} \left( \frac{k}{2} r_H^2 + \frac{r_H^4}{l^2} \right)^2 \right\},
$$

where $'$ denotes the derivative with respect to $r$. Then the entropy $S$ and energy $E$ have the following form:
When one has the following expressions:

\[ S = -\frac{dF}{dT_H} = -\frac{dF}{dr_H \ dT_H} = \frac{4\pi V_3 r_H^3}{\kappa^2} \left\{ 1 - \frac{16\epsilon}{l^2} \left( 1 + \frac{kl^2}{2r_H^2} + \frac{3k^2 l^4}{32r_H^2} \right) \left( 1 - \frac{c l^2}{4r_H^2} \right)^{-1} \right\}, \quad (70) \]

\[ E = F + TS \]

\[ = \frac{3V_3}{\kappa^2} \left\{ \frac{1}{2} k r_H^2 + \frac{r_H^4}{l^2} - \epsilon \left( \frac{18 r_H^4}{l^4} + \frac{31 k r_H^2}{2l^2} + \frac{9}{2} k^2 + \frac{5 k^3 l^2}{8 r_H^2} \right) \left( 1 - \frac{c l^2}{4r_H^2} \right)^{-1} \right\} \]

\[ = \frac{3V_3}{\kappa^2} \left[ \mu - \frac{\epsilon}{l^2} \left\{ k^3 l^4 + 8k l^2 \mu + \left( k^2 l^2 + 20 \mu \right) \sqrt{k^2 l^4 + 16l^2 \mu} \right\} \right. \]

\[ \times \left. \left( -2k l^2 + \sqrt{k^2 l^4 + 16l^2 \mu} \right)^{-1} \right]. \quad (71) \]

Hence, we described the thermodynamics of AdS BH where observable Universe may appear as corresponding brane. Some remarks are in order. It is remarkable that the entropy \( S \) is not proportional to the area of the horizon when \( k \neq 0 \) and the energy \( E \) is not to \( \mu \), either. We should note that the entropy \( S \) was proportional to the area and the energy \( E \) to \( \mu \) even in \( R^2 \)-gravity if there is no the squared Riemann tensor term \( (c = 0 \text{ in } (29)) \) [18], where we have the following expressions:

\[ F = -\frac{V_3}{8} r_H^2 \left( \frac{r_H^2}{l^2} - \frac{k}{2} \right) \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right), \quad (72) \]

\[ S = \frac{V_3}{2} \pi r_H^3 \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right), \quad (73) \]

\[ E = \frac{3V_3}{8} \mu \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right). \quad (74) \]

Here \( a \) and \( b \) are given in (29).

Several authors have investigated the entropy of higher derivative gravity using the first law of thermodynamics and the Noether current [22]. With the help of formula from [22], one gets the following expression of the entropy \( S \) corresponding to the action in (29):

\[ S = \frac{4\pi}{\kappa^2} \int_{\text{horizon}} d^d x \sqrt{g_H} \left\{ 1 + 2a k^2 \hat{R} + b k^2 \left( \hat{R} - g^{ij}_{H} \hat{R}_{ij} \right) \right. \]

\[ + 2c \left( \hat{R} - 2 g^{ij}_H \hat{R}_{ij} + g^{ij}_H g^{kl}_H \hat{R}_{ikjl} \right) \}. \quad (75) \]

Here \( g_H \) is the metric induced on the horizon and we follow the notations in this paper. When \( c = 0 \) and the curvatures are given by (41), we have \( \hat{R} = -\frac{20}{l^2} \) and \( g^{ij}_H \hat{R}_{ij} = -\frac{12}{l^2} \) for \( d = 4 \). Then Eq.(75) has the following form:

\[ S = \frac{V_3}{2} \pi r_H^3 \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right). \quad (76) \]

Therefore the result (73) is exactly reproduced. Eq.(73) tells that the entropy is proportional to the area of the horizon if \( c = 0 \) but Eq.(70) seems to tell that the entropy is not when \( c \neq 0 \). When \( a = b = 0 \) and \( c \neq 0 \) one gets
Then
\[
S = \frac{4\pi V_3 r_H^3}{\kappa^2} \left\{ 1 + \frac{c}{l^2} \left( 8 + \frac{6k l^2}{r_H^2} \right) \right\},
\]
which does not agree with (70). We should note, however, that, since we define \( S \) in (70) and \( E \) in (71) based on thermodynamics, it is clear that these quantities satisfy the first law:
\[
T \delta S = \delta E,
\]
for static (not rotating) case even if \( c \neq 0 \). The difference between the entropies might express the regularization (parametrization) dependence of the entropy. Indeed in our calculation, the traditional regularization method is applied. In the calculation of refs. [22] thermodynamics is defined on horizon, so in their case the regularization is implicit. Moreover, they mainly considered asymptotically flat backgrounds. Note that the entropy could be changed in general by the ambiguity in the choice of boundary terms (see section 3) which are necessary to make the variational principle well-defined and (or) to make the action finite. However the question remains why there was coincidence of two entropies when \( c = 0 \) in HD gravity? The answer is probably given in next section where we show that non-zero \( c \) HD gravity on such S-AdS BH has non-CFT field theory as dual one. The breaking of conformal invariance of dual QFT may cause the physical reason for such disagreement between entropies. Note in this connection that presumably logarithmic CFT (for recent discussion, see [23]) should be used to describe dual QFT near \( c = 0 \) barrier.

In order to clarify the situation let us consider the formula (75) for the entropy as well as (70) and (73) when \( b \neq 0 \) and/or \( c \neq 0 \) but \( a\kappa^2 \), \( bk^2 \) and \( c\kappa^2 \) are small. Combining (70) and (73), we have
\[
S = \frac{4\pi V_3 r_H^3}{\kappa^2} \left\{ 1 - \frac{40a\kappa^2}{l^2} - \frac{8b\kappa^2}{l^2} - \frac{4c\kappa^2}{l^2} - \frac{3c\kappa^2}{2l^2} \left( 4 + \frac{kl^2}{r_H^2} \right) \left( 2 + \frac{kl^2}{r_H^2} \right) \left( 1 - \frac{kl^2}{4r_H^2} \right)^{-1} \right\}.
\]
Note that the combination of the first 4 terms in the first line often appeared, for example, in (43) and (46), it might have a universal meaning. Eq.(80) can be further rewritten in the following form:
\[
S = \frac{4\pi V_3 r_H^3}{\kappa^2} \left\{ 1 - \frac{40a\kappa^2}{l^2} - \frac{8b\kappa^2}{l^2} + \frac{c\kappa^2}{l^2} \left( 8 + \frac{6k l^2}{r_H^2} \right) - \frac{12c\kappa^2}{l^2} \left( 2 + \frac{kl^2}{r_H^2} \right) \left( 1 - \frac{kl^2}{4r_H^2} \right)^{-1} \right\}.
\]
The last term depends on the curvature of the horizon \( k \). Since the scalar curvature \( R_H \) of the horizon is given by

\[
R_H = \frac{3k}{r_H^2}, \quad (82)
\]

by comparing (81) with (75), (76) and (78), we propose the following formula for the entropy:

\[
S = \frac{4\pi}{\kappa^2} \int_{\text{horizon}} d^3x \sqrt{g_H} \left\{ 1 + 2a\kappa^2 \hat{R} + b\kappa^2 \left( \hat{R} - g_{ij} \hat{R}_{ij} \right) + 2c \left( \hat{R} - 2g_{ij} \hat{R}_{ij} + g_{ij} g_{kl} \hat{R}_{ikjl} \right) - \frac{12c\kappa^2}{l^2} \left( 2 + \frac{R_H l^2}{3} \right) \left( 1 - \frac{R_H l^2}{12} \right)^{-1} \right\}. \quad (83)
\]

The last term diverges when \( R_H = \frac{12}{l^2} \). Since \( l \) is the length parameter of the 5d AdS space, this occurs when the size of the black hole is very large (of cosmological size). We do not expect that the perturbation with respect to \( c\kappa^2 \) is valid in this case. Therefore the singularity might not be real but apparent one. This consideration shows that small parameter expansion as done in this section has only limited region of validity.

