Remarks on the canonical quantization of noncommutative theories

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Abstract

Free noncommutative fields constitute a natural and interesting example of constrained theories with higher derivatives. The quantization methods involving constraints in the higher derivative formalism can be nicely applied to these systems. We study real and complex free noncommutative scalar fields where momenta have an infinite number of terms. We show that these expressions can be summed in a closed way and lead to a set of Dirac brackets which matches the usual corresponding brackets of the commutative case.

I. INTRODUCTION

Recently, there have been a great deal of interest in noncommutative fields. This interest started when it was noted that noncommutative spaces naturally arise in perturbative string theory with a constant background magnetic field in the presence of $D$-branes. In this limit, the dynamics of the $D$-brane can be described by a noncommutative gauge theory [1]. Besides their origin in strings and branes, noncommutative field theories are a very interesting subject by their own rights [2]. They have been extensively studied under several approaches [3,4]. To obtain the noncommutative version of a field theory one essentially replaces the usual product of fields by the Moyal product [1,2], which leads to an infinite number of spacetime derivatives over the fields. It can be directly verified that the Moyal product does not alter quadratic terms in the action, provided boundary terms are discarded. In this way, the noncommutativity does not affect the equations of motion for free fields. However, on the other hand, we know that momenta can be obtained as surface terms of a hypersurface orthogonal to the time direction [5]. This means that momenta are different in the versions with and without Moyal products. In fact, momenta in the version with Moyal products have an infinite number of terms.

Hence, noncommutative field theories provide us with an interesting and non academic example involving higher derivatives where the quantization rules for such systems can be nicely applied. We emphasize that this is a very peculiar situation. Usually, nontrivial examples of systems with higher derivatives are just academical and plagued with ghosts and nonunitarity problems. In fact, one can say that these systems have never constituted a confident test for the nonconventional quantization procedure involving higher derivatives, meanly in the cases where there are constraints. Free noncommutative theories, even though can be described without the Moyal product, are then an interesting theoretical laboratory for using the higher derivative formalism with constraints. This is precisely the purpose of our paper. We are going to study the quantization of the free noncommutative scalar theory without discarding the infinite higher derivative terms of the Lagrangian. We shall see that, regardless the completely different expressions of the momenta, the canonical quantization can be consistently developed in terms of a constraint formalism.

Our paper is organized as follows. In Sec. II we present a simple example involving the usual free scalar theory (without Moyal product) but writing the action in terms of higher derivatives. The purpose of considering this simple example is to present some particularities of the higher derivative quantization method we are going to use in the next sections. In Sec. III we deal with free noncommutative real scalar fields and in Sec. IV we consider the complex case, where the momentum expressions are still more evolved. We left Sec. V for our conclusions.

II. A SIMPLE EXAMPLE

Let us consider a simple example of a free scalar field involving higher derivatives, which is given by the action

\[ S = -\frac{1}{2} \int d^4x \phi \square \phi \]  

(2.1)

Of course, disregarding boundary terms at infinity we may rewrite this action as \( S = \frac{1}{2} \int d^4x \partial_{\mu} \phi \partial^{\mu} \phi \), from which one obtains the momentum \( \pi = \dot{\phi} \). The canonical quantization is obtained by transmogrifying the Poisson bracket \( \{ \phi(\vec{x}, t), \pi(\vec{y}, t) \} = \delta(\vec{x} - \vec{y}) \) in the commutator of field operators \( [\phi(\vec{x}, t), \pi(\vec{y}, t)] = i \delta(\vec{x} - \vec{y}) \). This leads to the commutator \( [\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = i \delta(\vec{x} - \vec{y}) \) which
together with the equation of motion $\Box \phi = 0$ permit us to calculate the propagator and develop all the process of quantization. However, let us keep the action as given by (2.1). For systems with higher derivatives, velocities are unconventionally assumed to be independent coordinates [6,7]. In the particular case of action (2.1), the independent coordinates are $\phi$ and $\dot{\phi}$. It is accepted that there is a conjugate momentum for each of them. These can be obtained by fixing the variation of fields and velocities at just one of the extreme times, say $\phi(\bar{x}, t_0) = 0 = \phi(\bar{x}, t_0)$ and keeping the other extreme free [5]. The momenta conjugate to $\phi$ and $\dot{\phi}$ are respectively, taken at time $t$ over a hypersurface orthogonal to $t$.

