D0-branes with non-zero angular momentum

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Abstract.
In my talk I shall consider the mechanism of self-expansion of a system of $N$ D0-branes into high-dimensional non-commutative world-volume investigated by Harmark and Savvidy in [1]. Here D2-brane is formed due to the internal angular momentum of D0-brane system. The idea is that attractive force of tension should be cancelled by the centrifugal motion preventing a D-brane system from collapse to a lower-dimensional one. I shall also present a new extended solution where a total of 9 space dimensions is used to embed a D0-brane system. In the last section, by performing linear analysis, the stability of the system is demonstrated.

INTRODUCTION

In the last years there has been increasing interest in dimensionally reduced supersymmetric Yang-Mills theories [2, 3, 4]. One of the reasons is that the reduction of ten-dimensional theory to $p+1$ dimension is relevant for the description of D$p$-branes, $p$-dimensional extended objects carrying Ramond-Ramond (RR) charges in type II superstring theories [5]. In the extreme reduction to zero dimensions it is believed to describe D0-branes, fundamental pointlike objects in type IIA superstring theory [6]. In certain energy regimes the dynamics of $N$ such particles can be described by the supersymmetric quantum mechanics of $N \times N$ Hermitian matrices obtained from dimensional reduction of $\mathcal{N} = 1, D = 10$ super-Yang-Mills theory down to $0+1$ dimensions 1.

It is believed that supersymmetric quantum mechanics of many D0-branes in type IIA superstring theory is equivalent to a partonic description of light-front M-theory [11], a more fundamental underlying M-theory [8, 9, 10]. The existence of matrix formulation of $M$-theory [11], the BRSS-conjecture, crucially relies on the existence within type-IIA string theory of a tower of massive BPS particles electrically charged with respect to the RR 1-form. These particles originally described as black holes in IIA supergravity were identified with D0-branes and can be interpreted as Kaluza-Klein particles of eleven-dimensional $M$-theory compactified on a circle. The existence of the $M$-theoretic Kaluza-Klein tower of states is equivalent to the statement that supersymmetric Yang-Mills quantum mechanics has exactly one bound state for each $N$ [12, 13, 14].

Remarkable aspect of matrix theory is that not only classical gravitational interactions can be produced in the large N-limit, but also the appearance of the superstring extended objects in terms of pointlike fundamental degrees of freedom. One example of such

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1 It was also proposed that $0+0$- dimensional matrix model should give a Poincare invariant non-perturbative definition of IIB superstring theory, the so-called IKKT model [7].
phenomena is the observation that a IIA superstring and a system of $N$ D0-branes can be blown-up to a D2-brane by placing it in background field [15, 16].

In my talk I shall consider another mechanism of self-expansion of a system of $N$ D0-branes into high dimensional non-commutative world-volume investigated by Harmark and Savvidy in [1], here D2-brane is formed due to the internal angular momentum of D0-brane system. The idea is that attractive force of tension should be cancelled by the centrifugal motion preventing a D-brane system from collapse to a lower-dimensional one [2]. I shall also present a new extended solution where a total of 9 space dimensions is used to embed a D0-brane system. In the last section, by performing linear analysis, the stability of the system is demonstrated.

In the $D$-brane formulation the dynamics of $N$ D0-branes can be described by the supersymmetric quantum mechanics of $N \times N$ Hermitian matrices obtained from dimensional reduction of $\mathcal{N} = 1$, $D = 10$ super-Yang-Mills theory to $0+1$ dimensions [6, 16, 20, 21] (the quantum mechanical model was originally studied in [4, 22, 23]). The effective action of $N$ D0-branes is the non-abelian SU($N$) Yang-Mills action plus the Chern-Simons action

$$S_{YM} = -T_0(2\pi l_s^2)^2 \int dt \text{Tr} \left( \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right),$$

(1)

where $F_{\mu \nu}$ is the non-abelian SU($N$) field strength in the adjoint representation and $T_0 = (g_s l_s)^{-1}$ is the D0-brane mass. To write this action in terms of coordinate matrices $X^i$, one has to use the dictionary [1]

$$A_i = \frac{1}{2 \pi l_s^2} X^i, \quad F_{0i} = \frac{1}{2 \pi l_s^2} \dot{X}^i, \quad F_{ij} = -\frac{i}{(2\pi l_s^2)^2} [X^i, X^j]$$