Hence we calculated thermodynamical quantities (entropy, free energy, Hawking temperature) of S-AdS BH in five dimensional HD gravity. BH under consideration contains the brane (four dimensional universe). Thermodynamical description of this section will be used in next section for comparison with FRW-description of brane (and dual QFT).

**V. BRANE EQUATION AS FRW EQUATION**

We now consider the brane equation (47) in five dimensional bulk space of black hole type. Such a brane equation can be regarded as FRW equation of the brane world. From FRW equation, we can find the energy and the entropy of the matter in the brane universe. One can compare these thermodynamical quantities with those in section 4, obtained from the bulk black hole. Since \( c = 0 \) case has been investigated in [18], we now concentrate on the situation when \( a = b = 0 \) and \( c \neq 0 \). The generalization to the case when all parameters \( a, b, c \) are non-zero is straightforward and does not change the qualitative conclusions.

Let us rewrite the metric (57) of Schwarzschild-anti de Sitter space with correction in a form of (31). If one chooses coordinates \( (q, \tau) \) as

\[
\begin{align*}
&l^2 e^{2A-2\rho} A^2_{,q} - e^{2\rho} l^2 q^2 = 1, \quad l^2 e^{2A-2\rho} A_{,q} A_{,\tau} - e^{2\rho} l^2 t_{,q} t_{,\tau} = 0, \\
&l^2 e^{2A-2\rho} A^2_{,\tau} - e^{2\rho} l^2 t^2 = -l^2 e^{2A}.
\end{align*}
\]

(84)

the metric takes the form (31). Here \( r = le^A \). Furthermore choosing a coordinate \( \tilde{t} \) by

\[d\tilde{t} = le^A d\tau,\]

the metric on the brane takes FRW form:

\[
dx_{\text{brane}}^2 = -d\tilde{t}^2 + l^2 e^{2A} \sum_{i,j=1}^{3} g_{ij} dx^i dx^j. \quad (85)
\]
Solving Eqs. (84), we have

$$H^2 = A^2_{,q} - \frac{e^{2\rho} e^{-2A}}{l^2}. \quad (86)$$

Here the Hubble constant $H$ is defined by $H = \frac{dA}{dt}$. Then using (57), one obtains the following equation:

$$H^2 = \frac{\mu}{r^4} - \frac{k}{2r^2} - \frac{2\mu^2\epsilon}{r^8}. \quad (87)$$

Especially, when $k = 2 > 0$, the spacial part of the brane has the shape of the three dimensional sphere and $r$ can be regarded as the radius of the spacial part of the brane universe. The last term in (87) is unusual, it did not appear even in the $R^2$-gravity with $c = 0$.

Eq. (87) can be rewritten in the form of the FRW equation (compare with [24]):

$$H^2 = -\frac{k}{2r^2} + \frac{\kappa_4^2}{6} \frac{\tilde{E}}{V}, \quad \tilde{E} = \frac{6V_3}{\kappa_4^3 r} \left( \mu - \frac{2\epsilon \mu^2}{r^4} \right), \quad V = r^3 V_3. \quad (88)$$

Here $V_3$ is the volume of the three dimensional sphere with a unit radius and $\kappa_4$ is the four dimensional gravitational coupling, which is given by

$$\kappa_4^2 = \frac{2\kappa^2}{l}, \quad \frac{1}{\kappa_4} = \frac{1}{\kappa^2} - \frac{4c}{l^2}. \quad (89)$$

Differentiating Eq. (88) with respect to $\tilde{t}$, since $H = \frac{1}{r} \frac{dr}{dt}$, we obtain the second FRW equation

$$\dot{H} = -\frac{\kappa_4^2}{4} \left( \frac{\tilde{E}}{V} + p \right) + \frac{k}{2r^2}, \quad p = \frac{2}{r^4 \kappa_4^3} \left( \mu - \frac{10\epsilon \mu^2}{r^4} \right), \quad (90)$$

if $H \neq 0$. We should note that Eq. (90) needs not to be satisfied for the static solution where $H = 0$. Here $p$ can be regarded as the pressure of the matter on the brane.

Note that when $r$ is large, the metric (57) has the following form:

$$ds_{AdS-S}^2 \to \frac{r^2}{l^2} \left( -dt^2 + l^2 \sum_{i,j} g_{ij} dx^i dx^j \right), \quad (91)$$

which tells that CFT time $\tilde{t}$ is equal to AdS time $t$ times the factor $\frac{r}{l}$:

$$t_{CFT} = \frac{r}{l} \tilde{t}. \quad (92)$$
Therefore the energy $\tilde{E}$ (88) in CFT should be related with the energy $E$ (71) in AdS BH by a factor $\frac{1}{r}$ [24]:

$$\tilde{E} = \frac{1}{r} E . \quad (93)$$

However, Eq.(93) is not satisfied in general. Using (89), the energy in (71) can be rewritten as:

$$E = \frac{6V_3}{\kappa^2 d} \left\{ \mu - \frac{\epsilon}{l^2} \left( k^2 l^2 + 16\mu \right) \left( k^4 l^4 + \sqrt{k^2 l^4 + 16l^2 \mu} \right) \right\} \times \left( -2k l^2 + \sqrt{k^2 l^4 + 16l^2 \mu} \right)^{-1} , \quad (94)$$

Then comparing (94) with (88), we find that Eq.(93) is satisfied when the brane exists at $r = r_{\text{br}}$:

$$\frac{1}{r_{\text{br}}^4} = \frac{1}{2l^2 \mu^2} \frac{(k^2 l^2 + 16\mu) \left( k^4 l^4 + \sqrt{k^2 l^4 + 16l^2 \mu} \right)}{\left( -2k l^2 + \sqrt{k^2 l^4 + 16l^2 \mu} \right)} = \frac{4 \left( 4r_H^2 + kl^2 \right)^2}{r_H^4 \left( 4r_H^2 - kl^2 \right)^2 (2r_H^2 - kl^2)} . \quad (95)$$

From Eqs.(88) and (90), one gets

$$-\frac{\tilde{E}}{V} + 3p = -\frac{48\epsilon \mu^2}{\kappa^2 r^8} , \quad (96)$$

which tells that the trace of the energy-stress tensor coming from the matter on the brane does not vanish:

$$T_{\mu}^{\mu} \neq 0 . \quad (97)$$

Therefore the conformal symmetry of the matter on the brane should be broken by the correction coming from the squared Riemann tensor term. The conformal symmetry could be restored when $r \to \infty$.

In [15], it was shown that FRW equation in $d$ dimensions can be regarded as a $d$ dimensional analogue of the Cardy formula of 2d conformal field theory (CFT) [25]:

$$\tilde{S} = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{k}{d - 2} \frac{c}{24} \right)} . \quad (98)$$

In the present case, identifying

$$\frac{2\pi \tilde{E} r}{d - 1} \Rightarrow 2\pi L_0 ,$$

$$\frac{(d - 2)V}{\kappa^2 d} \Rightarrow \frac{c}{24} ,$$

$$\frac{4\pi (d - 2) H V}{\kappa^2 d} \Rightarrow \tilde{S} , \quad (99)$$
the FRW-like equation (88) has the same form as (98) (for related discussion of this formula see papers [26]).

The total entropy of the universe could be conserved during the expansion. Then one can evaluate holographic (Hubble) entropy $\tilde{S}$ in (99) when the brane crosses the horizon $r = r_H$. When $r = r_H$, Eq. (88) tells that

$$H = \pm \frac{1}{\ell}.$$  \hspace{1cm} (100)

Here the plus sign corresponds to the expanding brane universe and the minus one to the contracting universe. Taking the expanding case and using (99), we find

$$\tilde{S} = \frac{4\pi(d - 2)V}{l\kappa_d^2} = \frac{2\pi(d - 2)r_H^{d - 1}V_{d - 1}}{\tilde{r}^2}. \hspace{1cm} (101)$$

When $c = 0$ but $a, b \neq 0$ in general, the entropy $\tilde{S}$ is identical with the black hole entropy $S$. When $c \neq 0$, however, there is a difference between the two kinds of entropies. For $d = 4$, using (70) and (101), we find

$$\tilde{S} - S = \frac{6\pi \epsilon V_3 r_H^3}{l^2 \kappa_4^2} \left( 4 + \frac{k l^2}{r_H^2} \right) \left( 2 + \frac{k l^2}{r_H^2} \right) \left( 1 - \frac{k l^2}{4 r_H^2} \right)^{-1}. \hspace{1cm} (102)$$

Formula (98) has been derived for the conformal field theory. Eq. (97), however, tells that the conformal invariance of dual theory is broken. The difference between $\tilde{S}$ and $S$ in (102) might express the fact that proposed dual QFT is not CFT. Then, the difference of entropies may serve as some measure for deviation from AdS/CFT correspondence. It would be interesting to analyze the physics behind this property. One possible speculation is that it could help in formulating of some sort of AdS/non-CFT correspondence.