Considering the variation of the action with just the extreme in $t_0$ kept fixed we have [8]

$$\delta S = \frac{1}{2} \int_{t_0}^{t} dt \int d^3 \bar{x} \left( \delta \phi \Box \phi + \phi \delta \phi \right)$$

$$= \int_{t_0}^{t} dt \int d^3 \bar{x} \left[ -\delta \phi \Box \phi + \frac{1}{2} \partial^\mu (\partial_\mu \phi \delta \phi - \phi \partial_\mu \delta \phi) \right]$$

$$= -\int_{t_0}^{t} dt \int d^3 \bar{x} \delta \phi \Box \phi + \frac{1}{2} \int d^3 \bar{x} \left( \phi \delta \dot{\phi} - \dot{\phi} \delta \phi \right)$$

$$= \frac{1}{2} \int d^3 \bar{x} \left( \phi \delta \dot{\phi} - \dot{\phi} \delta \phi \right)$$

(2.2)

where in the last step it was used the on-shell condition $\Box \phi = 0$. The quantities $\delta \phi$ and $\delta \dot{\phi}$ that appear in (2.2) are both taken at time $t$. From this boundary term we identify the conjugate momenta to $\phi$ and $\dot{\phi}$, which are the coefficients of $\delta \phi$ and $\delta \dot{\phi}$ respectively. We denote these momenta by

$$\pi = \frac{1}{2} \dot{\phi}$$

$$\pi^{(1)} = -\frac{1}{2} \dot{\phi}$$

(2.3)

(2.4)

We observe that even though the equation of motion is the same as the corresponding one of the case without higher derivatives, the momenta expressions are not.

Since now $\phi$ and $\dot{\phi}$ are independent coordinates, both expressions above are constraints [9]. This means that the commutators cannot be inferred from the fundamental Poisson brackets, namely

$$\{ \phi(\bar{x}, t), \pi(y, t) \} = \delta(\bar{x} - y)$$

$$\{ \phi(\bar{x}, t), \pi^{(1)}(y, t) \} = \delta(\bar{x} - y)$$

(2.5)

but from the Dirac ones. It is interesting to notice that the Poisson bracket $\{ \phi(\bar{x}, t), \dot{\phi}(\bar{y}, t) \}$ is zero. For the constraints given by (2.3) and (2.4) we obtain the Dirac brackets [8]

$$\{ \phi(\bar{x}), \pi(y) \}_D = \frac{1}{2} \delta(\bar{x} - y)$$

$$\{ \dot{\phi}(\bar{x}), \pi^{(1)}(y) \}_D = \frac{1}{2} \delta(\bar{x} - y)$$

(2.6)

from which we infer the commutators

$$[\phi(\bar{x}), \pi(y)] = \frac{i}{2} \delta(\bar{x} - y)$$

$$[\dot{\phi}(\bar{x}), \pi^{(1)}(y)] = \frac{i}{2} \delta(\bar{x} - y)$$

(2.7)

Using the expressions of the momenta given by (2.3) and (2.4) we observe that both relations above lead to the expected commutator $[\phi(\bar{x}), \dot{\phi}(\bar{y})] = i \delta(\bar{x} - \bar{y})$, which means that the quantization does not change, as it should be, even the boundary terms are contributing with different expressions for the momentum.

III. NONCOMMUTATIVE CASE

Let us consider the action

$$S = \frac{1}{2} \int_{t_0}^{t} dt \int d^3 \bar{x} \partial_\mu \phi \ast \partial^\mu \phi$$

(3.1)

where $\ast$ is the notation for the Moyal product, whose definition for two general fields $\phi_1$ and $\phi_2$ reads

$$\phi_1(x) \ast \phi_2(y) = \exp \left( \frac{i}{2} \theta^{\mu \nu} \partial_\mu \phi_1 \partial^\nu \phi_2 \right) \mid_{x=y}$$

(3.2)

and $\theta^{\mu \nu}$ is a constant antisymmetric matrix.

