On Massive High Spin Particles in (A)dS

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Abstract

In this Letter we consider the problem of partial masslessness and unitarity in (A)dS using gauge invariant description of massive high spin particles. We show that for $S = 2$ and $S = 3$ cases such formalism allows one correctly reproduce all known results. Then we construct a gauge invariant formulation for massive particles of arbitrary integer spin $s$ in arbitrary space-time dimension $d$. For $d = 4$ our results confirm the conjecture made recently by Deser and Waldron.

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Introduction

Massive high spin particles in (A)dS reveal a number of interesting and peculiar features, such as unitary forbidden mass range \([1, 2]\) and appearance of gauge invariance in partially massless theories \([3, 4, 5, 6]\). In this Letter we consider these properties using gauge invariant description of massive high spin particles in constant curvature space-time constructed in the same way as in the flat space case \([7, 8, 9]\). Being unitary and gauge invariant from the very beginning, such formalism appears to be very well suited for the investigation of unitarity, gauge invariance and partial masslessness. Moreover, such formalism could be useful for the problem of van Dam-Veltman-Zakharov discontinuity \([10, 11]\).

We start with the \(S = 2\) and \(S = 3\) cases and show that all known results concerning unitarity and partial masslessness can be correctly reproduced and understood in our approach. Then we consider massive particles of arbitrary integer spin \(s\) in arbitrary space-time dimension \(d\). In particular, we calculate all critical values for \(m^2\) corresponding to partially massless theories. For \(d = 4\) our results confirm the conjecture made recently by Deser and Waldron \([6]\).

1 Spin 2

As a warming up exercise, let us start with the most simple and rather well known case — massive spin-2 particle. For the gauge invariant description one needs three fields \((h_{\mu\nu}, A_\mu, \varphi)\) where the first one is symmetric. We begin with the Lagrangian:

\[
L_0 = L_0(h_{\mu\nu}) + L_0(A_\mu) + L_0(\varphi)
\]

\[
L_0(h_{\mu\nu}) = \frac{1}{2} D^\mu h^{\alpha\beta} D_\mu h_{\alpha\beta} - (Dh)^\mu (Dh)_\mu - \frac{1}{2} D^\mu h D_\mu h - (DDh)h
\]

\[
L_0(A_\mu) = -\frac{1}{2} D^\mu A^\nu D_\mu A_\nu + \frac{1}{2} (DA)(DA) + L_0(\varphi) = \frac{1}{2} D^\mu \varphi D_\mu \varphi
\]

which is the sum of usual massless Lagrangians for these fields where all derivatives are replaced by the covariant ones\(^1\). Here \((Dh)^\mu = D_\alpha h^{\alpha\mu}\), \((DDh) = D_\mu D_\nu h^{\mu\nu}\), \(h = g_{\mu\nu} h^{\mu\nu}\) and so on. Then we add all possible low derivative terms:

\[
\Delta L = a_2 A^\mu (Dh)_\mu + b_2 h(DA) - a_1 \varphi(DA)
  + d_2 h^{\mu\nu} h_{\mu\nu} + e_2 h^2 + f_2 \varphi^2 - d_1 A_\mu^2 + d_0 \varphi^2
\]

and require that the whole Lagrangian be invariant under the following gauge transformations:

\[
\delta h_{\mu\nu} = \frac{1}{2} (D_\mu \xi_\nu + D_\nu \xi_\mu) + \beta_2 g_{\mu\nu} \xi
\]

\[
\delta A_\mu = D_\mu \xi + \alpha_1 \xi_\mu
\]

\[
\delta \varphi = \alpha_0 \xi
\]

\(^1\)Because the covariant derivatives do not commute, there are slightly different choices for their ordering in kinetic terms. As a result, the structure of mass terms depends on the choice made.
In this, we use the normalization
\[ R_{\mu\nu,\alpha\beta} = -\Omega(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \quad \Omega = \frac{2\Lambda}{(d-1)(d-2)} \] (4)

The requirement of gauge invariance allows one to express all the parameters in the Lagrangian as well as
\[ a_2 = -2\alpha_1, \quad b_2 = -2\alpha_1, \quad a_1 = -\alpha_0, \quad \beta_2 = \alpha_1 \]
\[ d_2 = -\alpha_1^2 - \Omega, \quad e_2 = \alpha_1^2 - \frac{\Omega}{2}, \quad f_2 = -\alpha_1\alpha_0 \]
\[ 2d_1 = 6\alpha_1^2 - \alpha_0^2 - 3\Omega, \quad d_0 = 2\alpha_1^2 \]
and gives the relation on \( \alpha \)'s
\[ \alpha_0^2 = 6(\alpha_1^2 - \Omega) \] (5)

Here and further on, we will use the convention that \( \alpha_1 \) is the particle mass in the flat space limit (\( \Lambda = 0 \)). In this case it means
\[ \alpha_1^2 = \frac{m^2}{2} \] (6)
and consequently
\[ \alpha_0^2 = 3(m^2 - 2\Omega) \] (7)

From the last relation it is evident that consistent description exists in the unitary allowed region \( (m^2 \geq 2\Omega) \) only. It is easy to check that if one changes the sign of the scalar field \( \varphi \) kinetic term, then the gauge invariant formulation will be possible for \( m^2 < 2\Omega \), but unitarity will be lost the scalar field being the ghost one.