VI. THERMODYNAMICS OF S-ADS BH IN GAUSS-BONNET GRAVITY

In this section, we consider the situation that $R^2$-terms in 5d action are given by Gauss-Bonnet combination:

$$S = \int d^5 x \sqrt{-g} \left\{ c \left( \hat{R}^2 - 4 \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} + \hat{R}_{\mu \nu \xi \sigma} \hat{R}^{\mu \nu \xi \sigma} \right) + \frac{1}{\kappa^2} \hat{\hat{R}} - \Lambda \right\} \hspace{1cm} (103)$$

One chooses the coefficients $a = c, b = -4c$ in the previous action (29) after deleting the auxiliary fields. The Gauss-Bonnet combination is not topological one if the spacetime dimensions are not four. This combination, however, appears even in higher dimensions, for example, in the first order correction (on string tension) to the low energy effective action in the string theory [12]. Furthermore, the Gauss-Bonnet combination leads to special structure: if we choose it the FRW equations (11) and (13) are

$$0 = -\frac{(d - 1)k}{2\kappa^2 r^2} - \frac{(d - 1)(d - 2)}{2\kappa^2} H^2 - \rho$$

$$-\frac{1}{2} (d - 1)(d - 2)(d - 3)(d - 4)c H^4 - (d - 1)(d - 3)(d - 4) \frac{ckH^2}{r^2}$$

21
\[
0 = \frac{1}{\kappa^2} \left\{ \frac{(d-3)k}{2r^4} + \frac{(d-2)H_t}{r^2} + \frac{(d-1)(d-2)}{2}H^2 \right\} - p
\]

\[
+2(d-2)(d-3)(d-4)cH^2H_t + \frac{1}{2}(d-1)(d-2)(d-3)(d-4)cH^4
\]

\[
+ \frac{ck}{r^2} \left\{ 2(d-3)(d-4)H_t + (d-4)(d-3)^2H^2 \right\}
\]

\[
+ \frac{(d-1)(d-3)(d-4)c\kappa^2}{2(d-2)} \frac{ck^2}{r^4}.
\]

(105)

Since \( H = \frac{r'}{r} \), the derivative terms higher than \( r_t \) do not appear in (104) and the terms higher than \( r_{tt} \) do not in (105), either. In the sense of classical mechanics, if we give the initial values for \( r \) and \( r_t \), which satisfy the “constraint” (104), the time evolutions of these variables can be determined by the “equation of motion” (105). We should note that, in general \( R^2 \)-gravity, we need to give the initial values for \( r, r_t, r_{tt} \), and \( r_{ttt} \), which satisfy (11), in order to determine the time evolution by (13). This fact means that there is no ghost in the corresponding quantum theory which has \( R^2 \) terms as the Gauss-Bonnet combination (for very recent discussion of such theory see [27]), although the general higher derivative theory has ghost with negative norm.

We now treat the theory given by the action (103) perturbatively assuming \( ck^2 \) is small. Then the solution for the metric has the same form as obtained in \( c \neq 0 \) case (57):

\[
e^{2\rho} = \frac{1}{r^2} \left\{ -\mu + \frac{k}{2}r^2 + \frac{r^4}{l^2} + \frac{2\mu^2\epsilon}{r^2} \right\}, \quad \epsilon = ck^2.
\]

(106)

We should note, however, the length parameter \( l \) is given by

\[
\frac{24\epsilon}{\kappa^2l^4} - \frac{12}{\kappa^2l^2} - \Lambda = 0,
\]

(107)

instead of (56). Then the horizon radius \( r_H \) and the Hawking temperature \( T_H \) have the same forms as those obtained in (66) and (69):

\[
r_H = r_0 - \frac{c\mu^2k^2}{r_0^3} \left( \frac{2\mu - k}{2}r_0^2 \right), \quad r_0^2 = -\frac{k\mu^2}{4} + \frac{1}{2} \sqrt{\frac{k^2}{4} l^4 + 4\mu^2}.
\]

(108)

\[
T_H = \left. \left( \frac{e^{2\rho}}{4\pi} \right) \right|_{r=r_H}
\]

\[
= \frac{1}{4\pi} \left\{ \frac{4r_H}{l^2} + \frac{k}{r_H} - \frac{8\epsilon}{r_H} \left( \frac{k}{2} r_H^2 + \frac{r_H^4}{l^2} \right)^2 \right\},
\]

(109)

The free energy \( F \) looks as follows:

\[
F = \frac{V_3}{\kappa^2 l^2} \left[ \frac{l^2 k r_H^2}{2} + \epsilon \left\{ \frac{14r_H^4}{l^2} - 4r_H^2 k + \frac{l^2 k^2}{2} \\
+ 12l^2 \left( \frac{k}{2} r_H^2 + \frac{r_H^4}{l^2} \right)^2 \right\} \right],
\]

(110)
The entropy \( S \) and the energy \( E \) have the following form:

\[
S = \frac{4V_3\pi r_H^3}{\kappa^2} \left[ 1 - \frac{12c}{l^2} \left( 1 + \frac{c \kappa^2}{8l^2} \left( 4 + \frac{k l^2}{r_H^2} \right) \left( 2 + \frac{k l^2}{r_H^2} \right) \right) \times \left( 1 - \frac{k l^2}{4r_H^2} \right)^{-1} \right],
\]

\[
e = \frac{3V_3}{\kappa^2} \left[ \mu - \frac{\epsilon}{l^2} \left( 12\mu + \left( k^2 l^2 + 16\mu \right) \left( k l^2 + \sqrt{k^2 l^4 + 16l^2\mu} \right) \times \left( -2k l^2 + \sqrt{k^2 l^4 + 16l^2\mu} \right)^{-1} \right) \right].
\]

Now we again rewrite the metric (57) of Schwarzschild-anti de Sitter space with correction in the FRW form:

\[
\kappa^2 = \frac{2\kappa^2}{l}, \quad \frac{1}{\kappa^2} = \frac{1}{\kappa^2} - \frac{12c}{l^2},
\]

\[
H^2 = -\frac{k}{2r^2} + \frac{\kappa_4^2}{6V} \tilde{E}, \quad \tilde{E} = \frac{6V_3}{\kappa_4^2 r} \left( \mu - \frac{2\epsilon \mu^2}{r^4} \right),
\]

\[
\dot{H} = -\frac{\kappa_4^2}{4} \left( \frac{\tilde{E}}{V} + p \right) + \frac{k}{2r^2}, \quad p = \frac{2}{r^4\kappa_4^2} \left( \mu - \frac{10\epsilon \mu^2}{r^4} \right).
\]

These equations are identical with the previous ones (88) and (90) with \( a = b = 0 \) and \( c \neq 0 \) except the length parameter is given by (107). The obtained energy \( \tilde{E} \) in (114) is not identical with \( E \) in (112), in general, even if we take account of the factor \( \frac{1}{l} \) coming from the ratio of the time scale in 5d bulk space time and that on the brane:

\[
\tilde{E} \neq \frac{l}{r} E.
\]

One can find the two energies become equal to each other when the brane exists at \( r = r_{br} \), which is given by (95). Similarly, we can also obtain four dimensional entropy \( \tilde{S} \) by using generalized Cardy formula of two dimensional CFT (98) and (99). The obtained entropy \( S \) has the same form of that in \( a = b = 0 \) and \( c \neq 0 \) case, and especially, when the brane crosses the horizon, we have

\[
\tilde{S} = \frac{4\pi(d-2)V}{l\kappa_d^2} = \frac{2\pi(d-2)r_H^{d-1}V_{d-1}}{\kappa^2}.
\]

By putting \( d = 4 \), we find that the difference between \( S \) in (111), obtained from the bulk black hole, and \( \tilde{S} \) in (117) is again given by (102). The difference between \( \tilde{S} \) and \( S \) is again caused by the breaking of the conformal symmetry of QFT dual.