We notice that if one discards the surface terms, the derivatives of the Moyal product will not contribute for the action (3.1). Concerning to the equation of motion, this can be done without further considerations. However for the momenta we observe that one situation and another are completely different. Let us then follow a similar procedure to the previous section and make a variation of the action (3.1) by just keeping fixed the extreme in $t_0$. We obtain

$$\delta S = \frac{1}{2} \int_{t_0}^{t} dt \int d^3 \bar{x} \left( \partial_\mu \delta \phi \ast \partial^\mu \phi + \partial_\mu \phi \ast \partial^\mu \delta \phi \right)$$

$$= \frac{1}{2} \int_{t_0}^{t} dt \int d^3 \bar{x} \left( \delta \phi \partial_\mu \phi + \partial_\mu \phi \delta \phi \right)$$

$$= \frac{1}{2} \int d^3 \bar{x} \left( \delta \phi \ast \dot{\phi} + \dot{\phi} \ast \delta \phi \right)$$

(3.3)

where it was used the on-shell condition. The momenta shall be obtained from de development of (3.3). Using the definition of the Moyal product we have

$$\delta S = \int d^3 \bar{x} \left[ \dot{\phi} \delta \phi \right.$$}

$$+ \frac{1}{2} \left( \frac{i}{2} \right)^2 \theta^{\mu \nu} \theta^{\rho \sigma} \partial_\mu \partial_\nu \dot{\phi} \partial_\rho \partial_\sigma \delta \phi$$

$$+ \frac{1}{4!} \left( \frac{i}{2} \right)^4 \theta^{\mu \nu} \theta^{\rho \sigma} \theta^{\delta \gamma} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \dot{\phi} \partial_\delta \partial_\gamma \partial_\delta \partial_\gamma \delta \phi$$

$$+ \cdots$$

(3.4)
where we have used the short notation \( \bar{\theta} = \theta^{\delta} \partial_t \). Similarly, for the next term of the expression (3.4) we have
\[
\int d^3 \bar{x} \theta^{\mu
u} \theta^{\alpha\beta} \eta^{\alpha\beta} \bar{\phi} \partial_{\mu} \bar{\phi} \partial_{\nu} \partial_{\delta} \delta \partial \delta = \int d^3 \bar{x} \left( \bar{\theta}^2 \bar{\phi} \delta \phi + 2 \bar{\theta}^2 \bar{\phi} \delta \phi + \bar{\theta}^2 \bar{\phi} \delta \phi \right) \tag{3.5}
\]
where \( \bar{\theta} \) means \( n \)-time derivative over \( \phi \). We observe that odd terms in \( \theta^{\mu\nu} \) were canceled in the expression above. This was so because of the symmetric terms that appear in the first step of (3.3), actually necessary in the noncommutative case. In addition, due to the integration over \( d^3 \bar{x} \) only terms in \( \theta^{\delta i} \) will survive in the Moyal product \(^1\). For the quadratic term in \( \theta^{\mu\nu} \) we obtain
\[
\int d^3 \bar{x} \theta^{\mu
u} \theta^{\alpha\beta} \theta^{\gamma\delta} \bar{\eta} \partial_{\nu} \partial_{\delta} \partial_{\beta} \partial_{\gamma} \delta \phi = \int d^3 \bar{x} \left( \bar{\theta}^4 \phi \delta \phi + 4 \bar{\theta}^4 \phi \delta \phi + 6 \bar{\theta}^4 \phi \delta \phi + 4 \bar{\theta}^4 \phi \delta \phi + \bar{\theta}^4 \delta \phi + \bar{\theta}^4 \delta \phi \right) \tag{3.6}
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where \( \bar{\delta} \) means \( n \)-time derivative over \( \phi \). We observe that odd terms in \( \theta^{\mu\nu} \) were canceled in the expression above. This was so because of the symmetric terms that appear in the first step of (3.3), actually necessary in the noncommutative case. In addition, due to the integration over \( d^3 \bar{x} \) only terms in \( \theta^{\delta i} \) will survive in the Moyal product \(^1\). For the quadratic term in \( \theta^{\mu\nu} \) we obtain

\[^1\]It was pointed out by Gomis and Mehen [4], that the \( \theta^{\mu\nu} \) should be taken zero in the vertex terms in order to avoid causality and unitarity problems. However, the role played by these terms in the free case is not the same.
direct way (see Appendix A). The most relevant bracket for the obtainment of the propagator and the remaining of the quantization procedure is

$$\{\phi(\vec{x}, t), \phi(\vec{y}, t)\}_D = \delta(\vec{x} - \vec{y})$$

what means that the canonical quantization, even starting from the nontrivial momentum expressions (3.10) and (3.11), or (3.12) and (3.13), leads to same result of the corresponding free commutative theory.

**IV. COMPLEX SCALAR FIELDS**

We have seen in the previous analysis that there was a cancelation of odd terms in $\theta^{\mu \nu}$ in the $\delta S$ on-shell variation given by (3.3). This was so due to the symmetry of the real scalar fields in the action (3.1). In this section we are going to consider complex scalar fields where this symmetry does not exist and consequently that cancelation does not occur.