(2)

with $i, j = 1, 2, \ldots, 9$ and in $A_0 = 0$ gauge we have

$$S_{YM} = T_0 \int dt \text{Tr} \left( \frac{1}{2} \dot{X}^i \dot{X}^i - \frac{1}{4(2\pi l_s^2)^2} [X^i, X^j][X^i, X^j] \right).$$

(3)

The equations of motion are

$$\ddot{X}^i = -\frac{1}{(2\pi l_s^2)^2} [X^j, [X^j, X^i]]$$

(4)

and should be taken together with the Gauss constraint

$$[\dot{X}^i, X^i] = 0.$$  

(5)

The Chern-Simons action derived in [16, 20, 21] for the coupling of $N$ D0-branes to bulk RR $C^{(1)}$ and $C^{(3)}$ fields is

$$S_{CS} = T_0 \int dt \text{Tr} \left( C_0 + C_3 \dot{X}^i + \frac{1}{2\pi l_s^2} \left[ C^{(3)}_{0ij} [X^i, X^j] + C^{(3)}_{ijk} [X^i, X^j] \dot{X}^k \right] \right).$$

(6)

\[2\] Similar phenomena appear in the cases of rotating branes on spheres [17, 18], "giant gravitons", and in the cases when angular momentum is generated by crossed electric and magnetic BI fields [19].
and describes the interaction of $N$ D0-branes with slowly varying background fields of Type IIA supergravity. Even though the D0-brane world-volume is only one-dimensional a multiple D0-brane system can couple to a brane charges of higher dimension. Myers [16] considers the system of $N$ D0-branes in a constant external 4-form RR field strength $F_4 = dC^{(3)}$ and has found a stable solution, where the D0-branes are polarized and arranged into a static spherical configuration. A lower-dimensional object under the influence of higher-form RR fields may nucleate or be 'blown-up' into D2-brane. The background field imposes an external force that prevents the collapse of the D-brane to a lower-dimensional one.

Another way to get self-support against collapse is to allow D0-brane system to carry mechanical angular momentum [1]. This new kind of rotating solution of the system of $N$ D0-branes was constructed by Troels and Konstantin in [1] and in the subsequent articles [24, 25] it was demonstrated that this solution describes a stable D2-brane configuration. Below I shall concentrate mostly on this solution and on it generalizations presented in [25]. The basic idea in their construction is that the attractive force of tension should be cancelled by the centrifugal repulsion force. The earlier work where membrane solution appeared with non-zero angular momentum was [26].

**D2-BRANE FROM MULTIPLE ROTATING D0-BRANES**

I shall briefly review the spherical D2-brane configuration of type IIA string theory since a new solution for a rotating system of $N$ D0-branes presented below uses the essential elements of this construction. The solution is equivalent to the spherical membrane solution of M(atrix) theory [11, 29].

With the aim to construct a membrane with an $S^2$ geometry we shall embed the $S^2$ in a three-dimensional space spanned by the 123 directions and consider the ansatz

$$X_i(t) = \frac{2}{\sqrt{N^2-1}} L_i r_i(t), \quad i = 1, 2, 3$$

where the $N \times N$ matrices $L_1, L_2, L_3$ are the generators of the $N$ dimensional irreducible representation of $SU(2)$, with algebra

$$[L_i, L_j] = i \epsilon_{ijk} L_k .$$

and with the quadratic Casimir $\sum_{i=1}^3 L_i^2 = \frac{N^2-1}{4}$, so that $\text{Tr}(L_i^2) = \frac{N(N^2-1)}{12}$. For vanishing background fields the Hamiltonian is

$$H = \frac{N l_s}{3} \left[ \frac{1}{2} \sum_{i=1}^3 r_i^2 + \frac{\alpha^2}{2} \left( r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2 \right) \right],$$

where we have introduced a convenient parameter $\alpha = \frac{1}{2 \sqrt{N^2-1} l_s^2}$. This gives the equations of motion

$$\ddot{r}_1 = -\alpha^2 (r_2^2 + r_3^2) r_1, \quad \ddot{r}_2 = -\alpha^2 (r_1^2 + r_3^2) r_2, \quad \ddot{r}_3 = -\alpha^2 (r_1^2 + r_2^2) r_3.$$  