**VII. ASYMMETRICALLY WARPED SPACETIMES AND GRAVITATIONAL LORENTZ VIOLATION IN R\(^2\)-GRAVITY**

In this section we will be interested in so-called asymmetrically warped spacetimes for our theory. In [28] where such backgrounds were considered, it has been shown that the
apparent violation of the Lorentz invariance can occur. They have considered the brane in
the bulk space which is the charged black hole, whose metric is given by

\[
\begin{align*}
\text{for } e^{2\rho} = 1 \quad & \text{the solution in (66). In case of } k = 0, \text{ the solution } r_0 \text{ of (120) is given by } \\
\text{for } e^{2\rho} = 0 \text{ are given by } \\
\end{align*}
\]

\[
\begin{align*}
0 = -\frac{k}{2r^2} + \frac{\mu}{r^4} - \frac{2\epsilon\mu^2}{r^8} . \\

\end{align*}
\]

Note that there appear two horizons due to \( R^2 \)-term. The larger (outer) one \( r = r_{H^+} \) corresponds to the solution in (66). In case of \( k = 0 \), the solution \( r_0 \) of (120) is given by

\[
\begin{align*}
\text{for } e^{2\rho} = 0 \text{ are given by } \\
\text{for } e^{2\rho} = 0 \text{ are given by } \\
\end{align*}
\]

\[
\begin{align*}
\text{for } e^{2\rho} = 0 \text{ are given by } \\
\end{align*}
\]

The brane can exist due to the squared Riemann tensor term. Comparing Eq.(121) and
Eq.(122), we find that the brane lies inside the inner horizon.
Let us assume the brane is static and exists at $r = r_0$. Then the velocity $c_{\text{phn}}$ of the photon propagating on the brane is given by

$$c_{\text{phn}}(r_0) = \frac{e^{\rho(r_0)}}{r_0},$$  \hspace{1cm} (123)

which depends on $r_0$. On the other hand, the geodesic of the particle propagating the bulk is given by the Euler-Lagrange equation derived from the Lagrangian $L$

$$L = \frac{1}{2} \left\{ -e^{2\rho(r)} \dot{t}^2 + e^{-2\rho(r)} \dot{r}^2 + r^2 \sum_{i,j}^{3} g_{ij} \dot{x}^i \dot{x}^j \right\}.$$  \hspace{1cm} (124)

Here $\dot{}$ expresses the derivative with respect to the proper time $\tau$. If there is non-trivial solution of the geodesic, the particle can propagate faster than the photon on the brane. The Lagrangian has two kinds of the integrals, which correspond to the energy $E$ and the momentum $p_i$ of the particle:

$$-E = \frac{\partial L}{\partial \dot{t}} = -e^{2\rho(r)} \dot{t}, \quad p_i = \frac{\partial L}{\partial \dot{x}^i} = r^2 g_{ij} \dot{x}^j.$$  \hspace{1cm} (125)

For simplicity, one considers the case that the brane is flat: $g_{ij} = \delta_{ij}$ ($k = 0$). If the particle is massless, we have $(\frac{ds}{d\tau})^2 = 0$ on the geodesic and we find

$$0 = -E^2 e^{-2\rho(r)} + \dot{r}^2 e^{-2\rho(r)} + \frac{p^2}{r^2},$$  \hspace{1cm} (126)

which can be compared with the classical system of the particle with unit mass in the potential $V(r)$:

$$\frac{1}{2} \dot{r}^2 + V(r) = \hat{E},$$

$$V(r) = \frac{p^2 e^{2\rho(r)}}{2r^2} = \frac{p^2}{2} \left\{ -\frac{\mu}{r^4} + \frac{1}{l^2} + \frac{2\mu^2 \epsilon}{r^8} \right\},$$

$$\hat{E} = \frac{1}{2} \hat{E}^2.$$  \hspace{1cm} (127)

If $\epsilon = 0$ or $\epsilon < 0$, the potential is monotonically increasing function of $r$ and unbounded when $r \to 0$. Therefore if initially $\dot{r} \leq 0$ or the brane is the boundary of the bulk and there is no region with $r > r_0$, by the analogy with classical mechanics, the particle radiated from the brane cannot return to the brane but falls into the black hole singularity. The situation does not change if we consider the $R^2$-gravity without the Riemann tensor square term, that is, the gravity including the squares of the scalar curvature and Ricci tensor, where HD correction is included into the redefinition of the radius parameter $l$.

On the other hand, if $\epsilon > 0$, the potential increases when $r$ is small. Since we are treating $\epsilon$ as a parameter of the perturbation, we cannot say any definite thing when $r$ is small, where the correction becomes large, but the potential could be bounded and there would be non-trivial geodesic line, along which the particle can return to the brane, as in the case of the charged black hole [28]. Hence, HD gravity may lead to brane-worlds with apparent violations of Lorentz invariance as it occurs also in Einstein-Maxwell gravity [28].
VIII. BRANE SOLUTIONS IN DE SITTER SPACE

It is right time now to look to other type of bulk space, i.e. to de Sitter space. The motion of domain walls in de Sitter bulk has been recently discussed in ref. [5]. Moreover, there appeared recently attempt to formulate dS/CFT correspondence [29,30]. There were also earlier proposals on dS/CFT duality [31] and thermodynamics of de Sitter space was always under attention (for recent discussion, see [32]). Unfortunately, so far it did not appear any explicit example of dual CFT for de Sitter space (except not completely physical example by Hull [31]). Nevertheless, the attempts to find such example continue.

In [33], the quantum creation of four dimensional de Sitter brane universe in five dimensional de Sitter spacetime as in scenario of so-called Brane New World [6,7] is investigated. For $R^2$-gravity, if one of the solutions for $l^2$ in Eq.(42) is negative:

$$l_{ds}^2 = -l^2 > 0,$$

de Sitter space is exact solution. Moreover, even if the cosmological constant $\Lambda$ is negative, there can be a de Sitter space solution due to $R^2$ terms. By the Wick rotation, 5d de Sitter space becomes 5d sphere, whose metric is given by

$$ds_{S^5}^2 = dz^2 + l_{ds}^2 \sin^2 \frac{z}{l_{ds}} d\Omega_4^2.$$  \hspace{1cm} (129)

Here $d\Omega_4^2$ describes the metric of $S_4$ with unit radius. The coordinate $z$ is defined in $0 \leq z \leq l_{ds}\pi$. One also assumes the brane lies at $z = z_0$ and the bulk space is given by gluing two regions given by $0 \leq y < y_0$. Identifying

$$A = \ln \sin \frac{z}{l_{ds}}$$  \hspace{1cm} (130)

and using (47), we obtain the following equation:

$$0 = 1 - \sqrt{-1 + \frac{R^2}{l_{ds}^2}} - \frac{1}{l_{ds}}.$$  \hspace{1cm} (131)

Here $R \equiv l_{ds} \sin \frac{y_0}{l_{ds}}$ is the radius of the brane. In (131), the contribution from $R^2$-terms appears through $l$ by (42). When bulk is AdS, we need the quantum contribution from the matter on the brane in order that de Sitter brane existed. When bulk is de Sitter, even in classical case that there is no quantum contribution from the matter on the brane, Eq.(131) has a solution:

$$R^2 = R_0^2 \equiv \frac{l_{ds}^2}{2} \text{ or } \frac{y_0}{l_{ds}} = \frac{\pi}{6}, \quad \frac{5\pi}{6}.$$  \hspace{1cm} (132)

