Quantum stochastic equation for the low density limit

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Abstract

A new derivation of quantum stochastic differential equation for the evolution operator in the low density limit is presented. We use the distribution approach and derive a new algebra for quadratic master fields in the low density limit by using the energy representation. We formulate the stochastic golden rule in the low density limit case for a system coupling with Bose field via quadratic interaction. In particular the vacuum expectation value of the evolution operator is computed and its exponential decay is shown.

1 Introduction

There are many studies of the large time behaviour in quantum theory. One of the powerful methods is the stochastic limit. Many important physical models have been investigated by using this method (see [1, 2, 3] for more discussions). This method, however, is restricted to the studying of the long time behaviour of the models in the weak coupling case, i.e. when coupling constant is a small parameter, and it can not be applied directly to important class of models which contain terms in the interaction without small coupling constant. This last class of models includes models in which the small parameter is the density. Such models naturally arise in the low density limit (LDL) [4].

In this paper a derivation of quantum stochastic differential equation for the evolution operator in the low density limit is presented. The equation obtained is equivalent to the stochastic equation which has been derived in [4] but we use a new method. We use the distribution approach [2, 3] and derive a new algebra for quadratic master fields in the low density limit by using the energy representation. An advantage of this method is the simplicity of derivation of quantum stochastic equations and computation of correlation functions. We formulate the stochastic golden rule in the low density limit case for a system coupling with Bose field via quadratic interaction. In particular the vacuum expectation value of the evolution operator is computed and its exponential decay is shown.

Main results of the paper are quantum stochastic differential equation (14), the new algebra of commutation relations for the master field (Theorem 1) and the expression for the expectation value of the evolution operator (19).

Let us explain our notations. We shall restrict ourselves to the case of a Boson reservoir in this paper. Let $H_S$ be a Hilbert space of the system with the Hamiltonian $H_S$. The
reservoir is described by the Boson Fock space $\Gamma(\mathcal{H}_1)$ over the Hilbert space $\mathcal{H}_1$. Moreover, the Hamiltonian of the reservoir is given by $H_R := \hbar \Gamma(H_1)$ (the second quantization of the one particle Hamiltonian $H_1$) and the total Hamiltonian is given by a self-adjoint operator on the total Hilbert space $\mathcal{H}_S \otimes \Gamma(\mathcal{H}_1)$, which has the form

$$H_{\text{tot}} := H_S \otimes 1 + 1 \otimes H_R + H_{\text{int}} =: H_{\text{free}} + H_{\text{int}}.$$ 

Here $H_{\text{int}}$ is the interaction Hamiltonian between the system and the reservoir. The evolution operator at time $t$ is given by:

$$U_t := e^{itH_{\text{free}}} \cdot e^{-itH_{\text{tot}}}.$$ 

Obviously, it satisfies the differential equation

$$\partial_t U_t = -i H_{\text{int}}(t) U_t$$

where the quantity $H_{\text{int}}(t)$ will be called the evolved interaction and defined as

$$H_{\text{int}}(t) = e^{itH_{\text{free}}} H_{\text{int}} e^{-itH_{\text{free}}}.$$ 

The interaction Hamiltonian will be assumed to have the following form:

$$H_{\text{int}} := D \otimes A^+(g_0)A(g_1) + D^+ \otimes A^+(g_1)A(g_0)$$

where $D$ is a bounded operator in $\mathcal{H}_S$, $D \in \mathcal{B}(\mathcal{H}_S)$, $A$ and $A^+$ are annihilation and creation operators and $g_0, g_1 \in \mathcal{H}_1$ are form-factors describing the interaction of the system with the reservoir. Therefore, with the notion

$$S^0_t := e^{itH_1} ; \quad D(t) := e^{itH_S} D e^{-itH_S}$$

the evolved interaction can be written in the form

$$H_{\text{int}}(t) := D(t) \otimes A^+(S^0_t g_0)A(S^0_t g_1) + D^+(t) \otimes A^+(S^0_t g_1)A(S^0_t g_0)$$

(1)

Let us consider a gauge invariant quasi-free state $\varphi^{(\xi)}$ ($\xi = \lambda^2$ is the fugacity which has in the LDL the same asymptotics as the density), i.e. for each $f \in \mathcal{H}_1$,

$$\varphi^{(\xi)}(W(f)) = \exp \left(-\frac{1}{2} < f, (1 + \xi e^{-\beta H})(1 - \xi e^{-\beta H})^{-1} f > \right)$$

(2)

Here $W(f)$ is the Weyl operator and $H$ is a self-adjoint operator and will be supposed in the present article to be commutative with the one particle Hamiltonian.

Throughout the paper, for simplicity, the following technical condition is assumed: the two test functions in the interaction Hamiltonian have disjoint supports in the energy representation. Thus the disjointness is invariant under the action of any function of $H$. This assumption means that the two test function $g_0, g_1$ in the interaction Hamiltonian satisfy:

$$< g_0, S_t e^{-\beta H} g_1 > = 0 \quad \forall t \in \mathbb{R}.$$
Also the rotating wave approximation condition will be assumed. This condition means that
\[ e^{itH} D e^{-itH} = e^{-it\omega_0} D, \]
where \( \omega_0 \) is a real number.

We will fix a projection operator \( P_0 \) in \( \mathcal{H}_1 \) commuting with \( H_1 \) and \( H \) and such that
\[ P_0 g_0 = g_0 \quad \text{and} \quad P_0 g_1 = 0. \]

Using this projection let us define the group \( \{ S_t; t \in \mathbb{R} \} \) of unitary operators on \( \mathcal{H}_1 \) by
\[ S_t = S_0^0 e^{-it\omega_0 P_0} = e^{it(H_1 - \omega_0 P_0)}. \]
The infinitesimal generator \( H'_1 \) of \( S_t \) is given by
\[ H'_1 = H_1 - \omega_0 P_0. \]

Following Palmer [5] we realize the representation space as the tensor product of a Fock and anti-Fock representations. Then the expectation values with respect to the state \( \varphi^{(\xi)} \) for the model with the interaction Hamiltonian (1) can be conveniently represented as the vacuum expectation values in the Fock-anti-Fock representation for the modified Hamiltonian.

