Numerical relativity describes a discrete initial value problem for general relativity. A choice of coordinate gauge is provided by slicing space-time into space-like hypersurfaces. This introduces past and future gauge relative to the hypersurface of present time. Here, we propose solving the discretized Einstein equations with a choice of gauge in the future and a dynamical gauge in the past. The method is illustrated on a polarized Gowdy wave.

The initial value problem for general relativity is receiving much attention in the prediction of wave-forms from candidate sources for the upcoming gravitational wave detectors LIGO/VIRGO [1,2]. The structure of gravitational waves has recently been elucidated in new hyperbolic formulations of general relativity (see, e.g., [3] for references), which holds promise for accurate integration schemes. Long-time integrations also require accurate conservation of the associated elliptic constraints, representing energy and momentum conservation [4]. While analytically these constraints are conserved under dynamical evolution, the nonlinear nature of the equation typically tends to introduce some numerical departure from exact conservation.

In numerical relativity the coordinate gauge is chosen by slicing space-time into space-like hypersurfaces. Data from one hypersurface are evolved numerically onto the next, for example using a hyperbolic formulation. Algebraic slicing is particularly illustrative, which considers prescribed lapse and shift functions \( N_a = g_{ta} \) for the metric tensor \( g_{ab} \).

Here, we propose an new approach for numerical relativity with preservation of the constraints by introducing a dynamical gauge in the past. With prescribed gauge and dynamical
space-like components of the metric in the future, this obtains a complete system of ten equa-
tions in ten unknowns. In a sense, a dynamical gauge in the past epitomizes gauge freedom:
the notion that all physically relevant quantities assume gauge-invariance. A choice of gauge
is necessary, however, for a specific representation of the structure of space-time is required
for detailed calculations.

We illustrate this approach on the vacuum Einstein equations, described by the vanishing
Ricci tensor

\[ R_{ab} = 0. \]  \hspace{1cm} (1)

The Ricci tensor \( R_{ab} \) is a second-order expression in the metric \( g_{ab} \). Hence, (1) defines a
relationship between metric data \((g^{n-1}_{ab}, g^n_{ab}, g^{n+1}_{ab})\) on a triple of time-slices \( t_{n-1} < t_n < t_{n+1} \):

\[ R_{ab} \left( g^{n+1}_{ab}, g^n_{ab}, g^{n-1}_{ab} \right) = 0. \]  \hspace{1cm} (2)

Here, \( R_{bd} = R^a_{bcd} \) derived from the Riemann tensor

\[ R^a_{bcd} = \partial_d \Gamma^a_{bc} - \partial_c \Gamma^a_{bd} + \Gamma^e_{bc} \Gamma^a_{ed} - \Gamma^e_{bd} \Gamma^a_{ec}. \]  \hspace{1cm} (3)

This expression (3) can be discretized by finite differencing on a triple of time-slices, while
preserving the exact symmetry \( R_{ab} = R_{ba} \).

Algebraic gauge-fixing takes the form of specifying the components \( N_a = g_{ta} \) in coor-
dinates \( \{x^a\}_{a=1}^4 \) with \( t = x^1 \) time-like. A gauge-choice on a triple of time-slices, therefore,
amounts to a choice of \((N^{n-1}_a, N^n_a, N^{n+1}_a)\). Recall that this gauge-choice in the metric arises
explicitly in the Gauss-Codacci relations for energy-momentum conservation. The com-
ponents \( h_{ij} = g_{ij} \), where \( i, j = 2, 3, 4 \) refer to the metric in the time-slice \( t = \text{const.} \), which
contain the dynamical part of the metric. This \((h_{ij}, N_a)\) reflects the mixed hyperbolic-elliptic
structure in numerical relativity and (1) represents ten evolution equations in these variables
on a triple of time-slices.