In Eq.(131), the first term corresponds to the gravity, which makes the radius $R$ larger. On the other hand, the second term corresponds to the tension, which makes $R$ smaller. When $R < R_0$, gravity becomes larger than the tension and when $R > R_0$, vice versa. Then both of the solutions in (132) are stable. Although it is not clear from (131), $R = l_{ds} \left(\frac{\pi}{l_{ds}} = \frac{\pi}{2}\right)$ corresponds to the local maximum.
When one considers brane in de Sitter space, there is no any essential difference between Einstein gravity and $R^2$-gravity. Then we now consider the black hole solution when $c \neq 0$ and the FRW type equation (88) exists:

$$H^2 = \frac{2}{l_{\text{dS}}^2} - \frac{1}{r^2} + \frac{\mu}{r^4} - \frac{2\epsilon \mu^2}{r^8}.$$  

(133)

The last term in (133) comes from $R^2$-term. One cannot embed the brane with the shape of flat plane or hyperboloid in the bulk sphere, then it is reasonable to consider brane with the shape of sphere $k = 2$. The static solution corresponds to $H = 0$. When $\epsilon = 0$ ($c = 0$), the static solution

$$r^2 = r^2_0 = \frac{l_{\text{dS}}^4 \pm \sqrt{l^4 - 8\mu l_{\text{dS}}^2}}{4}$$  

(134)

if

$$8\mu \leq l_{\text{dS}}^2.$$  

(135)

The case of $8\mu = l_{\text{dS}}^2$ corresponds to the extremal case. When $\epsilon \neq 0$, where $\epsilon$ is small, using the perturbation with respect to $\epsilon$ one gets

$$r^2 = r^2_0 \pm \frac{2\epsilon \mu^2}{r^4_0 - 2\mu r^2_0}$$  

(136)

$$= \frac{l_{\text{dS}}^2 \pm \sqrt{l^4_{\text{dS}} - 8\mu l_{\text{dS}}^2}}{4} + \frac{16\epsilon^2}{2l^4_{\text{dS}} - 16\mu l^2_{\text{dS}} \pm (2l^2_{\text{dS}} - 8\mu) \sqrt{l^4_{\text{dS}} - 8\mu l^2_{\text{dS}}}},$$

when $8\mu < l^2_{\text{dS}}$ and

$$r^2 = \frac{l^4}{4} + \sqrt{2\epsilon},$$  

(137)

when $8\mu = l^2_{\text{dS}}$. In (136), when $\mu$ is small

$$r^2 - r^2_0 = \frac{4\epsilon \mu^2}{l^4}, \quad -\epsilon.$$  

(138)

Therefore if $\epsilon$ is positive, the correction from $R^2$ (Riemann tensor)-term makes the radius of the outer (inner) horizon larger (smaller).

We will finish with this explicit example of dS brane in dS bulk. Of course, one can go to details and to discuss brane FRW-dynamics as in section 5, etc. For our purposes the explicit demonstration of existence of dS brane in dS bulk is enough.

**IX. CONFORMAL ANOMALY FROM $R^2$-GRAVITY IN DS/CFT CORRESPONDENCE**

As it was mentioned in previous section, the explicit example of dS/CFT correspondence still did not appear. Nevertheless, it has been demonstrated in ref. [29] and in third paper
from [30] that supposing dS/CFT correspondence the central charge (conformal anomaly) of dual CFT may be derived from dS gravity.

In [17] conformal anomaly from $R^2$-gravity in AdS/CFT correspondence has been found. In this section, using similar method we derive conformal anomaly in dS/CFT correspondence. Let us write $l_{\text{dS}}$ in (128) as $l$, therefore instead of (42), $l$ should satisfy the following equation:

$$0 = \frac{80a}{l^4} + \frac{16b}{l^4} + \frac{8c}{l^4} + \frac{12}{\kappa^2 l^2} - \Lambda .$$

In order to calculate the conformal anomaly from bulk dS gravity, one considers the fluctuations around de Sitter space, whose metric can be expressed as

$$ds_{\text{dS}}^2 = -\frac{l^2}{4} \rho^{-2} d\rho^2 + \rho^{-1} \sum_{i=1}^d \left(dx^i\right)^2 .$$

Then analogously to AdS case [34], we assume the metric has the following form:

$$ds^2 \equiv \hat{G}_{\mu
u} dx^\mu dx^\nu = -\frac{l^2}{4} \rho^{-2} d\rho^2 + \sum_{i=1}^d \hat{g}_{ij} dx^i dx^j ,$$

$$\hat{g}_{ij} = \rho^{-1} g_{ij} .$$

Suppose that there is a brane at $\rho = 0$. Note that there is a redundancy in the expression of (141). In fact, if we reparametrize the metric:

$$\delta \rho = \delta \sigma \rho , \quad \delta g_{ij} = \delta \sigma g_{ij} .$$

by a constant parameter $\delta \sigma$, the expression (141) is invariant. The transformation (142) is nothing but the scale transformation on the brane.

In the parametrization (141), scalar curvature $\hat{R}$

$$\hat{R} = \frac{d^2 + d}{l^2} + \rho R - \frac{2(d-1)\rho}{l^2} \hat{g}^{ij} \hat{g}_{ij} - \frac{3}{l^2} \hat{g}^{ij} \hat{g}_{ik} \hat{g}^{kl} \hat{g}_{ij}$$

$$+ \frac{4\rho^2}{l^2} \hat{g}_{ij} \hat{g}''_{ij} + \frac{\rho^2}{l^2} \hat{g}^{ij} \hat{g}^{kl} \hat{g}_{ij} \hat{g}_{kl}$$

Ricci tensor $\hat{R}_{\mu
u}$

$$\hat{R}_{\rho\rho} = -\frac{d}{4\rho^2} - \frac{1}{2} \hat{g}^{ij} \hat{g}_{ij}'' + \frac{1}{4} \hat{g}^{ik} \hat{g}^{jl} \hat{g}_{kl} \hat{g}_{ij} ,$$

$$\hat{R}_{ij} = R_{ij} + \frac{2\rho}{l^2} \hat{g}_{ij}'' - \frac{2}{l^2} \hat{g}^{kl} \hat{g}_{ki} \hat{g}''_{lj} + \frac{\rho}{l^2} \hat{g}_{ij} \hat{g}^{kl} \hat{g}_{kl}$$

$$+ \frac{2 - d}{l^2} \hat{g}_{ij} - \frac{1}{l^2} \hat{g}_{ij} \hat{g}^{kl} \hat{g}_{kl} + \frac{d}{l^2 \rho} \hat{g}_{ij} ,$$

$$\hat{R}_{\rho i} = \hat{R}_{\rho i}$$

$$= \frac{1}{2} \hat{g}^{jk} \hat{g}_{kj,i} - \frac{1}{2} \hat{g}^{kj} \hat{g}_{j,k,i} + \frac{1}{2} \hat{g}^{ij} \hat{g}_{ki} + \frac{1}{4} \hat{g}^{kl} \hat{g}_{ij} \hat{g}^{jm} \hat{g}_{jm,k} - \frac{1}{4} \hat{g}_{ij} \hat{g}_{jk} ,$$

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Riemann tensor $\hat{R}_{\mu \nu \rho \sigma}$

\[
\hat{R}_{\mu \nu \rho \sigma} = 0 , \\
\hat{R}_{\rho \mu \nu \pi} = R_{\rho \mu \nu \pi} = R_{\rho \pi \mu \nu} = R_{\pi \rho \mu \nu} = 0 , \\
\hat{R}_{i j \rho \pi} = \hat{R}_{i \rho j \pi} = 0 , \\
\hat{R}_{i \rho j \rho} = \hat{R}_{i \rho j \rho} = -\hat{R}_{i \rho j \rho} = \hat{R}_{\rho i j \rho} \\
= -\frac{1}{4 \rho^3} g_{ij} + \frac{1}{4 \rho} g^{kl} g_{k l} g_{ij} - \frac{1}{2 \rho} g_{ij}^n , \\
\hat{R}_{i j k l} = -\hat{R}_{i j k l} = \hat{R}_{j k i l} = -\hat{R}_{j k i l} \\
= \frac{1}{4 \rho} \{ 2 g_{i j, k} - 2 g_{i j, k} - g_{l m} (g_{i m, k} + g_{k m, i} - g_{i k, m}) g_{i j} \\
+ g_{l m} (g_{i m, j} + g_{j m, i} - g_{i j, m}) g_{i k} \} , \\
\hat{R}_{i j k l} = \frac{1}{\rho} R_{i j k l} \\
+ \frac{1}{\rho^2 l^2} \left\{ \left( g_{i l} - \rho g_{i j}^l \right) (g_{i k} - \rho g_{i j} k) - \left( g_{j k} - \rho g_{j i} k \right) (g_{i l} - \rho g_{i i l}) \right\} .
\]

(145)

may be easily calculated. Here "\( \prime \)" expresses the derivative with respect to \( \rho \) and \( R, R_{ij}, R_{ijkl} \) are scalar curvature, Ricci and Riemann tensors, respectively, on \( M_d \).