Denote by \( \mathcal{H}'_1 \) the conjugate of \( \mathcal{H}_1 \), i.e.
\[ \iota : \mathcal{H}_1 \rightarrow \mathcal{H}_1, \quad \iota(\lambda f) := \bar{\lambda} \iota(f) \]
\[ <\iota(f), \iota(g)> := <g, f> \]
then, \( \mathcal{H}'_1 \) is a Hilbert space. The corresponding Fock space \( \Gamma(\mathcal{H}'_1) \) is called in this context the anti-Fock space.

It was shown in [4] that that with notations \( D_0 = D, D_1 = D^+ \) the modified Hamiltonian acting in \( \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}'_1) \) has the form
\[ H_{\lambda}(t) = \sum_{\varepsilon=0,1} D_\varepsilon \otimes (A^+(S_t g_\varepsilon) A(S_t g_{1-\varepsilon}) \otimes 1+ \lambda(A(S_t g_{1-\varepsilon}) \otimes A(S_t L g_\varepsilon) + A^+(S_t g_\varepsilon) \otimes A^+(S_t L g_{1-\varepsilon})). \]
Here \( A \) and \( A^+ \) are Bose annihilation and creation operators acting in the Fock spaces \( \Gamma(\mathcal{H}_1) \) and \( \Gamma(\mathcal{H}'_1) \) and \( L := e^{-\beta H/2}. \)

Moreover, it will be assumed that there exists a subset \( \mathcal{K} \) (which includes \( g_0, g_1 \)) of the one particle Hilbert space \( \mathcal{H}_1 \), such that
\[ \int_{\mathbb{R}} |<f, S_t g>| dt < \infty \quad \forall f, g \in \mathcal{K}. \]

The interaction Hamiltonian determines the evolution operator \( U_t^{(\lambda)} \) which is the solution of the Schrödinger equation in interaction representation:
\[ \partial_t U_t^{(\lambda)} = -i H_{\lambda}(t) U_t^{(\lambda)}. \]
with initial condition
\[ U_0^{(\lambda)} = 1. \]
One has the following integral equation for the evolution operator.

\[ U_t^{(\lambda)} = 1 - i \int_0^t dt' H_\lambda(t') U_{t'}^{(\lambda)}. \]

### 2 Energy representation

We will investigate the limit of the evolution operator when \( \xi \to +0 \) after the time rescaling \( t \to t/\xi \), where \( \xi = \lambda^2 \). After this time rescaling the equation for the evolution operator becomes

\[ \partial_t U_{t/\lambda^2}^{(\lambda)} = -i \sum_{\varepsilon = 0, 1} D_\varepsilon \otimes (N_{\varepsilon, 1-\varepsilon, \lambda}(t) + B_{1-\varepsilon, \varepsilon, \lambda}(t) + B_{\varepsilon, 1-\varepsilon, \lambda}(t)) U_{t/\lambda^2}^{(\lambda)} \]

where we introduced the notations:

\[ N_{\varepsilon_1, \varepsilon_2, \lambda}(t) = \frac{1}{\lambda^2} A^+(S_{t/\lambda^2} g_{\varepsilon_1}) A(S_{t/\lambda^2} g_{\varepsilon_2}) \otimes 1 \]

\[ B_{\varepsilon_1, \varepsilon_2, \lambda}(t) = \frac{1}{\lambda} A^+(S_{t/\lambda^2} g_{\varepsilon_1}) \otimes A^+(S_{t/\lambda^2} L g_{\varepsilon_2}) \]

Let us introduce the energy representation for the creation and annihilation operators by the formulae

\[ A_E^+(g) = A^+(P_E g) = \int dk (P_E g)(k) a^+(k) = \int dk \delta(H_1' - E) g(k) a^+(k) \]

\[ A_E(g) = A(P_E g) \]

Here

\[ P_E = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt S_t e^{-itE} = \delta(H_1' - E). \]

It has the properties

\[ S_t = \int dE P_E e^{itE} \]

\[ P_E P_{E'} = \delta(E - E') P_E \]

\[ P_E^* = P_E \]

In the case when \( \mathcal{H}_1 = L^2(\mathbb{R}^d) \) the one-particle Hamiltonian is the multiplication operator to the function \( \omega(k) \) and acts on any element \( f \in L^2(\mathbb{R}^d) \) as \( H_1 f(k) = \omega(k) f(k) \).

It is easy to check that

\[ [A_E(f), A_{E'}^+(g)] = \delta(E - E') < f, P_E g >. \]
Here $\langle \cdot, \cdot \rangle$ means the scalar product in $\mathcal{H}_1$.

Using the energy representation one gets

$$N_{\varepsilon_1, \varepsilon_2, \lambda}(t) = \int dE_1 dE_2 N_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t)$$

$$B_{\varepsilon_1, \varepsilon_2, \lambda}(t) = \int dE_1 dE_2 B_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t)$$

where

$$N_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t) := \frac{e^{it(E_1 - E_2)/\lambda^2}}{\lambda^2} A_{E_1}^+(g_{\varepsilon_1}) A_{E_2}(g_{\varepsilon_2}) \otimes 1$$

$$B_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t) := \frac{e^{it(E_2 - E_1)/\lambda^2}}{\lambda} A_{E_1}(g_{\varepsilon_1}) \otimes A_{E_2}(Lg_{\varepsilon_2}).$$

Let us also denote

$$\gamma_{\varepsilon}(E) := \int_0^\infty <g_{\varepsilon}, S_t g_{\varepsilon}> e^{-itE} dt.$$  