This system is otherwise known as $0+1$ dimensional classical SU(2) YM mechanics [22, 23]. Let us for simplicity take all radii to be equal to each other: $r_1 = r_2 = r_3 = r$. 
With this we have from (7) the physical radius of the membrane \( R^2 = X_1^2 + X_2^2 + X_3^2 = I \, r^2 \), where \( I \) is the \( N \times N \) identity matrix. The last formula shows that the \( N \) D0-branes are constrained to lie on an \( S^2 \) sphere of radius \( r \). The equations of motion (9) in this case reduce to the equation \( \ddot{r} = -2\alpha^2 r^3 \) and the solution is \( r(t) = R_0 \sin(R_0 \alpha t + \phi) \) oscillating between \( R_0 \) and \(-R_0\). One can trust this solution if \( |r(t)| \ll l_s \sqrt{N} \), \( |r(t)| \ll 1 \), \( |\ddot{r}(t)| \ll l_s^{-1} \). Since we also require \( |r(t)| \gg l_s \) we must have \( N \gg 1 \). Thus it is necessary in order to have a large amount of D0-branes to build a macroscopic spherical membrane.

In order to construct the rotating ellipsoidal membrane, viewed as a non-commutative collection of moving D0-branes we shall take previous configuration of the non-commutative fuzzy sphere in the 135 directions, and set it to rotate in the transverse space along three different axis, i.e. in the 12, 34 and 56 planes. We thus use a total of 6 space dimensions to embed our D0-brane system. The corresponding ansatz is [1]

\[
\begin{align*}
X_1(t) &= \frac{2}{\sqrt{N^2 - 1}} L_1 r_1(t) , & X_2(t) &= \frac{2}{\sqrt{N^2 - 1}} L_1 r_2(t) , \\
X_3(t) &= \frac{2}{\sqrt{N^2 - 1}} L_2 r_3(t) , & X_4(t) &= \frac{2}{\sqrt{N^2 - 1}} L_2 r_4(t) , \\
X_5(t) &= \frac{2}{\sqrt{N^2 - 1}} L_3 r_5(t) , & X_6(t) &= \frac{2}{\sqrt{N^2 - 1}} L_3 r_6(t) .
\end{align*}
\] (10)

where the \( N \times N \) matrices \( L_1, L_2, L_3 \) are the generators of the \( N \)-dimensional irreducible representation of \( SU(2) \). In this ansatz the matrix structure is such that the coordinate matrices are proportional to the \( SU(2) \) generators in pairs and the Gauss constraint (5) is identically satisfied. It is also the only finite-dimensional subalgebra of the group of diffeomorphisms of \( S^2 \), the \( S_{\text{diff}}(S^2) \) [28]. That is why the \( SU(2) \) ansatz is unique: it is the only type of solution that carries over to the supermembrane without modification [30].

Substituting the ansatz into (3) gives the Hamiltonian \( H = \frac{N T_6}{3} \left( \frac{1}{2} \sum_{i=1}^{6} \dot{r}_i^2 + \frac{\alpha^2}{2} \left[ (r_1^2 + r_2^2)(r_3^2 + r_4^2) + (r_1^2 + r_5^2)(r_3^2 + r_6^2) + (r_2^2 + r_4^2)(r_5^2 + r_6^2) \right] \right) \) and the corresponding equations of motion

\[
\begin{align*}
\ddot{r}_1 &= -\alpha^2 (r_1^2 + r_2^2 + r_3^2 + r_6^2) r_1 , & \ddot{r}_2 &= -\alpha^2 (r_3^2 + r_4^2 + r_5^2 + r_6^2) r_2 , \\
\ddot{r}_3 &= -\alpha^2 (r_1^2 + r_2^2 + r_3^2 + r_6^2) r_3 , & \ddot{r}_4 &= -\alpha^2 (r_1^2 + r_2^2 + r_5^2 + r_6^2) r_4 , \\
\ddot{r}_5 &= -\alpha^2 (r_1^2 + r_2^2 + r_3^2 + r_4^2) r_5 , & \ddot{r}_6 &= -\alpha^2 (r_1^2 + r_2^2 + r_3^2 + r_4^2) r_6 .
\end{align*}
\] (11)