As in the previous papers [34] on holographic conformal anomaly, we expand the metric \( g_{i j} \) as a power series with respect to \( \rho \),

\[
g_{i j} = g(0)_{i j} + \rho g(1)_{i j} + \rho^2 g(2)_{i j} + \cdots .
\]

Substituting (146) into (143), (144) and (145), one gets \( \rho \):

\[
\sqrt{-g} = \frac{l}{2} \rho^{-\frac{d}{2} - 1} \sqrt{g(0)} \left\{ 1 + \frac{\rho}{2} g(1)_{i j} \right. \\
- \frac{1}{4} g(0)_{i j} g(0)_{k l} g(1)_{i k} g(1)_{j l} + \frac{1}{8} \left( g(1)_{i j} \right)^2 + O(\rho^3) \} , \\
\sqrt{-g} R = \frac{l}{2} \rho^{-\frac{d}{2} - 1} \sqrt{g(0)} \left\{ \frac{d^2 + d}{l^2} + \rho \left( R(0) - \frac{-d^2 + 3d - 4}{2 l^2} g(1)_{i j} \right) \\
+ \rho^2 \left( -g(1)_{i j} R(0) + \frac{1}{2} R(0) g(1)_{i j} g(1)_{i j} - \frac{-d^2 + 7d - 24}{2 l^2} g(1)_{i j} \right) \\
- \frac{d^2 - 7d + 20}{4 l^2} g(0)_{i j} g(0)_{k l} g(1)_{i k} g(1)_{j l} - \frac{-d^2 + 7d - 16}{8 l^2} \left( g(0)_{i j} g(1)_{i j} \right)^2 + O(\rho^3) \} , \\
\sqrt{-g} R^2 = \frac{l}{2} \rho^{-\frac{d}{2} - 1} \sqrt{g(0)} \left\{ \frac{d^2 (d + 1)}{l^4} + \rho \left( \frac{2d (d + 1)}{l^2} R(0) \\
+ \frac{d^4 - 6d^3 + d^2 + 8d}{2 l^4} g(1)_{i j} g(1)_{i j} \right) + \rho^2 \left( R(0) - \frac{2d (d + 1)}{l^2} g(1)_{i j} \right) \right. \\
- \frac{-d^2 + 3d - 4}{l^2} R(0) g(1)_{i j} + \frac{-d^2 - 14d^3 + 33d^2 + 48d}{2 l^4} g(1)_{i j} \right\} .
\]

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\begin{align*}
&+ \frac{-d^4 + 14d^3 - 25d^2 - 40d}{4l^4} g_{ij}^{kl} g_{(0)ij}^{(1)kl} + \frac{d^4 - 14d^3 + 49d^2 - 32d + 32}{8l^4} \left( g_{(0)ij}^{(1)ij} \right) + O(\rho^3) \right) , \\
\sqrt{-G} \hat{R}_{\mu \nu} \hat{R}^{\mu \nu} &= \frac{1}{2} \rho^{-\frac{d}{2} - 1} \sqrt{g_{(0)}} \left\{ \frac{d^2(d + 1)}{l^4} + \rho \left( \frac{2d}{l^2} R_{(0)} + \frac{d^3 - 7d^2 + 8d}{2l^4} g_{ij}^{kl} g_{(0)ij}^{(1)ij} \right) \rho^2 \left( R_{(0)}^{ij} R_{(0)}^{(0)ij} - \frac{4d - 4}{l^2} g_{(1)ij} R_{(0)}^{ij} - \frac{d - 2}{l^2} R_{(0)} g_{(0)} g_{(1)ij} \right) \\
&- \frac{-d + 2}{l^2} R_{(0)} g_{(0)} g_{(1)ij} + \frac{d^3 - 15d^2 + 48d}{2l^4} g_{ij}^{kl} g_{(0)ij}^{(1)ij} g_{(0)} g_{(2)ij} \\
&+ \frac{d^3 - 15d^2 + 56d - 32}{8l^4} \left( g_{ij}^{(0)} g_{(1)ij} \right) + O(\rho^3) \right) , \\
\sqrt{-G} \hat{R}_{\mu \nu \rho \sigma} \hat{R}^{\mu \nu \rho \sigma} &= \frac{1}{2} \rho^{-\frac{d}{2} - 1} \sqrt{g_{(0)}} \left\{ \frac{2d(d + 1)}{l^4} + \rho \left( \frac{4}{l^2} R_{(0)} + \frac{d^2 - 7d + 8}{2l^4} g_{ij}^{kl} g_{(0)ij}^{(1)ij} \right) \rho^2 \left( R_{(0)}^{ijkl} R_{(0)}^{ij} R_{(0)}^{kl} - \frac{4d - 4}{l^2} g_{(1)ijkl} R_{(0)}^{ij} + \frac{2}{l^2} R_{(0)} g_{ij}^{kl} g_{(0)} g_{(1)ij} \right) \\
&+ \frac{d^2 - 15d + 48}{2l^4} g_{ij}^{kl} g_{(0)ij}^{(1)ij} + \frac{-d^2 + 23d - 56}{2l^4} g_{ij}^{kl} g_{(0)ij}^{(1)ij} + \frac{d^2 - 15d + 48}{4l^4} \left( g_{ij}^{(0)} g_{(1)ij} \right) + O(\rho^3) \right) .
\end{align*}

We regard $g_{(0)ij}$ in (146) as independent field on the brane or boundary, which we denote by $M_d$. One can solve $g_{(1)ij} (l = 1, 2, \cdots)$ with respect to $g_{(0)ij}$ using equations of motion. When substituting the expression (146) or (147) into the classical action (1), the action contains the divergence from $\rho \to 0$ in general situation. We regularize the divergence by introducing a cutoff parameter $\epsilon$:

$$
\int d^{d+1}x \to \int d^d x \int_\epsilon d\rho , \int_{M_d} d^d x (\cdots) \to \int d^d x (\cdots) \bigg|_{\rho = \epsilon} .
$$

Then the action (1) can be expanded as a power series of $\epsilon$:

$$
S = S_0 (g_{(0)ij}) \epsilon^{-\frac{d}{2}} + S_1 (g_{(0)ij}, g_{(1)ij}) \epsilon^{-\frac{d}{2} - 1} + \cdots + S_n \ln \epsilon - S_\frac{d}{2} + O(\epsilon^\frac{d}{2}) .
$$

The term $S_n$ proportional to $\ln \epsilon$ appears when $d$ =even. In (149), the terms proportional to the inverse power of $\epsilon$ in the regularized action are invariant under the scale transformation

$$
\delta g_{(0)\mu \nu} = 2 \delta \sigma g_{(0)\mu \nu} , \delta \epsilon = 2 \delta \sigma \epsilon .
$$