For the critical value \( (m^2 = 2\Omega) \) scalar field completely decouples, while the rest fields \( (h_{\mu\nu}, A_{\mu}) \) describe unitary partially massless theory with the helicities \( (\pm \frac{3}{2}, \pm \frac{1}{2}) \). The Lagrangian in this case has a form:
\[ L = L_0(h_{\mu\nu}) + L_0(A_{\mu}) - 2MA^\mu(Dh)_\mu - 2Mh(DA) 
- 2M^2h_{\mu\nu}h_{\mu\nu} + \frac{M^2}{2}h^2 + \frac{3M^2}{2}A_{\mu}^2 \] (8)
where \( M^2 = \frac{\Lambda}{3} \) and it is invariant under the gauge transformations:
\[ \delta h_{\mu\nu} = \frac{1}{2}(D_\mu\xi_\nu + D_\nu\xi_\mu) + Mg_{\mu\nu}\xi, \]
\[ \delta A_{\mu} = D_{\mu}\xi + M\xi_\mu \] (9)

Indeed we have \( 10 + 4 = 14 \) independent components and \( 4 + 1 = 5 \) gauge parameters leaving \( 14 - 2 \times 5 = 4 \) physical degrees of freedom.

As is well known, in flat space massive spin-2 particle in the massless limit breaks into the massless particles with spins 2, 1 and 0. In AdS space the situation is different. If we consider \( m \to 0 \) limit (which is consistent for \( \Lambda < 0 \) only) we obtain massless spin-2 particle as well as massive spin-1 particle described by the the fields \( (A_{\mu}, \varphi) \) with the Lagrangian:
\[ L = L_0(A_{\mu}) + L_0(\varphi) + M\varphi(DA) + \frac{M^2}{4}A_{\mu}^2 \quad M^2 = -2\Lambda \] (10)
which is invariant under gauge transformations:
\[ \delta A_{\mu} = D_{\mu}\xi, \quad \delta \varphi = M\xi \] (11)
2 Spin 3

Let us turn to more interesting and instructive case of massive spin-3 particles. For gauge invariant description of such particle one needs four fields $\Phi_{\mu\nu\lambda}$, $h_{\mu\nu}$, $A_\mu$ and $\varphi$, the first two being symmetric ones. Again we start with the sum of covariantized massless Lagrangians for all these fields:

$$L_0 = L_0(\Phi_{\mu\nu\lambda}) + L_0(h_{\mu\nu}) + L_0(A_\mu) + L_0(\varphi)$$

where Lagrangians for $h_{\mu\nu}$, $A_\mu$ and $\varphi$ are the same as in the previous section, $\tilde{\Phi}_\lambda = g^{\mu\nu}\Phi_{\mu\nu\lambda}$, and add all possible low derivative terms:

$$\Delta L = -a_3 h^{\mu\nu}(D\Phi)_{\mu\nu} + b_3 \tilde{\Phi}_\mu (Dh)_\mu - c_3 (D\tilde{\Phi}) h$$
$$+ a_2 A_\mu (Dh)_\mu + b_2 h (DA) - a_1 \varphi (DA)$$
$$- d_3 \Phi_{\mu\nu\lambda}^2 - e_3 \tilde{\Phi}_\mu^2 - f_3 \tilde{\Phi}_\mu A_\mu + d_2 h_{\mu\nu}^2$$
$$+ e_2 h^2 + f_2 h \varphi - d_1 A_\mu^2 + d_0 \varphi^2$$