The special solution of these equations, describing a rotating ellipsoidal membrane with three distinct principal radii \( R_1, R_2 \) and \( R_3 \) is [1]

\[
\begin{align*}
\dot{r}_1(t) &= R_1 \cos(\omega_1 t + \phi_1) , & \dot{r}_2(t) &= R_1 \sin(\omega_1 t + \phi_1) ,
\end{align*}
\]

3 Clearly the spherical membrane will be classically point-like at the nodes of the \( \sinus \). Thus the membrane solution will break down after a finite amount of time, since the classical solution may not be valid at substringy distances, and possibly decay into a Schwarzschild black hole [29].
This particular functional form of the solution ensures that the highly non-linear equations for any of the components \( r_i \) are reduced to a harmonic oscillator. The solution (12) keeps \( r_1^2 + r_2^2 = R_1^2 \), \( r_2^2 + r_4^2 = R_2^2 \) and \( r_3^2 + r_6^2 = R_3^2 \) fixed, which allows us to say that the object described by (12) rotates in six spatial dimensions as a whole without changing its basic shape. At any point in time one can always choose a coordinate system in which the object spans only three space dimensions.

Using the equations of motion (11), the three angular velocities are determined by the radii, and do not necessarily have to coincide: \( \omega_1 = \alpha \sqrt{R_2^2 + R_3^2} \), \( \omega_2 = \alpha \sqrt{R_1^2 + R_3^2} \), \( \omega_3 = \alpha \sqrt{R_1^2 + R_2^2} \). This dependence of the angular frequency on the radii is such that the repulsive force of rotation has to be balanced with the attractive force of tension in order for (12) to be a solution. Thus the radii \( R_1 \), \( R_2 \) and \( R_3 \) parameterize (12) along with the three phases \( \Phi_i \), to produce altogether a six-parameter family of solutions. In order to exhibit the properties of the solution (12) one shall evaluate the energy, the components of angular momentum and to find out nonzero D-brane currents. The non-zero components of \( M_{ij} \) and we have to impose anti-symmetrization with respect to \( i j k \) indices. For our solution the non-zero components of \( Q \) are: \( Q_{135} = \frac{1}{2} \sqrt{\alpha} R_1 R_2 R_3 \), \( Q_{246} = \frac{1}{2} \sqrt{\alpha} R_1 R_2 R_3 \) \( \cos \omega_1 t \cos \omega_3 t \), \( \cos \omega_5 t \), together with \( Q_{246} \), \( Q_{146} \), \( Q_{235} \), \( Q_{236} \), \( Q_{125} \), \( Q_{245} \). From this it is easy to obtain the corresponding \( J \)'s by differentiation with respect to time. Thus the Chern-Simons action (6) shows that the coupling of this system to \( F_{0123} \) is non-vanishing and that the spherical membrane solution has a D2-brane dipole moment. The higher currents are equal to zero and our system does not carry D4-D8-charges.

In addition I shall present "breathing" brane solutions. For that it is convenient to introduce polar coordinates \(( \rho \cos \phi, \rho \sin \phi \) so that the Hamiltonian \( \hat{H} = \frac{3M}{N T_0} \) takes the form: \( \hat{H} = \frac{1}{2} \sum_{i=1}^{3} \left[ \rho_i^2 \dot{\phi}_i^2 + \rho_i^2 \dot{\phi}_i^2 \right] + \frac{1}{2} \rho_1^2 \rho_2^2 + \rho_2^2 \rho_3^2 + \rho_3^2 \rho_1^2 \). The conservation integrals are: \( \rho_1^2 \dot{\phi}_1 = \tilde{M}_1 \), \( \rho_2^2 \dot{\phi}_2 = \tilde{M}_2 \), \( \rho_3^2 \dot{\phi}_3 = \tilde{M}_3 \), where \( \tilde{M}_i = \frac{3M_i}{N T_0} \) and the Hamiltonian takes the