3 The limiting commutation relations

Besides the operators $B_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t)$ and $N_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t)$ defined above let us consider the following operators:

$$N_{\varepsilon_1, \varepsilon_2, \lambda}^t(E_1, E_2, t) = \frac{e^{it(E_2 - E_1)/\lambda^2}}{\lambda^2} 1 \otimes A_{E_1}^+(Lg_{\varepsilon_1}) A_{E_2}(Lg_{\varepsilon_2})$$

with $A_{E_1}^+(Lg_{\varepsilon_1})$ has been defined in (3). The commutators of these operators are:

$$[B_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t), B_{\varepsilon_3, \varepsilon_4, \lambda}^+(E_3, E_4, t')] =$$

$$\frac{e^{i(t'-t)(E_1 - E_2)/\lambda^2}}{\lambda^2} \left( \delta_{\varepsilon_1, \varepsilon_3} \delta_{\varepsilon_2, \varepsilon_4} \delta(E_1 - E_3) \delta(E_2 - E_4) <g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > <g_{\varepsilon_2}, P_{E_2} L^2 g_{\varepsilon_2}> + \right.$$

$$\left. \lambda^2 \delta_{\varepsilon_1, \varepsilon_3} \delta(E_1 - E_3) <g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > N_{\varepsilon_4, \varepsilon_2, \lambda}^t(E_4, E_2, t') + \right.$$

$$\lambda^2 \delta_{\varepsilon_2, \varepsilon_4} \delta(E_2 - E_4) <g_{\varepsilon_2}, P_{E_2} L^2 g_{\varepsilon_2} > N_{\varepsilon_3, \varepsilon_1, \lambda}^t(E_3, E_1, t') \right)$$

$$[B_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t), N_{\varepsilon_3, \varepsilon_4, \lambda}^t(E_3, E_4, t')] =$$

$$\delta_{\varepsilon_1, \varepsilon_3} \frac{e^{i(t'-t)(E_1 - E_2)/\lambda^2}}{\lambda^2} \delta(E_1 - E_3) <g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > B_{\varepsilon_4, \varepsilon_2, \lambda}(E_4, E_2, t')$$

$$[N_{\varepsilon_1, \varepsilon_2, \lambda}(E_1, E_2, t), N_{\varepsilon_3, \varepsilon_4, \lambda}(E_3, E_4, t')] =$$

$$\delta_{\varepsilon_2, \varepsilon_3} \frac{e^{i(t'-t)(E_3 - E_1)/\lambda^2}}{\lambda^2} \delta(E_2 - E_3) <g_{\varepsilon_2}, P_{E_2} g_{\varepsilon_2} > N_{\varepsilon_1, \varepsilon_4, \lambda}(E_1, E_4, t') - \left.$$
\[ \delta_{\epsilon_1, \epsilon_4} (E_1 - E_4) < g_{\epsilon_1}, P_{E_1} g_{\epsilon_1} > N_{\epsilon_3, \epsilon_2}(E_3, E_2, t) \} \]

Notice that in the sense of distributions one has the limit
\[
\lim_{\lambda \to 0} \frac{e^{i(t' - t)(E_1 - E_2)/\lambda^2}}{\lambda^2} = 2\pi \delta(t' - t) \delta(E_1 - E_2)
\]
and, in the sense of distributions over the standard simplex (cf. [3]) one has the limit
\[
\lim_{\lambda \to 0} \frac{e^{i(t' - t)(E_1 - E_2)/\lambda^2}}{\lambda^2} = \delta_+(t' - t) \frac{1}{i(E_1 - E_2 - i0)}
\]

The following theorem describes the algebra of commutation relations for the master field in the LDL.

**Theorem 1** The limits
\[ X_{\epsilon_1, \epsilon_2}(E_1, E_2, t) := \lim_{\lambda \to 0} X_{\epsilon_1, \epsilon_2, \lambda}(E_1, E_2, t), \quad X = B, N \]
exist in the sense of convergence of matrix elements and satisfy the (causal) commutation relations
\[ [B_{\epsilon_1, \epsilon_2}(E_1, E_2, t), B_{\epsilon_3, \epsilon_4}^+(E_3, E_4, t')] = \]
\[
\delta_{\epsilon_1, \epsilon_4} \delta_{\epsilon_2, \epsilon_3} \delta_+(t' - t) \delta(E_3 - E_2) \delta(E_2 - E_4) \frac{< g_{\epsilon_1}, P_{E_1} g_{\epsilon_1} >}{i(E_1 - E_2 - i0)} < g_{\epsilon_2}, P_{E_2} L^2 g_{\epsilon_2} > \quad (5)
\]
\[
[B_{\epsilon_1, \epsilon_2}(E_1, E_2, t), N_{\epsilon_3, \epsilon_4}(E_3, E_4, t')] = \]
\[
\delta_{\epsilon_1, \epsilon_3} \delta_+(t' - t) \delta(E_1 - E_3) \frac{< g_{\epsilon_1}, P_{E_1} g_{\epsilon_1} >}{i(E_1 - E_2 - i0)} B_{\epsilon_4, \epsilon_2}(E_4, E_2, t') \quad (6)
\]
\[
[N_{\epsilon_1, \epsilon_2}(E_1, E_2, t), N_{\epsilon_3, \epsilon_4}(E_3, E_4, t')] = \delta_+(t' - t) \frac{1}{i(E_3 - E_1 - i0)}
\]
\[
\left\{ \delta_{\epsilon_2, \epsilon_3} \delta(E_2 - E_3) < g_{\epsilon_2}, P_{E_2} g_{\epsilon_2} > N_{\epsilon_1, \epsilon_4}(E_1, E_4, t') - \right.
\]
\[
\delta_{\epsilon_1, \epsilon_4} \delta(E_1 - E_4) < g_{\epsilon_1}, P_{E_1} g_{\epsilon_1} > N_{\epsilon_3, \epsilon_2}(E_3, E_2, t) \right\} \quad (7)
\]
The commutation relations of the master field are obtained by (5), (6), (7) replacing the factor \( \delta_+(t' - t) \) by \( \delta(t' - t) \) and \( (i(E_1 - E_2 - i0))^{-1} \) by \( 2\pi \delta(E_1 - E_2) \).