The invariance is caused by (142). The subtraction of these terms proportional to the inverse power of $\epsilon$ does not break the invariance. When $d$ is even, however, there appears the term
$S_{\ln}$ proportional to $\ln \epsilon$. The subtraction of the term $S_{\ln}$ breaks the invariance under the transformation (150). The reason is that the variation of the $\ln \epsilon$ term under the scale transformation (150) is finite when $\epsilon \to 0$ since $\ln \epsilon \to \ln \epsilon + \ln(2\delta \sigma)$. Therefore the variation should be cancelled by the variation of the finite term $S_{d2}$ (which does not depend on $\epsilon$)

$$
\delta S_{d2} = \ln(2\sigma)S_{\ln} \tag{151}
$$

since the original total action (1) is invariant under the scale transformation. Since the action $S_{d2}$ can be regarded as the action renormalized by the subtraction of the terms which diverge when $\epsilon \to 0$, the $\ln \epsilon$ term $S_{\ln}$ gives the conformal anomaly $T$ of the renormalized theory on the boundary $M_d$:

$$
S_{\ln} = \frac{1}{2} \int d^d x \sqrt{-g(0)} T. \tag{152}
$$

When $d = 4$, by substituting the expressions in (147) into the action (1) and using the regularization in (148), we find

$$
S_{\ln} = -\frac{1}{2} \int d^d x \sqrt{-g(0)} \left[ l \left( AR_{(0)}^{ij} + B R_{(0)ij} + C R_{(0)ijkl} R_{ijkl} \right) + \left( 40a + \frac{8b}{l^3} + \frac{4c}{l^3} + \frac{6}{l \kappa^2} - \frac{l \Lambda}{2} \right) g_{(0)}^{ij} g_{(2)ij} \right. \\
- \left( 40a + \frac{12b}{l} + \frac{12c}{l} + \frac{l \kappa^2}{l^3} \right) g_{(1)ij} R_{(0)}^{ij} \right. \\
- \left( -\frac{8a}{l} - \frac{2b}{l} - \frac{2c}{l} - \frac{l}{2 \kappa^2} \right) R_{(0)} g_{(0)ij} g_{(1)ij} \right. \\
+ \left( \frac{20a}{l^3} + \frac{8b}{l^3} + \frac{10c}{l^3} - \frac{2}{l \kappa^2} + \frac{l \Lambda}{4} \right) g_{(0)g_{(0)}g_{(1)ij} g_{(1)ij}} \right. \\
+ \left( \frac{6a}{l^3} + \frac{2b}{l^3} + \frac{c}{l^3} + \frac{1}{2l \kappa^2} - \frac{l \Lambda}{8} \right) \left( g_{(0)g_{(1)ij}} \right)^2 \left. \right] . \tag{153}
$$

All this reasoning is very similar to the one in AdS/CFT correspondence.

Other terms proportional to $\epsilon^{-1}$ or $\epsilon^{-2}$ which diverge when $\epsilon \to 0$ can be subtracted without loss of the general covariance and scale invariance. The equation obtained by the variation over $g_{(1)ij}$ is given by

$$
0 = AR_{(0)}^{ij} + B g_{(0)g_{(0)}g_{(1)ij}} + 2C g_{(0)g_{(0)}g_{(1)ijkl} g_{(1)kl}} + 2D g_{(0)g_{(0)}g_{(1)ijkl}} , \tag{154}
$$

where

\begin{align*}
A &\equiv - 40a - \frac{12b}{l} - \frac{4c}{l} - \frac{l \kappa^2}{l^3} , \\
B &\equiv \frac{8a}{l} + \frac{2b}{l} + \frac{2c}{l} + \frac{l \kappa^2}{2l^2} , \\
C &\equiv \frac{20a}{l^3} + \frac{8b}{l^3} + \frac{10c}{l^3} - \frac{2}{l \kappa^2} + \frac{l \Lambda}{4} = \frac{40a}{l^3} + \frac{12b}{l^3} + \frac{12c}{l^3} + \frac{1}{l \kappa^2} , \\
D &\equiv \frac{6a}{l^3} + \frac{2b}{l^3} + \frac{c}{l^3} + \frac{1}{2l \kappa^2} - \frac{l \Lambda}{8} = -\frac{4a}{l^3} - \frac{1}{l \kappa^2} .
\end{align*}
Here (139) is used in order to rewrite the expressions of $C$ and $D$ and to remove the cosmological constant $\Lambda$. Multiplying $g^{(1)ij}$ with (154), we obtain

$$g^{(1)ij}g^{(1)ij} = \frac{A + 4B}{2(C + 4D)} R^{(0)} .$$

(155)

Substituting (155) into (154), we can solve (154) with respect to $g^{(1)ij}$ as follows:

$$g^{(1)ij} = -\frac{A}{2C} R^{(0)ij} + \frac{AD - BC}{2C(C + 4D)} R^{(0)}g^{(0)ij} .$$

(156)

Substituting (156) into (153), one finds the following expression for the anomaly $T$

$$T = -\left( al + \frac{A^2 D - 4B^2 C - 2ABC}{4C(C + 4D)} \right) R^{(0)}$$

$$-\left( bl - \frac{A^2}{4C} \right) R^{(0)ij} R^{ij} - c R^{(0)ijkl} R^{ijkl} .$$

(157)

Substituting (154) into (157), one gets

$$T = -\left( \frac{l_3^3}{8\kappa^2} + 5al + bl \right) (G - F) - \frac{cl}{2} (G + F) .$$

(158)

Here Gauss-Bonnet invariant $G$ and the square of Weyl tensor $F$, which are given in (18) appeared.

We should note that the relative signs in (158) of the term coming from the Einstein term proportional to $\frac{1}{\kappa^2}$ and the terms coming from $R^2$ terms proportional to $a, b$ or $c$ are different from the expression of the anomaly [17] in AdS/CFT. The apparent reason is that the length parameter $l_{dS}$ in de Sitter space is related with that $l_{AdS}$ in AdS by $l_{dS}^2 = -l_{AdS}^2$. Of course the difference can be absorbed into the redefinition of the parameters $a, b$ and $c$. We also note that there is an ambiguity of the sign when we relate, in (149) the action in the bulk with Lorentzian signature with that in the boundary with Euclidean signature.

As some speculative example we take ${\mathcal N} = 2$ SCFT theory with $n_V$ vector multiplets and $n_H$ hypermultiplets where QFT conformal anomaly is given by

$$T = \frac{1}{24 \cdot 16\pi^2} \left[ -\frac{1}{3} (11n_V + n_H) R^2 
+12n_V R_{ij} R^{ij} + (n_H - n_V) R_{ijkl} R^{ijkl} \right] ,$$

(159)

and especially, when the gauge group is $Sp(N)$,

$$T = \frac{1}{24 \cdot 16\pi^2} \left[ -\left( 8N^2 + 6N - \frac{1}{3} \right) R^2 
+(24N^2 + 12N) R_{ij} R^{ij} + (6N - 1) R_{ijkl} R^{ijkl} \right] .$$

(160)

In the framework of AdS/CFT correspondence, the ${\mathcal N} = 2$ theory with the gauge group $Sp(N)$ arises as the low-energy theory on the world volume on $N$ D3-branes sitting inside 8
D7-branes at an O7-plane [35]. The string theory dual to this theory has been conjectured to be type IIB string theory on $\text{AdS}_5 \times X^5$ where $X_5 = S^5/Z_2$ [36], whose low energy effective action is given by
\[
S = \int_{\text{AdS}_5} d^5x \sqrt{G} \left\{ \frac{N^2}{4\pi^2} (R - 2\Lambda) + \frac{6N}{24 \cdot 16\pi^2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\}.
\] (161)

In case of AdS/CFT, the conformal anomaly (160) can be reproduced from the action (161) [17] to the next-to-leading order of $\frac{1}{N}$. For the expression (158) derived in the framework of dS/CFT, the anomaly (160) can be reproduced to the next-to-leading order of $\frac{1}{N}$ if we choose
\[
\frac{1}{\kappa^2} = \frac{N^2}{4\pi^2}, \quad a = b = 0, \quad c = -\frac{6N}{24 \cdot 16\pi^2}.
\] (162)

Furthermore we also choose
\[
\Lambda = \frac{12N^2}{12\pi^2},
\] (163)
to make the length scale $l$ be unity in the leading order of $\frac{1}{N}$. Substituting (162) and (163) into (139), we find
\[
\frac{1}{l^2} = 1 - \frac{2ck^2}{3} + O \left( (ck^2)^2 \right).
\] (164)

Then conformal anomaly looks
\[
T = \left( -\frac{1}{8\kappa^2} + \frac{c}{8} \right) (G - F) + \frac{c}{2} (G + F) + \frac{1}{\kappa^2} O \left( (ck^2)^2 \right)
\] 
\[
= \frac{N^2}{16\pi^2} \left( \frac{1}{3} R_{(0)}^2 + R_{(0)ij} R_{(0)}^{ij} \right) + \frac{6N}{24 \cdot 16\pi^2} \left( \frac{3}{4} R_{(0)}^2 - \frac{13}{4} R_{(0)ij} R_{(0)}^{ij} + R_{(0)ijkl} R_{(0)}^{ijkl} \right) + O(1).
\] (165)

In the first line in (165), we substituted (164) and putted $a = b = 0$ and in the second line, (162), (163) and the explicit expression for $G$ and $F$ in (18) are substituted. Eq. (165) exactly reproduces the result in (159) as in [17]. We should note, however, the parameters in (162) and (163) correspond to the action
\[
S = \int_{\text{dS}_5} d^5x \sqrt{G} \left\{ \frac{N^2}{4\pi^2} (R - 2\Lambda) - \frac{6N}{24 \cdot 16\pi^2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\}.
\] (166)

Comparing (166) with (161), one sees the sign on front of $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ term is different besides the sign of the cosmological constant.