\[
\begin{align*}
    r_3(t) &= R_2 \cos(\omega_2 t + \phi_2), \quad r_4(t) = R_2 \sin(\omega_2 t + \phi_2), \\
    r_5(t) &= R_3 \cos(\omega_3 t + \phi_3), \quad r_6(t) = R_3 \sin(\omega_3 t + \phi_3).
\end{align*}
\]
form: \[ \hat{H} = \frac{1}{2} \sum_{i=1}^{3} \left[ \dot{\rho}_i^2 + \frac{M_i^2}{\rho_i^2} \right] + \frac{1}{2} [\rho_1^2 \rho_2^2 + \rho_2^2 \rho_3^2 + \rho_3^2 \rho_1^2]. \] The equations of motion are:

\[ \dot{\rho}_i = -\rho_i (\ddot{\rho}_i - \rho_i^2) + \frac{\ddot{M}_i}{\rho_i^3}, \quad i = 1, 2, 3 \quad (15) \]

and our previous solution (12) is \( \rho_i = R_i = \text{Const}, i = 1, 2, 3 \) and \( \dot{\rho}_i^2 = \omega_i^2 = \ddot{M}_i^2/R_i^4 = R_{i+1}^2 + R_{i+2}^2. \) In the special case when all coordinates are equal to each other \( \rho_1 = \rho_2 = \rho_3 = \rho(t) \) and can depend on time, the "breathing" brane solution, then \( \hat{H} = \frac{3}{2} [\dot{\rho}^2 + \frac{M_i^2}{\rho^2} + \rho^4] \) and corresponding equation can be integrated. The new solution is elliptic function \( \rho = \rho(t) [24] \).

In order to increase the number of parameters of the rotating N D0-brane system, I shall take previous configuration and set it to rotate in the transverse spaces along three different axis, \( i.e. \) in the 123, 456 and 789 planes. Thus I shall use a total of 9 space dimensions to embed D0-brane system. The corresponding ansatz is

\[
\begin{align*}
X_1(t) &= \frac{2}{\sqrt{N^2 - 1}} L_1 r_1(t) ,& X_2(t) &= \frac{2}{\sqrt{N^2 - 1}} L_1 r_2(t) , & X_3(t) &= \frac{2}{\sqrt{N^2 - 1}} L_1 r_3(t) ,
X_4(t) &= \frac{2}{\sqrt{N^2 - 1}} L_2 r_4(t) ,& X_5(t) &= \frac{2}{\sqrt{N^2 - 1}} L_2 r_5(t) , & X_6(t) &= \frac{2}{\sqrt{N^2 - 1}} L_2 r_6(t) ,
X_7(t) &= \frac{2}{\sqrt{N^2 - 1}} L_3 r_7(t) ,& X_8(t) &= \frac{2}{\sqrt{N^2 - 1}} L_3 r_8(t) , & X_9(t) &= \frac{2}{\sqrt{N^2 - 1}} L_3 r_9(t) [16].
\end{align*}
\]

The coordinate matrices are again proportional to the \( SU(2) \) generators and the Gauss constraint (5) is identically satisfied. Substituting the last ansatz into (3) gives the Hamiltonian:

\[ H = \frac{N T_6}{2} \left( \frac{1}{2} \sum_{i=1}^{9} r_i^2 + \alpha^2 \left[ (r_1^2 + r_2^2 + r_3^2)(r_4^2 + r_5^2 + r_6^2) + (r_1^2 + r_2^2 + r_3^2) (r_4^2 + r_5^2 + r_6^2) \right] \right) \]

and the equations

\[
\begin{align*}
\ddot{r}_i &= -\alpha^2 (r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2) r_i , \quad i = 1, 2, 3 \\
\ddot{r}_j &= -\alpha^2 (r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2) r_j , \quad j = 4, 5, 6 \\
\ddot{r}_k &= -\alpha^2 (r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2) r_k , \quad k = 7, 8, 9 . \quad (17)
\end{align*}
\]