**Proof.** The proof of the theorem follows immediately from the commutation relations (4).
4 The master space and the associated white noise

Let $\mathcal{H}_{0,1}$ denote the closed subspace of $\mathcal{H}_1 = L^2(\mathbb{R}^d)$ spanned by the vectors

$$S_t g_\varepsilon, \quad \varepsilon \in \{0, 1\}, \quad t \in \mathbb{R}.$$ 

Let $K$ be a non zero subspace of $\mathcal{H}_{0,1}$ such that $g_\varepsilon \in K \ (\varepsilon = 0, 1)$ and

$$\int_{-\infty}^{\infty} |\langle f, S_t g \rangle| dt < \infty \quad \forall f, g \in K.$$ 

This assumption implies that the sesquilinear form $(\cdot | \cdot) : K \times K \rightarrow \mathbb{C}$ defined by

$$(f | g) = \int_{-\infty}^{\infty} \langle f, S_t g \rangle dt, \quad f, g \in K$$

is well defined. Moreover it defines a pre-scalar product on $K$. We denote $\{K, (\cdot | \cdot)\}$ or simply $K$, the completion of the quotient of $K$ by the zero $(\cdot | \cdot)$-norm elements.

Define then Hilbert space $K_{0,1} = K \otimes_{\beta} K$ as the completion of the algebraic tensor product $K \otimes K^*$ with respect to the scalar product

$$(f_0 \otimes_{\beta} f_1 | f_0' \otimes_{\beta} f_1') := \int_{-\infty}^{\infty} \langle f_0, S_t f_0' \rangle \langle S_t f_1, L^2_f f_1' \rangle dt = \int_{-\infty}^{\infty} \langle f_0, S_t f_0' \rangle \langle f_1', S_{-t} L^2 f_1 \rangle dt.$$ 

Bounded operators acts naturally on $K_{0,1}$ by

$$(A \otimes_{\beta} B)(f_0 \otimes_{\beta} f_1) = A f_0 \otimes_{\beta} B f_1 \quad \forall A, B \in B(K).$$ 

The limit reservoir (or master) space is the space

$$\mathcal{F}(L^2(\mathbb{R}) \otimes K_{0,1}).$$

The (non-causal) commutation relations (5),..., (7) mean that operators $B_{\varepsilon_1, \varepsilon_2}(E_1, E_2, t)$ are the white noise operators $b_t(\cdot)$ in $\mathcal{F}(L^2(\mathbb{R}) \otimes K_{0,1})$:

$$B_{\varepsilon_1, \varepsilon_2}(E_1, E_2, t) =: b_t(P_{E_1} g_{\varepsilon_1} \otimes_{\beta} P_{E_2} g_{\varepsilon_2}).$$

The number operator is

$$N_{\varepsilon_1, \varepsilon_2}(E_1, E_2, t) = \sum_{\varepsilon'=0,1} n_{\varepsilon'}(E_1) b^+_{\varepsilon_1}(g_{\varepsilon_1} \otimes_{\beta} P_{E_1} g_{\varepsilon'}) b_t(P_{E_2} g_{\varepsilon_2} \otimes_{\beta} P_{E_1} g_{\varepsilon'}) =$$

$$= \sum_{\varepsilon'=0,1} \int dE n_{\varepsilon'}(E_1) B^+_{\varepsilon_1, \varepsilon'}(E, E_1, t) B_{\varepsilon_2, \varepsilon'}(E_2, E_1, t)$$

(8)
where we denoted
\[ n_{\varepsilon}(E) := \frac{1}{\langle g_{\varepsilon}, P_E L^2 g_{\varepsilon} \rangle}. \]

By identifying the element of the algebraic tensor product \( f \otimes g \in K \otimes K^* \) with the operator
\[ |f \rangle \langle g| : \xi \in \mathcal{H}_1 \rightarrow \langle g, \xi \rangle f \in \mathcal{H}_1 \]
so that
\[ (|f \rangle \langle g|)^* = |g \rangle \langle f| \]
and introducing the scalar product of such operators \( X, Y \) (notice that \( L^2 = e^{-\beta H} \)) by
\[ <Y, X> := Tr \int dt e^{-\beta H} X S_t S_t^* = 2\pi Tr \int dE e^{-\beta H} P_E X P_E \]
one can rewrite white noise \( b_t(g \otimes \beta f) \) as \( b_t(|g \rangle \langle f|) \) with commutation relations defined by
\[ [b_t(Y), b_t^+(X)] = \delta(t' - t) <Y, X>. \]

Let us introduce for simplicity
\[ B_{\varepsilon_1, \varepsilon_2}(E, t) := \int dE' B_{\varepsilon_1, \varepsilon_2}(E', E, t) \]
\[ N_{\varepsilon_1, \varepsilon_2}(E, t) := \int dE' N_{\varepsilon_1, \varepsilon_2}(E, E', t) \]
with (causal) commutation relations:
\[ [B_{\varepsilon_1, \varepsilon_2}(E, t), B_{\varepsilon_3, \varepsilon_4}^+(E', t')] = \delta_{\varepsilon_1, \varepsilon_3} \delta_{\varepsilon_2, \varepsilon_4} \delta_+(t' - t) \delta(E - E') \gamma_{\varepsilon_1}(E) < g_{\varepsilon_2}, P_E L^2 g_{\varepsilon_2} > \]
\[ [B_{\varepsilon_1, \varepsilon_2}(E, t), N_{\varepsilon_3, \varepsilon_4}(E', t')] = \delta_{\varepsilon_1, \varepsilon_3} \delta_+(t' - t) \frac{< g_{\varepsilon_1}, P_{E'} g_{\varepsilon_1} >}{i(E' - E - i0)} B_{\varepsilon_4, \varepsilon_2}(E, t'). \]

In these notations the limiting Hamiltonian acts on \( \mathcal{H}_S \otimes \mathcal{F}(L^2(\mathbb{R}) \otimes K_{0,1}) \) as
\[ H(t) = \int dE \sum_{\varepsilon=0,1} D_{\varepsilon} \otimes (N_{\varepsilon,1-\varepsilon}(E, t) + B_{1-\varepsilon,\varepsilon}(E, t) + B_{\varepsilon,1-\varepsilon}^+(E, t)). \]

5 Emergence of the drift term and annihilation process

The results of the preceeding section allow us to write the equation for the evolution operator in the stochastic limit
\[ \partial_t U_t = -i H(t) U_t = -i \sum_{\varepsilon=0,1} D_{\varepsilon} \otimes (N_{\varepsilon,1-\varepsilon}(t) + B_{1-\varepsilon,\varepsilon}(t) + B_{\varepsilon,1-\varepsilon}^+(t)) U_t \quad (9) \]
In order to bring it to the normally ordered form one needs to compute the commutator

\[-iD_{\varepsilon}[B_{1-\varepsilon,\varepsilon}(t), U_t] = -iD_{\varepsilon} \int dE [B_{1-\varepsilon,\varepsilon}(E, t), U_t].\]