Hence, even in situation where we do not know the string theoretical framework for dS/CFT correspondence as in AdS/CFT set-up, we constructed 5d de Sitter HD gravity action, which might be dual to 4d $Sp(N)$ $\mathcal{N} = 2$ super-Yang-Mills theory. This speculation may suggest the strategy for search of corresponding string (M-theory) configurations.
In the present paper we constructed number of cosmological and BH brane-worlds for general model of HD gravity (mainly in five dimensions). The essential feature of the theory under consideration is the presence of Riemann tensor square term (non-zero $c$ case) and not only of curvature square and Ricci tensor square terms as usually occurs in four dimensions. Thermodynamics of S-AdS BH which is exact solution of theory only when $c = 0$ is described in detail (perturbation on $c$ is used). The entropy, energy and free energy are calculated. It is demonstrated that if $c$ is not zero the entropy (energy) is not proportional to the area (mass). Moreover, in such a case the entropies found by different regularization methods do not coincide (usually, in Einstein gravity or HD gravity where $c$ is zero they coincide).

The brane equation of motion (bulk is BH) is presented as FRW equation. Using AdS/CFT correspondence in the form presented by Verlinde it is shown that dual QFT is not conformal theory when Riemann tensor term presents in the theory. As a result the holographic (Hubble) entropy does not coincide with BH entropy when $c$ is not zero (usually in AdS/CFT set-up they do coincide). The conformal symmetry breaking probably is responsible for unusual behaviour of BH entropy.

Asymmetrically warped spacetime (charged black hole) induced by $c$-term effect which plays the role of charge is constructed. It is interesting that Lorentz invariance violation for such background occurs. It is also presented cosmological dS brane playing the role of the boundary which connects two bulk dS spaces. The radius of such dS brane which may be useful for construction of inflationary universe is found in terms of parameters of HD gravity. Such brane may find the applications in framework of proposed dS/CFT correspondence. The holographic conformal anomaly from five dimensional dS HD gravity is also evaluated.

In general, the results of our work demonstrate how one can find different brane-world solutions and describe their properties for HD gravity. Of course, there are many interesting topics left for future investigation. For example, how to find the dynamical entropy bounds for dS brane in the way similar to refs. [15,26]? This may help in better understanding of dS/CFT correspondence (if it exists). Another interesting question is related with the effect of parameter $c$ to graviton propagator and the corresponding trapping of HD gravity.

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APPENDIX A: COMPARISON WITH $C = 0 \, R^2$-GRAVITY

In this Appendix for completeness we summarize the results with $c = 0$ case in the action (29) based on [18].

When $c = 0$, Schwarzschild-anti de Sitter space is an exact solution:

$$ds^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu = -e^{2\rho_0} dt^2 + e^{-2\rho_0} dr^2 + r^2 \sum_{i,j} g_{ij} dx^i dx^j ,$$

$$e^{2\rho_0} = \frac{1}{r^{d-2}} \left( -\mu + \frac{k r^{d-2}}{d-2} + \frac{r^d}{l^2} \right) . \quad (A1)$$

The curvatures have the following form:

$$\hat{R} = -\frac{d(d+1)}{l^2} , \quad \hat{R}_{\mu\nu} = -\frac{d}{l^2} \hat{G}_{\mu\nu} . \quad (A2)$$

In (A1), $\mu$ is the parameter corresponding to mass and the scale parameter $l$ is given by solving the following equation:

$$0 = \frac{d^2(d+1)(d-3)a}{l^4} + \frac{d^2(d-3)b}{l^4} - \frac{d(d-1)}{\kappa^2 l^2} - \Lambda . \quad (A3)$$

We also assume $g_{ij}$ corresponds to the Einstein manifold, defined by $r_{ij} = k g_{ij}$, where $r_{ij}$ is the Ricci tensor defined by $g_{ij}$ and $k$ is the constant. By using the method parallel with section 4, we found the following thermodynamical quantities:

$$F = -\frac{V_3}{8} r_H^2 \left( \frac{r_H^2}{l^2} - \frac{k}{2} \right) \left( \frac{8}{\kappa^2} - \frac{320 a}{l^2} - \frac{64 b}{l^2} \right) , \quad (A4)$$

$$S = \frac{V_3 \pi r_H^3}{2} \left( \frac{8}{\kappa^2} - \frac{320 a}{l^2} - \frac{64 b}{l^2} \right) , \quad (A5)$$

$$E = \frac{3V_3 \mu}{8} \left( \frac{8}{\kappa^2} - \frac{320 a}{l^2} - \frac{64 b}{l^2} \right) , \quad (A6)$$

which seem to tell that the contribution from the $R^2$-terms can be absorbed into the redefinition:

$$\frac{1}{\kappa^2} = \frac{1}{\kappa^2} + \frac{40a}{l^2} - \frac{8b}{l^2} , \quad (A7)$$

although this is not true for $c \neq 0$ case.

The FRW type equations for $c = 0$ case correspond to (87) and (90):

$$H^2 = -\frac{k}{(d-2)r^2} + \frac{\kappa_d^2}{(d-1)(d-2)V} \tilde{E} ,$$

$$\dot{H} = -\frac{\kappa_d^2}{2(d-2)} \left( \frac{\tilde{E}}{V} + p \right) + \frac{k}{(d-2)r^2} ,$$

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\[
\tilde{E} = \frac{(d-1)(d-2)\mu V_{d-1}}{\kappa_d^2 r},
\]
\[
p = \frac{(d-2)\mu}{r^d\kappa_d^2},
\]
\[
V = r^{d-1}V_{d-1}.
\]
(A8)

Here \(\kappa_d\) is \(d\) dimensional gravitational coupling, which is given by

\[
\kappa_d^2 = \frac{2\tilde{\kappa}^2}{l}.
\]
(A9)

In case of the Einstein gravity \((a = b = c = 0\) in (1)), the equation corresponding to (A9) has the form

\[
\kappa_{(\text{Ein})d}^2 = \frac{2\kappa^2}{l}.
\]
(A10)

Then the effects of the higher derivative terms, when \(c = 0\), appear through the redefinition of \(\kappa^2\) to \(\tilde{\kappa}^2\).

When \(d = 4\), using (6) we find

\[
\tilde{E} = \frac{l}{r^2}E.
\]
(A11)

Here \(E\) is given by (A6). The factor \(\frac{l}{r}\) can be understood as the inverse of the factor for the time variables between the brane and the bulk in (92).

Using Cardy-Verlinde formula (98), one has the expression for the entropy \(\tilde{S}\) for \(c = 0\) case by using (99). Especially we can evaluate holographic (Hubble) entropy \(\tilde{S}\) when the brane crosses the horizon \(r = r_H\):

\[
\tilde{S} = \frac{4\pi(d-2)V}{l\kappa_d^2} = \frac{2\pi(d-2)r_{H^{d-1}}}{\tilde{\kappa}^2}V_{d-1}.
\]
(A12)

For \(d = 4\), the entropy \(\tilde{S}\) is identical with the black hole entropy in (5)

\[
\tilde{S} = S.
\]
(A13)

The relation (A13) breaks when \(c \neq 0\), as shown in (102).
REFERENCES


[29] A. Strominger, hep-th/0106113;