The noncommutative action for complex fields reads

$$S = \int d^4x \partial_\mu \phi^* \ast \partial^\mu \phi$$

We could have also written here a symmetric quantity by adding a term with $\partial_\mu \phi \ast \partial^\mu \phi^*$ into the Lagrangian of the action (4.1). We are going to see that this is nonetheless necessary because the action with $\partial_\mu \phi \ast \partial^\mu \phi^*$, even though having different momenta expressions, leads to the same quantum result of the one given by (4.1). What is important to notice is that the Lagrangian of the action (4.1) does not have any problem related to hermiticity, i.e., $(\partial_\mu \phi^* \ast \partial^\mu \phi)^* = \partial_\mu \phi^* \ast \partial^\mu \phi$, and discarding boundary terms, the action (4.1) leads to the usual free case $S = \int d^4x \partial_\mu \phi^* \partial^\mu \phi$.

Following the same steps as those of the previous section, we get

$$\delta S = \int d^3\vec{x} (\delta \phi^* \ast \phi + \phi^* \ast \delta \phi)$$

whose a similar development permit us to obtain the momenta

$$\pi = \sum_{p,n=0}^{\infty} \frac{1}{p!(2n)!} (-\frac{i}{2} \sqrt{\nabla^2})^{p+2n} \phi^*$$

$$\pi^{(1)} = \sum_{p,n=0}^{\infty} \frac{1}{p!(2n+1)!} (-\frac{i}{2} \sqrt{\nabla^2})^{p+2n+1} \phi^*$$

$$\pi^* = \sum_{p,n=0}^{\infty} \frac{1}{p!(2n)!} (-\frac{i}{2} \sqrt{\nabla^2})^{p+2n} \phi$$

$$\pi^{(1)*} = \sum_{p,n=0}^{\infty} \frac{1}{p!(2n+1)!} (-\frac{i}{2} \sqrt{\nabla^2})^{p+2n+1} \phi$$

respectively conjugate to $\phi$, $\dot{\phi}$, $\dot{\phi}^*$, and $\dot{\phi}^*$. Also here, these sums lead to closed expressions

$$\pi = \frac{1}{2} [1 + \cosh(i\vec{\partial} \sqrt{\nabla^2})] \phi^*$$

$$\pi^{(1)} = \frac{1}{2} [-1 + \cosh(i\vec{\partial} \sqrt{\nabla^2})] \phi^*$$

$$\pi^* = \frac{1}{2} [1 + \cosh(i\vec{\partial} \sqrt{\nabla^2})] \phi$$

$$\pi^{(1)*} = \frac{1}{2} [-1 + \cosh(i\vec{\partial} \sqrt{\nabla^2})] \phi$$

where the $\sinh$-operators come from the odd terms in $\theta^{\mu \nu}$. All the relations above are constraints. The calculation of the Dirac brackets is a kind of direct algebraic work (see Appendix B). The important point is that the brackets

$$\{\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)\}_D = \delta(\vec{x} - \vec{y})$$

$$\{\phi^*(\vec{x}, t), \dot{\phi}^*(\vec{y}, t)\}_D = \delta(\vec{x} - \vec{y})$$

are obtained, which means that the canonical quantization is correctly achieved.

To conclude this section, let us mention that we could have started from

$$\tilde{S} = \int d^4x \partial_\mu \phi \ast \partial^\mu \phi^*$$

instead of the action (4.1). Considering the on-shell variation of $\tilde{S}$ and keeping one of the extreme times fixed we have

$$\delta \tilde{S} = \int d^3\vec{x} (\delta \phi \ast \dot{\phi}^* + \dot{\phi} \ast \delta \phi^*)$$

which leads to expressions for the momenta similar to (4.7)-(4.10) with a changing in the sign of the $\sinh$-terms. Even though the expressions for the momenta are not equivalent in the two cases, we can trivially show that the constrained canonical procedure leads to the same Dirac brackets given by (4.11).
V. CONCLUSION

We have studied the free noncommutative scalar theory by using the constrained canonical formalism in the appropriate form for dealing with higher order derivative theories. This means that we have considered the momenta as defined as the coefficients of $\delta \phi$ and $\delta \phi$ calculated on the hypersurface orthogonal to the time direction. We have shown that the evolved expressions coming from the momenta definitions can be summed in a closed way making it possible to be harmoniously applied in the Dirac constrained formalism. We have also considered the complex scalar fields, where the momentum expressions are still more evolved.