Notice that $D_{\varepsilon}D_{1-\varepsilon}$ is a positive self-adjoint operator. Therefore one can assume that for each $E \in \mathbb{R}$, the inverse operator

\[T_{\varepsilon}(E) := (1 + (\gamma_{\varepsilon} \gamma_{1-\varepsilon})(E)D_{\varepsilon}D_{1-\varepsilon})^{-1}\]

always exists. Notice also that, since $D_{\varepsilon}D_{1-\varepsilon}$ commutes with $T_{\varepsilon}(E)$, one has

\[1 - D_{\varepsilon}D_{1-\varepsilon}(\gamma_{\varepsilon} \gamma_{1-\varepsilon})(E)T_{\varepsilon}(E) = T_{\varepsilon}(E).\]

Therefore

\[iD_{\varepsilon}(1 - D_{1-\varepsilon}(\gamma_{\varepsilon} \gamma_{1-\varepsilon})(E)T_{\varepsilon}(E)D_{\varepsilon}) = iT_{\varepsilon}(E)D_{\varepsilon}\]

**Theorem 2** For the model described above one has

\[-iD_{\varepsilon}[B_{1-\varepsilon,\varepsilon}(t), U_t] = - \int dE \gamma_{1-\varepsilon}(E)D_{\varepsilon}D_{1-\varepsilon}T_{\varepsilon}(E) \times \]

\[\left< g_{\varepsilon}, P_E L^2 g_{\varepsilon} > U_t - iD_{\varepsilon} \gamma_{\varepsilon}(E) U_t B_{1-\varepsilon,\varepsilon}(E, t) + U_t B_{\varepsilon,\varepsilon}(E, t) \right>\]

**Proof.** Using the integral equation for the evolution operator and the commutation relations (5),(6), one gets

\[-iD_{\varepsilon}[B_{1-\varepsilon,\varepsilon}(E, t), U_t] = \]

\[- \sum_{\varepsilon' = 0,1} D_{\varepsilon}D_{\varepsilon'} \int dE' \int_0^t dt_1 [B_{1-\varepsilon,\varepsilon}(E, t), N_{\varepsilon',1-\varepsilon'}(E', t_1) + B_{\varepsilon',1-\varepsilon'}(E', t_1)] U_{t_1} = \]

\[-D_{\varepsilon}D_{1-\varepsilon} \gamma_{1-\varepsilon}(E) \left< g_{\varepsilon}, P_E L^2 g_{\varepsilon} > + B_{\varepsilon,\varepsilon}(E, t) \right> U_t \]

Notice that the first equality in (12) holds because, due to the time consecutive principle

\[[B_{\varepsilon,\varepsilon'}(E, t), U_{t_1}] = 0.\]

Similarly one computes the commutator

\[[B_{\varepsilon,\varepsilon}(E, t), U_t] = -i \int dE' \int_0^t dt_1 [B_{\varepsilon,\varepsilon}(E, t), N_{\varepsilon',1-\varepsilon'}(E', t_1)] U_{t_1} = \]

\[-iD_{\varepsilon} \gamma_{\varepsilon}(E) B_{1-\varepsilon,\varepsilon}(E, t) U_t \]

After substitution of this commutator into (12) one gets

\[-iD_{\varepsilon}[B_{1-\varepsilon,\varepsilon}(E, t), U_t] = -D_{\varepsilon}D_{1-\varepsilon} \gamma_{1-\varepsilon}(E) \times \]

9
\[
\left( < g_\varepsilon, P_E L^2 g_\varepsilon > U_t - i D_\varepsilon \gamma_\varepsilon(E)([B_{1-\varepsilon,\varepsilon}(E,t),U_t] + U_t B_{1-\varepsilon,\varepsilon}(E,t)) + U_t B_{\varepsilon,\varepsilon}(E,t) \right)
\]

Then for
\[
f_\varepsilon(E,t) := -iD_\varepsilon[B_{1-\varepsilon,\varepsilon}(E,t),U_t]
\]
one has
\[
(1 + (\gamma_\varepsilon \gamma_{1-\varepsilon})(E)D_\varepsilon D_{1-\varepsilon}) f_\varepsilon(E,t) = -\gamma_\varepsilon(E)D_\varepsilon D_{1-\varepsilon} \left( < g_\varepsilon, P_E L^2 g_\varepsilon > U_t - iD_\varepsilon \gamma_\varepsilon(E)U_t B_{1-\varepsilon,\varepsilon}(E,t) + U_t B_{\varepsilon,\varepsilon}(E,t) \right).
\]

Since the inverse operator \((1 + (\gamma_\varepsilon \gamma_{1-\varepsilon})(E)D_\varepsilon D_{1-\varepsilon})^{-1}\) exists we can solve the equation above for \(f_\varepsilon(E,t)\). Using this solution we find
\[
-iD_\varepsilon[B_{1-\varepsilon,\varepsilon}(t),U_t] = \int dE f_\varepsilon(E,t) = -\int dE \gamma_\varepsilon(E)D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) \times \left( < g_\varepsilon, P_E L^2 g_\varepsilon > U_t - iD_\varepsilon \gamma_\varepsilon(E)U_t B_{1-\varepsilon,\varepsilon}(E,t) + U_t B_{\varepsilon,\varepsilon}(E,t) \right).
\]

6 Emergence of the number and creation processes

In order to bring equation (9) to the normally ordered form one needs also to move the annihilation operators in \(N_{\varepsilon,1-\varepsilon}(t)\) to the right of the evolution operator. Using (8) this leads to
\[
-iD_\varepsilon N_{\varepsilon,1-\varepsilon}(t)U_t = -iD_\varepsilon \int dE \sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B_{\varepsilon,\varepsilon'}^+(E,t)B_{1-\varepsilon,\varepsilon'}(E,t)U_t =
\]
\[
-iD_\varepsilon \int dE \sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B_{\varepsilon,\varepsilon'}^+(E,t) \left( [B_{1-\varepsilon,\varepsilon'}(E,t),U_t] + U_t B_{1-\varepsilon,\varepsilon'}(E,t) \right) \tag{13}
\]
and the commutator is evaluated using (12) and (12a).