In algebraic gauge-fixing, we prescribe \( N^{n+1}_a \) as a future gauge in computing \( h^{n+1}_{ij} \) on
a future hypersurface \( t = t_{n+1} \) from data at present and past hypersurfaces \( t = t_{n-1} \) and
\( t = t_n \). We propose closure by re-introducing \( N_{a}^{n-1} \) as dynamical gauge in the past, leaving \( h_{ij}^{n-1} \) fixed. Combined, this defines an advanced hyperbolic-retarded elliptic evolution of the metric. The discrete problem in \((h_{ij}^{n+1}, N_{a}^{n-1})\) takes the implicit form of ten equations

\[
R_{ab}(h_{ij}^{n+1}, N_{a}^{n-1}, \cdots) = 0. \tag{4}
\]

Here the dots refer to the remaining data \((h_{ij}^{n-1}, h_{ij}^{n}, N_{a}^{n}, N_{a}^{n+1})\), which are kept fixed while solving for \((h_{ij}^{n+1}, N_{a}^{n-1})\). Thus, (4) which takes into account all ten Einstein equations with no reduction of variables. Time-stepping by (4) evolves the metric into the future with dynamical gauge in the past, in an effort to satisfy the constraint equations.

The presented approach can be illustrated on a polarized Gowdy wave. Gowdy cosmologies are an extensively studied class of universes with compact space-like hypersurfaces with two Killing vectors \( \partial_{\sigma} \) and \( \partial_{\delta} \). With cyclic boundary conditions, the space-like hypersurfaces are homeomorphic to the three-torus as a model universe collapsing into a singularity. The associated line-element is (see, e.g., [5])

\[
ds^2 = e^{(\tau-\lambda)/2} \left(-e^{-2\tau} d\tau^2 + d\theta^2\right) + d\Sigma^2, \tag{5}\]

where \( \lambda = \lambda(\tau, \theta) \) and \( d\Sigma \) denotes the surface element in the space supported by the two Killing vectors. Polarized Gowdy waves form a special case, which permit a reduction to

\[
d\Sigma^2 = e^{-\tau} \left(e^P d\sigma^2 + e^{-P} d\delta^2\right). \tag{6}\]

Here \( P \) satisfies a linear wave-equation \( P_{\tau\tau} = e^{-2\tau} P_{\theta\theta}; \) a long wave-length solution is

\[
P_0(\tau, \theta) = Y_0(e^{-\tau}) \cos \theta, \tag{7}\]

where \( Y_0 \) is the Bessel function of the second kind of order zero. This leaves

\[
\lambda(\tau, \theta) = \frac{1}{2} Y_0(e^{-\tau}) Y_1(e^{-\tau}) e^{-\tau} \cos 2\theta + \frac{1}{2} \int_{e^{-\tau}}^{1} \left(Y_0^2(s) + Y_1^2(s)\right) sds. \tag{8}\]

An spectrally accurate numerical integration is described in [6].

The implicit equation (4) for the dynamical variables \((h_{ij}^{n+1}, N_{a}^{n-1})\) can be implemented numerically. We have done so by solving for the five components \((h_{ii}^{n+1}, N_{i}^{n-1}, N_{\theta}^{n-1})\) using
Newton iterations on the four diagonal plus the $\tau\theta$–component of (4). The choice of future gauge $N^{n+1}_a$ is provided by the analytical solution (5-8).

Figure 1 shows numerical results for evolution of initial data on $0 \leq \tau \leq 4$. The results show that all Einstein equations in the form of $R_{ab} = 0$ are satisfied with arbitrary precision, while the metric components are solved accurately to within one percent.

In summary, a dynamical gauge in the past completes the degrees of freedom in the future hypersurface, thereby permitting all ten Einstein equations to be satisfied with arbitrary precision. The simulation of a nonlinear one-dimensional Gowdy wave serves to illustrate a numerical implementation. It would be of interest to consider this approach in higher dimensions, including a self-consistent integration of any of the modern hyperbolic formulations and efficient elliptic solvers.

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Figure Captions

**FIGURE 1.** Shown is the simulation for $0 \leq \tau \leq 4$ of the polarized Gowdy wave. The solutions $P(\tau, \theta)$ and $\lambda(\tau, \theta)$ are displayed as a function of $(\tau, \theta)$ (upper windows). The middle windows display the solutions for $\tau = 4$, wherein the circles denote the numerical solution and the solid lines the analytical solution. The $\tau$–evolution of the errors (lower windows) are computed relative to the analytical solution to Gowdy’s reduced wave equation. The simulations discretize $\theta$ by $m_1 = 64$ points and the $\tau$–interval by $m_2 = 1024$ time-steps. Particular to the proposed numerical algorithm is a dynamical gauge in the past and a prescribed gauge in the future. This permits satisfying all of the discretized Einstein equations $R_{ab} = 0$ to within arbitrary precision by Newton iterations.


FIGURE 1