The special solution of these equations, which ensures that the highly non-linear equations for any of the components \( r_i \) are reduced to a harmonic oscillator, is:

\[
\begin{align*}
\dot{r}_1(t) &= R_1 \cos(\omega_1 t + \phi_1) , & \dot{r}_2(t) &= R_1 \sin(\theta_1) \cdot \sin(\omega_1 t + \phi_1) , & \dot{r}_3(t) &= R_1 \cos(\theta_1) \cdot \sin(\omega_1 t + \phi_1) , \\
\dot{r}_4(t) &= R_2 \cos(\omega_2 t + \phi_2) , & \dot{r}_5(t) &= R_2 \sin(\theta_2) \cdot \sin(\omega_2 t + \phi_2) , & \dot{r}_6(t) &= R_2 \cos(\theta_2) \cdot \sin(\omega_2 t + \phi_2) , \\
\dot{r}_7(t) &= R_3 \cos(\omega_3 t + \phi_3) , & \dot{r}_8(t) &= R_3 \sin(\theta_3) \cdot \sin(\omega_3 t + \phi_3) , & \dot{r}_9(t) &= R_3 \cos(\theta_3) \cdot \sin(\omega_3 t + \phi_3) . \end{align*}
\]

The object described by (18) rotates in nine spatial dimensions as a whole without changing its basic shape. The radii \( R_1, R_2 \) and \( R_3 \) parameterize (18) along with

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4 A somewhat similar ansatz was proposed in [31], where some of the features of the solution were foreseen.
the six phases $\theta_i$ and $\phi_i$, to produce altogether a nine-parameter family of solutions. The energy and the angular momentum are the same (13). The interaction with the $C^{(5)}$-field is governed by the Chern-Simons action derived in [16, 20, 21]

$$\mathcal{T}_n \frac{1}{(2\pi l_s^2)^2} \int dt S \text{Tr} \left( C^{(5)}_{ijklm} [X^i, X^j] [X^k, X^l] [X^m, X^n] \right).$$

We denote the corresponding current by $j^{ijklm}$

$$j^{ijklm} = \frac{1}{(2\pi l_s^2)^2} S \text{Tr} \left( [X^i, X^j] [X^k, X^l] [X^m, X^n] \right). \quad (19)$$

One can be convinced that all components of $J$ are equal to zero. Thus the Chern-Simons action shows that the coupling of this system to $F_{ijklm}$ is vanishing and that the extended solution does not carry $D4$-brane charge.

**STABILITY ANALYSIS WITHIN SU(2) AND FULL SU(N) GROUP**

The purpose of this section is to present a complete stability analysis of the fluctuations in the neighborhood of the rotating $D0$-brane solution of [1]. Initially in [1] were analyzed perturbations that do not modify the original $SU(2)$ ansatz. In [25, 24] this analysis was extended to the case when perturbations are in the full $SU(N)$ algebra directions. In the full $SU(N)$ case there are exactly $N^2 + 12$ zero-modes, of which $N^2 - 1$ are the consequence of the global color rotation symmetry of the solution, and 6 are associated with global space rotations. All other modes are completely stable and execute harmonic oscillations around the original trajectory.

Let me present the stability analysis of the $SU(2)$ ($l = 1$) perturbation of the system [1] and then move to more general $SU(N)$ ($l = 2, 3, 4, \ldots$) perturbations [25, 24]. The equations of variation which follow from equations in polar coordinates (15) are:

$$\delta \rho_1 = -4(R_2^2 + R_3^2) \delta \rho_1 - 2R_1 R_2 \delta \rho_2 - 2R_1 R_3 \delta \rho_3 \quad (20)$$

$$\delta \rho_2 = -2R_2 R_1 \delta \rho_1 - 4(R_1^2 + R_3^2) \delta \rho_2 - 2R_2 R_3 \delta \rho_3 \quad (21)$$

$$\delta \rho_3 = -2R_3 R_1 \delta \rho_1 - 2R_3 R_2 \delta \rho_2 - 4(R_1^2 + R_2^2) \delta \rho_3, \quad (22)$$

and have only positive modes [24]

$$\Omega_1^2 = 4(R_1^2 + R_2^2 + R_3^2), \quad (23)$$

$$\Omega_{2,3}^2 = 2(R_1^2 + R_2^2 + R_3^2) \pm \sqrt{2((R_1^2 - R_2^2)^2 + (R_2^2 - R_3^2)^2 + (R_3^2 - R_1^2)^2)}. \quad (24)$$