**Theorem 3** For the model described above one has
\[
-iD_\varepsilon N_{\varepsilon,1-\varepsilon}(t)U_t = -\int dE \left( \gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) - iD_\varepsilon \gamma_\varepsilon(E)B_{1-\varepsilon,\varepsilon'}^+(E,t) + B_{\varepsilon,\varepsilon}(E,t) \right) U_t + \\
\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E) \left( iT_\varepsilon(E)D_\varepsilon B_{\varepsilon,\varepsilon'}^+(E,t) U_t B_{1-\varepsilon,\varepsilon'}(E,t) + \\
\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) B_{\varepsilon,\varepsilon'}^+(E,t) U_t B_{\varepsilon,\varepsilon'}(E,t) \right) 
\]

**Proof.** From (12) and (12a) it follows that
\[
-iD_\varepsilon[B_{1-\varepsilon,\varepsilon'}(E,t),U_t] = -\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) \\
n_{\varepsilon'}^{-1}(E) (\delta_{\varepsilon,\varepsilon'} - \delta_{1-\varepsilon,\varepsilon'} iD_\varepsilon \gamma_\varepsilon) U_t - iD_\varepsilon \gamma_\varepsilon(E) U_t B_{1-\varepsilon,\varepsilon'}(E,t) + U_t B_{\varepsilon,\varepsilon'}(E,t).
\]
After substitution of these commutators in (13) and using (10) one finishes the proof of the theorem.
7 The normally ordered equation

Theorems (2) and (3) allow us to obtain immediately the normally ordered equation for the evolution operator in the LDL. This procedure of deduction of quantum stochastic differential equation is being called a stochastic golden rule. The normally ordered equation has the form

$$\partial_t U_t = -\sum_{\varepsilon=0,1} \int dE \left\{ i T_\varepsilon(E) D_\varepsilon \sum_{\varepsilon'=0,1} n_{\varepsilon'}(E) B_{\varepsilon',\varepsilon}(E,t) U_t B_{1-\varepsilon,\varepsilon'}(E,t) + \right.$$}

$$\gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) \sum_{\varepsilon'=0,1} n_{\varepsilon'}(E) B_{\varepsilon',\varepsilon}(E,t) U_t B_{\varepsilon,\varepsilon'}(E,t) +$$

$$iT_\varepsilon(E) D_\varepsilon B_{\varepsilon,1-\varepsilon}(E,t) U_t + \gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) B_{\varepsilon,\varepsilon'}(E,t) U_t +$$

$$iT_\varepsilon(E) D_\varepsilon U_t B_{1-\varepsilon,\varepsilon'}(E,t) + \gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) U_t B_{\varepsilon,\varepsilon'}(E,t) +$$

$$\gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) < g_\varepsilon, P_E L^2 g_\varepsilon > U_t \right\}$$

Equation (14) is equivalent to the quantum stochastic differential equation

$$dU_t = -\sum_{\varepsilon=0,1} \int dE \left\{ i T_\varepsilon(E) D_\varepsilon dN_\varepsilon(2\pi|g_\varepsilon > < g_{1-\varepsilon}|P_E \otimes \beta P_E) + \right.$$}

$$\gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) dN_\varepsilon(2\pi|g_\varepsilon > < g_{1-\varepsilon}|P_E \otimes \beta P_E) +$$

$$iT_\varepsilon(E) D_\varepsilon dB_t^+ (g_\varepsilon \otimes \beta P_E g_{1-\varepsilon}) + \gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) dB_t^+(g_\varepsilon \otimes \beta P_E g_\varepsilon) +$$

$$iT_\varepsilon(E) D_\varepsilon dB_t (g_{1-\varepsilon} \otimes \beta P_E g_\varepsilon) + \gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) dB_t (g_{1-\varepsilon} \otimes \beta P_E g_\varepsilon) +$$

$$\gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) < g_{1-\varepsilon}, P_E L^2 g_{1-\varepsilon} > dt \right\} U_t \tag{14a}$$

**Theorem 4** Equation (14a) is equivalent to the quantum stochastic differential equation

**Proof.** Let us consider the following term in (14)

$$\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E) B_{\varepsilon',\varepsilon}(E,t) U_t B_{1-\varepsilon,\varepsilon'}(E,t) =$$

$$n_{1-\varepsilon}(E) b_t^+ (g_\varepsilon \otimes \beta P_E g_{1-\varepsilon}) U_t b_t (g_{1-\varepsilon} \otimes \beta P_E g_{1-\varepsilon}) +$$

$$n_{\varepsilon}(E) b_t^+ (g_\varepsilon \otimes \beta P_E g_\varepsilon) U_t b_t (g_{1-\varepsilon} \otimes \beta P_E g_{1-\varepsilon}) \tag{15}$$
The matrix element of this expression on the exponential vectors \( \psi(f), \psi(f') \) is (we use Dirac’s notations also for bra- and ket-vectors from \( K_{0,1} \))

\[
(n_{1-\varepsilon}(E) < f | g_{\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon} > < g_{1-\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon} | f' > + n_{\varepsilon}(E) < f | g_{\varepsilon} \otimes \beta P_{E} g_{\varepsilon} > < g_{1-\varepsilon} \otimes \beta P_{E} g_{\varepsilon} | f' > ) < \psi(f) | U_{t} | \psi(f') > =
\]

\[
< f | T_{\varepsilon,1-\varepsilon}(E) | f' > < \psi(f) | U_{t} | \psi(f') > \quad (15a)
\]

which is the time derivative of the matrix element \(< \psi(f) | dN_{t}(T_{\varepsilon,1-\varepsilon}(E)) | \psi(f') >\) where \( N_{t}(T_{\varepsilon,1-\varepsilon}(E)) \) is the number process with intensity

\[
T_{\varepsilon,1-\varepsilon}(E) = n_{1-\varepsilon}(E) | g_{\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon} > < g_{1-\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon} | + n_{\varepsilon}(E) | g_{\varepsilon} \otimes \beta P_{E} g_{\varepsilon} > < g_{1-\varepsilon} \otimes \beta P_{E} g_{\varepsilon} |
\]