These examples naturally obtained from noncommutative theories make possible to verify the consistency of the constrained canonical quantization procedure involving higher derivatives which is in some sense a controversial subject in the literature.

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APPENDIX A

DIRAC BRACKETS FOR THE REAL SCALAR CASE

Let us denote constraints (3.12) and (3.13) in a simplified notation like

\begin{align}
T^0 &= \pi + K \dot{\phi} \\
T^1 &= \pi^{(1)} + L \phi
\end{align}

where $K$ and $L$ are the operators

\begin{align}
K &= -\frac{1}{2} \left[ 1 + \cosh(i\bar{\partial}\sqrt{\bar{\nabla}^2}) \right] \\
L &= \frac{1}{2} \left[ 1 - \cosh(i\bar{\partial}\sqrt{\bar{\nabla}^2}) \right]
\end{align}

Using the fundamental Poisson brackets given by (3.14) and (3.15) we have

\begin{align}
\{T^0(\bar{x}, t), T^1(\bar{y}, t)\} &= -L_y \delta(\bar{x} - \bar{y}) + K_x \delta(\bar{x} - \bar{y}) \\
&= -\delta(\bar{x} - \bar{y}) \\
&= -\{T^1(\bar{x}, t), T^0(\bar{y}, t)\}
\end{align}

where in the last step there was a providential cancelation of the even operators $\cosh(i\bar{\partial}\sqrt{\bar{\nabla}^2})$ acting on $\delta(\bar{x} - \bar{y})$. Since the remaining Poisson brackets of the constraints are zero, we have that the corresponding Poisson brackets matrix is given by

\[ M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta(\bar{x} - \bar{y}) \]  

whose inverse is directly obtained, as is also the Dirac brackets (3.16).

APPENDIX B

DIRAC BRACKETS FOR THE COMPLEX CASE

Let us denote the corresponding constraints by

\begin{align}
T^0 &= \pi + K_1 \dot{\phi} + K_2 \dot{\phi}^* \\
T^1 &= \pi^{(1)} + L_1 \dot{\phi} + L_2 \dot{\phi}^* \\
T^2 &= \pi^* + K_1 \dot{\phi} - K_2 \dot{\phi} \\
T^0 &= \pi^{(1)*} - L_1 \dot{\phi} + L_2 \dot{\phi}
\end{align}

where $K_1$, $K_2$, $L_1$, and $L_2$ are short notations for the operators

\begin{align}
K_1 &= -\frac{1}{2} \left[ 1 + \cosh(i\bar{\partial}\sqrt{\bar{\nabla}^2}) \right] \\
K_2 &= \frac{1}{2} \sinh(i\bar{\partial}\sqrt{\bar{\nabla}^2}) \sqrt{\bar{\nabla}^2} \\
L_1 &= \frac{1}{2} \sinh(i\bar{\partial}\sqrt{\bar{\nabla}^2}) \frac{1}{\sqrt{\bar{\nabla}^2}} \\
L_2 &= \frac{1}{2} \left[ 1 - \cosh(i\bar{\partial}\sqrt{\bar{\nabla}^2}) \right]
\end{align}

The Poisson brackets for these constraints are

\begin{align}
\{T^0(\bar{x}, t), T^2(\bar{y}, t)\} &= (K_{2y} + K_{2x}) \delta(\bar{x} - \bar{y}) = 0 \\
\{T^0(\bar{x}, t), T^3(\bar{y}, t)\} &= (-L_{2y} + K_{1x}) \delta(\bar{x} - \bar{y}) \\
&= -\delta(\bar{x} - \bar{y}) \\
\{T^1(\bar{x}, t), T^2(\bar{y}, t)\} &= (K_{1y} + L_{2x}) \delta(\bar{x} - \bar{y}) \\
&= \delta(\bar{x} - \bar{y}) \\
\{T^1(\bar{x}, t), T^3(\bar{y}, t)\} &= (L_{1y} + L_{1x}) \delta(\bar{x} - \bar{y}) = 0
\end{align}

The remaining brackets are trivially zero. It is interesting to observe the harmonious cancelation among the different operators acting on the delta function. Now, we can easily construct the matrix of the Poisson brackets of the constraints and calculate the relevant Dirac brackets given by (4.11).

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