Then we require that the whole Lagrangian be invariant under the following gauge transformations:

$$\delta \Phi_{\mu\nu\lambda} = \frac{1}{3} (D_\mu \xi_{\nu\lambda} + D_\nu \xi_{\mu\lambda} + D_\lambda \xi_{\mu\nu}) + \beta_3 (g_{\mu\nu} \xi_\lambda + g_{\mu\lambda} \xi_\nu + g_{\nu\lambda} \xi_\mu)$$
$$\delta h_{\mu\nu} = \frac{1}{2} (D_\mu \xi_\nu + D_\nu \xi_\mu) + \beta_2 g_{\mu\nu} \xi + \alpha_2 \xi_{\mu\nu}$$
$$\delta A_\mu = D_\mu \xi + \alpha_1 \xi_\mu$$
$$\delta \varphi = \alpha_0 \xi$$

where parameter $\xi_{\mu\nu}$ is symmetric and traceless. The requirement of gauge invariance allows one to express all the parameters in the Lagrangian as well as $\beta_{2,3}$

$$a_3 = -3 \alpha_2, \quad b_3 = -6 \alpha_2, \quad c_3 = -\frac{3}{2} \alpha_2, \quad \beta_3 = \frac{\alpha_2}{4}$$
$$a_2 = -2 \alpha_1, \quad b_2 = -2 \alpha_1, \quad a_1 = -\alpha_0, \quad \beta_2 = \alpha_1$$
$$2d_3 = -3 \alpha_2^2 + \Omega, \quad 2e_3 = 9 \alpha_2^2 - 18 \Omega, \quad f_3 = -3 \alpha_2 \alpha_1$$
$$2d_2 = -2 \alpha_1^2 + \frac{15}{2} \alpha_2^2 - 2 \Omega, \quad 2e_2 = 2 \alpha_1^2 - 3 \alpha_2^2 - \Omega, \quad f_2 = -\alpha_1 \alpha_0$$
$$2d_1 = -\alpha_0^2 + 6 \alpha_1^2 - 3 \Omega, \quad d_0 = 2 \alpha_1^2$$

and gives two relations on $\alpha$’s

$$\alpha_1^2 = \frac{15}{4} \alpha_2^2 - 5 \Omega$$
$$\alpha_0^2 = \frac{9}{2} \alpha_2^2 + 6 \alpha_1^2 - 6 \Omega$$
Our convention that \( m \) is the particle mass in the flat space gives now \( \alpha_1^2 = \frac{5}{4}m^2 - 4\Omega \) and one finds:

\[
\begin{align*}
\alpha_1^2 &= \frac{5}{4}(m^2 - 4\Omega) \\
\alpha_2^2 &= 6(m^2 - 6\Omega)
\end{align*}
\]

One can see that now we have two critical values for the cosmological constant. At the first one \( m^2 = 6\Omega \) scalar field \( \varphi \) completely decouples, while the rest fields describe unitary partially massless theory with helicities \( \pm 3, \pm 2, \pm 1 \). Indeed, we have total \( 20 + 10 + 4 = 34 \) independent field components and \( 9 + 4 + 1 = 14 \) gauge parameters which gives \( 34 - 2 \times 14 = 6 \) physical degrees of freedom. In general for \( m^2 < 6\Omega \) the theory becomes inconsistent, but at the second critical value \( m^2 = 4\Omega \) one obtains two completely decoupled systems. The one with fields \( \Phi_{\mu\nu\lambda} \), \( h_{\mu\nu} \) describes unitary partially massless theory with helicities \( \pm 3, \pm 2, \pm 1 \). Indeed, we have total \( 20 + 10 = 30 \) independent components, \( 9 + 4 = 13 \) gauge parameters and \( 30 - 2 \times 13 = 4 \) physical degrees of freedom.

If we consider massless \( m \to 0 \) limit in AdS space \( \Lambda < 0 \) we obtain massless spin-3 particle as well as the system of \( (h_{\mu\nu}, A_{\mu}, \varphi) \) fields giving gauge invariant description of massive spin-2 particle exactly as in the previous section.

3 Arbitrary integer spin

Now we are ready to consider general case of massive particle with arbitrary integer spin \( s \) in arbitrary space-time dimension \( d \). In order to avoid complex expressions with indices, we will use condensed notations [9]. Let us denote \( \Phi^s \) completely symmetric tensor of rank \( s \), which is double traceless \( \tilde{\Phi} = 0 \), where \( \Phi = Tr(\Phi^s), \tilde{\Phi} = Tr(Tr(\Phi^s)) \) and so on, \( Tr \) is a contraction of two indices by metric tensor. For the description of massless particles with spin \( s \) we will use the Lagrangian [12]:

\[
\mathcal{L}_{\Phi^s} = (-1)^s \left\{ \frac{1}{2} (\partial_{\mu} \Phi^s)(\partial^{\mu} \Phi^s) - \frac{s}{2} (\partial \cdot \Phi^s)(\partial \cdot \Phi^s) - \frac{s(s-1)}{4}(\partial_{\mu} \Phi^s)(\partial^{\mu} \Phi^s) - \frac{s(s-1)}{2}(\partial \cdot \Phi^s)\Phi^s - \frac{1}{8}s(s-1)(s-2)(\partial \cdot \Phi^s)(\partial \cdot \Phi^s) \right\}
\]
where \((\partial \cdot \Phi^s) \equiv \partial_\mu \Phi^{\mu \nu_2 \ldots \nu_s}\), \((\partial \cdot \partial \cdot \Phi^s) \equiv \partial_\mu \partial_\nu \Phi^{\mu \nu_2 \ldots \nu_s}\), and point denotes contraction of all indices between tensor objects, for example, \(\Phi^s \cdot \Phi^s \equiv \Phi_{\nu_1 \ldots \nu_s} \Phi^{\mu_1 \ldots \mu_s}\). In flat space this Lagrangian is invariant under the following gauge transformations:

\[
\delta \Phi^s = \frac{1}{s} \{ \partial \xi^{s-1} \}_s.
\]  

(17)

where \(\{ \ldots \}_s\) means symmetrization over all indices (without normalization) and \(\xi^{s-1}\) is symmetric traceless tensor of rank \(s-1\).

To obtain a description for massive particle we start with the sum of massless Lagrangians for fields \((\Phi^s, \Phi^{s-1}, \ldots, \Phi^1, \Phi^0)\):

\[
L_0 = \sum_{k=0}^s L_0(\Phi^k)
\]

(18)

where all derivatives now are covariant and add the most general low derivative terms:

\[
\Delta L = \sum_k (-1)^k \left[ a_k \Phi^{k-1}(D \cdot \Phi^k) + b_k \Phi^k(D \Phi^{k-1}) + c_k (D \cdot \Phi^k) \Phi^{k-1} + d_k (\Phi^k)^2 + e_k (\tilde{\Phi}^k)^2 + f_k \tilde{\Phi}^k \Phi^{k-2} \right]
\]

(19)

Then we require that the whole Lagrangian be invariant under the following gauge transformations:

\[
\delta \Phi^k = \frac{1}{k} \{ \partial \xi^{k-1} \}_s + \alpha_k \xi^k + \beta_k \{ g^2 \xi^{k-2} \}_s.
\]

(20)

where the first term is absent at \(k = 0\) and the third one at \(k < 2\), while \(g^2\) is a metric tensor. This allows one to express all the parameters in the Lagrangian as well as \(\beta\)'s:

\[
a_k = -k\alpha_{k-1}, \quad b_k = -(k-1)\alpha_{k-1}, \quad c_k = -\frac{k(k-1)(k-2)}{4} \alpha_{k-1},
\]

\[
2d_k = \frac{2(k+1)(2k+d-3)}{(2k+d-4)} \alpha_k^2 - k\alpha_{k-1}^2 - \Omega[(k-1)(k-4) + (k-2)(d-1)], \quad k \geq 1
\]

\[
d_0 = \frac{d}{d-2} \alpha_1^2, \quad \beta_k = \frac{2\alpha_{k-1}}{(k-1)(2k+d-6)},
\]

\[
2e_k = -\frac{k(k^2-1)(2k+d)}{4(2k+d-4)} \alpha_k^2 + \frac{k^2-1}{2} \alpha_{k-1}^2 - \Omega \frac{k(k-1)}{2} [k(k-3) + (k-1)(d-1)]
\]

\[
2f_k = -k(k-1)\alpha_{k-1} \alpha_{k-2}
\]

and gives recurrent relation on \(\alpha\)'s:

\[
(k-1)\alpha_{k-2}^2 = -\frac{(k+1)(2k+d-2)}{2k+d-4} \alpha_k^2 + \frac{2k(2k+d-5)}{(2k+d-6)} \alpha_{k-1}^2 - 2\Omega(2k+d-5), \quad k \geq 2
\]

(21)

In order to solve this relation we use our usual convention on mass normalization which in this case gives \(\alpha_{s-1}^2 = m^2/s\). We get:

\[
\alpha_k^2 = \frac{(s-k)(s+k+d-3)}{(k+1)(2k+d-2)} [m^2 - \Omega(s-k-1)(s+k+d-4)], \quad 0 \leq k \leq s-2
\]

(22)
In some sense the last formula is the main result of our work. It gives all the critical values for the cosmological constant at which partial masslessness appears. Indeed, it’s easy to see that then, say, $\alpha_k = 0$ the whole system decompose onto two subsystems, one of them (with the fields $\Phi^s, \Phi^{s-1}, ..., \Phi^{k+1}$) describing unitary partially massless theory. For $d = 4$ our results coincide with the conjecture made recently by Deser and Waldron [6].

**Acknowledgments**

I am grateful to S. M. Klishevich for collaboration on earlier stages of this work.
References