Let us now prove that

\[
T_{\varepsilon,1-\varepsilon}(E) = 2\pi | g_{\varepsilon} > < g_{1-\varepsilon} | P_{E} \otimes \beta P_{E} \quad (15b)
\]

To this goal let us consider the action of the \( T_{\varepsilon,1-\varepsilon}(E) \) on vectors of the form

\[
| f > = | P_{E_{1}} g_{\varepsilon_{1}} \otimes \beta P_{E_{2}} g_{\varepsilon_{2}} > .
\]

One has

\[
T_{\varepsilon,1-\varepsilon}(E) | f > =
\]

\[
2\pi \delta_{\varepsilon_{1}-\varepsilon_{1}} \delta(E_{1}-E) \delta(E_{2}-E) < g_{1-\varepsilon}, P_{E} g_{1-\varepsilon} > \left( \delta_{\varepsilon_{1},1-\varepsilon_{1}} | g_{\varepsilon_{1}} \otimes \beta P_{E} g_{1-\varepsilon} > + \delta_{\varepsilon_{1},1-\varepsilon_{2}} | g_{\varepsilon_{1}} \otimes \beta P_{E} g_{\varepsilon_{2}} > \right) =
\]

\[
2\pi \delta_{\varepsilon_{1},1-\varepsilon_{1}} < g_{1-\varepsilon}, P_{E} g_{1-\varepsilon} > \delta(E_{1}-E) \delta(E_{2}-E) \left( \delta_{\varepsilon_{1},1-\varepsilon_{1}} | g_{\varepsilon_{2}} \otimes \beta P_{E} g_{1-\varepsilon} > + \delta_{\varepsilon_{1},1-\varepsilon_{2}} | g_{\varepsilon_{2}} \otimes \beta P_{E} g_{\varepsilon_{2}} > \right) =
\]

\[
2\pi \delta_{\varepsilon_{1},1-\varepsilon_{1}} < g_{1-\varepsilon}, P_{E} g_{1-\varepsilon} > \delta(E_{1}-E) \delta(E_{2}-E) | g_{\varepsilon} \otimes \beta P_{E} g_{\varepsilon_{2}} > =
\]

\[
2\pi | g_{\varepsilon} > < g_{1-\varepsilon} | P_{E} \otimes \beta P_{E} | f > \quad (15c)
\]

Therefore (15b) holds and the term (15) corresponds to the number process

\[
dN_{t}(2\pi | g_{\varepsilon} > < g_{1-\varepsilon} | P_{E} \otimes \beta P_{E})
\]

Computing the same matrix element for the term

\[
\sum_{\varepsilon' = 0, 1} n_{\varepsilon'}(E) B_{\varepsilon,\varepsilon'}^{+}(E, t) U_{t} B_{\varepsilon,\varepsilon'}(E, t) =
\]

\[
n_{1-\varepsilon}(E) b_{1}^{+}(g_{\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon}) U_{t} b_{1}(g_{\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon}) + n_{\varepsilon}(E) b_{1}^{+}(g_{\varepsilon} \otimes \beta P_{E} g_{\varepsilon}) U_{t} b_{1}(g_{\varepsilon} \otimes \beta P_{E} g_{\varepsilon}) \quad (16)
\]

one finds an expression like (15a) with \( T_{\varepsilon,1-\varepsilon} \) replaced by the operator

\[
T_{\varepsilon,\varepsilon}(E) = n_{1-\varepsilon}(E) | g_{\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon} > < g_{\varepsilon} \otimes \beta P_{E} g_{1-\varepsilon} | + n_{\varepsilon}(E) | g_{\varepsilon} \otimes \beta P_{E} g_{\varepsilon} > < g_{\varepsilon} \otimes \beta P_{E} g_{\varepsilon} |
\]

and a calculation similar to the one done in (15c) leads to the conclusion that

\[
T_{\varepsilon,\varepsilon}(E) = 2\pi | g_{\varepsilon} > < g_{\varepsilon} | P_{E} \otimes \beta P_{E}
\]

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Therefore the term (16) corresponds to the number process
\[ dN_t(2\pi|g_\varepsilon><g_\varepsilon|_{P_E} \otimes_\beta P_E) \]
This finishes the proof of the Lemma.
Let us introduce the notations:
\[ R_{\varepsilon,\varepsilon}(E) = -\gamma_{1-\varepsilon}(E)D_{\varepsilon}D_{1-\varepsilon}T_{\varepsilon}(E) \]
\[ R_{\varepsilon,1-\varepsilon}(E) = -iT_{\varepsilon}(E)D_{\varepsilon}. \]
In these notations the quantum stochastic differential equation for the evolution operator can be rewritten as
\[
\begin{align*}
    dU_t &= \sum_{\varepsilon=0,1} \int dE \left\{ R_{\varepsilon,1-\varepsilon}(E)dN_t(2\pi|g_\varepsilon><g_\varepsilon|_{P_E} \otimes_\beta P_E) + \\
    &\quad R_{\varepsilon,\varepsilon}(E)dN_t(2\pi|g_\varepsilon><g_\varepsilon|_{P_E} \otimes_\beta P_E) + \\
    &\quad R_{\varepsilon,1-\varepsilon}(E)dB_t^+(g_\varepsilon \otimes_\beta P_E g_{1-\varepsilon}) + R_{\varepsilon,\varepsilon}(E)dB_t^+(g_\varepsilon \otimes_\beta P_E g_\varepsilon) + \\
    &\quad R_{\varepsilon,1-\varepsilon}(E)dB_t(g_{1-\varepsilon} \otimes_\beta P_E g_\varepsilon) + R_{\varepsilon,\varepsilon}(E)dB_t(g_\varepsilon \otimes_\beta P_E g_\varepsilon) + \\
    &\quad R_{\varepsilon,\varepsilon}(E) <g_\varepsilon, P_E L^2 g_\varepsilon > dt \right\} U_t.
\end{align*}
\]
Notice that the quantum stochastic differential equation (14a) can be written also in the Frigerio-Maassen form [7]. In order to prove this recall that for any pair of Hilbert spaces \(X_0, X_1\) if \(N, A\) denote the number and annihilation processes on the Fock space \(\mathcal{F}(X_1)\) then for \(X_0 \in B(X_0), X_1 \in B(X_1), x \in X_1, \) Frigerio and Maassen [7] introduced the notation:
\[
\begin{align*}
    N(X_0 \otimes X_1) := X_0 \otimes N(X_1) \\
    A(X_0 \otimes X_1 x) := X_0 \otimes A(X_1 x) \\
    <x, X_0 \otimes X_1 x> := X_0 \otimes 1 < x, X_1 x >
\end{align*}
\]
Let us also introduce an operator \(T_3(E)\) acting on the triple \(H_S \otimes K \otimes_\beta K\) (this is the reason for introducing index 3) as
\[
T_3(E) := 2\pi \sum_{\varepsilon,\varepsilon'=0,1} R_{\varepsilon,\varepsilon'}(E) \otimes |g_\varepsilon><g_{\varepsilon'}|_{P_E} \otimes_\beta P_E
\]
and the vector \(\xi(E) \in K \otimes_\beta K\)
\[
\xi(E) := \frac{1}{2\pi} \sum_{\varepsilon=0,1} \frac{1}{<g_\varepsilon, P_E g_\varepsilon>} g_\varepsilon \otimes_\beta g_\varepsilon.
\]
In these notations equation (14a) can be written as
\[
\begin{align*}
    dU_t &= \int dE \left( dN_t(T_3(E)) + dB_t^+(T_3(E)\xi(E)) + \\
    &\quad dB_t(T_3(E)\xi(E)) + <\xi(E), T_3(E)\xi(E) > dt \right) U_t.
\end{align*}
\]
8 Connection with scattering theory

Here we consider relation between the evolution operator and scattering theory. Because of the number conservation, the closed subspace of $\mathcal{H}_S \otimes \mathcal{F}$ generated by vectors of the form $u \otimes A^+(f)\Phi$ ($u \in \mathcal{H}_S$, $f \in \mathcal{H}_1 = L^2(\mathbb{R}^d)$) which is naturally isomorphic to $\mathcal{H}_S \otimes \mathcal{H}_1$, is globally invariant under the time evolution operator $\exp[i(H_S \otimes 1 + 1 \otimes H_R + V)t]$. Therefore the restriction of the time evolution operator to this subspace corresponds to an evolution operator on $\mathcal{H}_S \otimes \mathcal{H}_1$ given by

$$\exp[i(H_S \otimes 1 + 1 \otimes H_1 + V_1)t]$$

where

$$V_1 = \sum_{\varepsilon=0,1} D_\varepsilon \otimes |g_\varepsilon><g_{1-\varepsilon}| \quad (17)$$

The 1-particle Møller wave operators are defined as

$$\Omega_\pm = s - \lim_{t \to \pm \infty} \exp[i(H_S \otimes 1 + 1 \otimes H + V_1)t]\exp[-i(H_S \otimes 1 + 1 \otimes H)t]$$

and the 1-particle $T$-operator is defined as

$$T = V_1 \Omega_+ \quad (18)$$

From (17) it follows that

$$\Omega_\pm = s - \lim_{t \to \pm \infty} U^{(1)}_t$$

where $U^{(1)}_t$ is the solution of

$$\partial_t U^{(1)}_t = -i \left( \sum_{\varepsilon=0,1} D_\varepsilon \otimes |S_t g_\varepsilon><S_t g_{1-\varepsilon}| \right) U^{(1)}_t, \quad U^{(1)}_0 = 1.$$

In order to make a connection between the stochastic process $U_t$ and scattering theory notice that the operator $T_3(E)$ can be written as

$$T_3(E) = 2\pi T(E) \otimes_\beta P_E$$

where operator $T(E)$ acts on $\mathcal{H}_S \otimes K$ as

$$T(E) = \sum_{\varepsilon,\varepsilon'=0,1} R_{\varepsilon,\varepsilon'}(E) \otimes |g_\varepsilon><g_{\varepsilon'}|P_E.$$

In [4] it was proved that $T$-operator defined by (18) connected with $T(E)$ by the following formula

$$T = \int dET(E).$$
9 Vacuum expectation value

For the vacuum matrix element of the evolution operator from (14) one immediately gets

\[ < U(t) >_{\text{vac}} = e^{-\Gamma t}. \]  

(19)

The operator \( \Gamma \) acts in \( \mathcal{H}_S \) as

\[ \Gamma = \sum_{\epsilon=0,1} \int dE \gamma_{1-\epsilon}(E) D_{\epsilon} D_{1-\epsilon} T_\epsilon(E) < g_\epsilon, P_E L^2 g_\epsilon > . \]

**Theorem 5** The operator \( \Gamma \) has a non-negative real part (i.e. this operator describes the damping).

**Proof.** From the definition of \( T_\epsilon(E) \) we know that

\[ \gamma_{1-\epsilon}(E) D_{\epsilon} D_{1-\epsilon} T_\epsilon(E) = \frac{D_{\epsilon} D_{1-\epsilon}}{\gamma_{1-\epsilon}(E)} + \gamma_{1-\epsilon} D_{\epsilon} D_{1-\epsilon} \]

But \( \gamma_\epsilon \) and \( \gamma_{1-\epsilon} \) (hence also \( 1/\gamma_{1-\epsilon} \)) have positive real part and \( D_{\epsilon} D_{1-\epsilon} \) is positive self–adjoint.

Hence the above expression has a positive real part because it is of the form:

\[ \frac{H}{z_1 + z_2 H} = \frac{H(\text{Re} z_1 + H \text{Re} z_2)}{|z_1 + z_2 H|^2} - i \frac{H(\text{Im} z_1 + H \text{Im} z_2)}{|z_1 + z_2 H|^2} \]

where \( H \) is positive self–adjoint and \( z_1, z_2 \) have a positive real part. Since \( < g_\epsilon, P_E L^2 g_\epsilon > \geq 0 \) the thesis follows.

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