To consider perturbations in all directions of underlying $SU(N)$ group we should represent $SU(N)$ generators $Y^l_m$ as higher order monomials in the $N \times N$ matrix generators $L_i$, $i = 1, 2, 3$ of the $SU(2)$ group

$$Y^l_m = \sum_{i_1, \ldots, i_l} e^l_m (i_1, \ldots, i_l) L_{i_1} \cdots L_{i_l}. \quad (25)$$

The total number of generators is then $\sum_{l=1}^{N-1} (2l + 1) = N^2 - 1$ as it should be for $SU(N)$. The general explicit construction of the $Y^l_m$ is due to Schwinger (see also [32, 33, 27, 34]).
Let us represent the rotating D0-brane solution in a more convenient form: \( X^i(t) = L_i R \cos(\omega t), \) \( \dot{X}^i(t) = X^{i+3}(t) = L_i R \sin(\omega t), \) \( i = 1, 2, 3. \) In what follows we will set \( R = 1, \) with \( \omega \) equal to \( \omega^2 = 2R^2 = 2. \) In the basis provided by the spherical operators \( Y_m^l \) (25) we have [28, 30],

\[
\begin{align*}
[L_z, Y_m^l] &= m Y_m^l & \text{for} \; l = 1, \ldots, N - 1 \\
[L_{\pm}, Y_m^l] &= \sqrt{(l \mp m)(l \pm m + 1)} Y_{m \pm 1}^l. \quad (26)
\end{align*}
\]

We will not use the explicit form of these matrices, as the defining relations (26) is all that is needed. The properties under Hermitian conjugation can be summed up as \( Y_{i,m}^1 = (-1)^m Y_{l,-m}. \) I shall first consider the perturbations which are parallel in space to one of the directions of our solution, that is in the directions 123456 and then in the directions 789. Let us decompose the fluctuation fields in the basis defined by \( Y_m^l, \) where \( l \) runs from 1 to \( N - 1 \)

\[
\delta X^i = \sum_{m=-l}^{l} Y_m^l \xi_m^i, \quad \delta \dot{X}^i = \sum_{m=-l}^{l} Y_m^l \eta_m^i \quad i = 1, 2, 3. \quad (27)
\]

We do not explicitly show an \( l \) index on the \( \eta, \xi. \) We should also impose \( \xi_m^* = (-1)^m \xi_{-m} \) and \( \eta_m^* = (-1)^m \eta_{-m} \) for all \( m = -l, \ldots, l. \) The variational equations of motion are

\[
-\delta \dot{X}^i = [\delta X^j, [X^j, X^i]] + [X^j, [\delta X^j, X^i]] + [X^j, [\delta X^i, X^j]] + [X^i, [\delta X^j, X^j]] + [X^i, [\delta X^i, X^j]] + [\delta X^j, [\delta X^i, X^j]]. \quad (28)
\]

The constraint equation looks like

\[
\sum_{i,m} [\delta \dot{X}^i, X^i] + [\dot{X}^i, \delta X^i] + [\delta \dot{X}^i, \dot{X}^i] + [\dot{X}^i, \delta \dot{X}^i] = 0. \quad (29)
\]

Using the commutation relations (26) we get for the constraint \( \sum_l L_{nm}^l (\cos(\omega) \xi_m^i + \omega \sin(\omega) \xi_m^i + \sin(\omega) \eta_m^i - \omega \cos(\omega) \eta_m^i) = 0, \) where \( L_{nm}^l, \) are now the \( SU(2) \) generators in the \((2l + 1) \times (2l + 1)\) representation. In the co-moving coordinates

\[
u_m^i = \cos(\omega) \xi_m^i + \sin(\omega) \eta_m^i \quad \text{and} \quad \xi_m^i = -\sin(\omega) \xi_m^i + \cos(\omega) \eta_m^i. \quad (30)
\]

the constraint looks simpler,

\[
\sum_{l,m} L_{nm}^l (\nu_m^i - 2\omega v_m^i) = 0. \quad (31)
\]

The variational equation of motion (28) after substituting the fields (27) is

\[
\dot{\xi}_m^i + (l + 1) \xi_m^i = \cos(\omega) \left( L_{mj}^j L_{nm}^{jm} + i \varepsilon_{jk} L_{nm}^{jk} \right) \left( \cos(\omega) \xi_m^i + \sin(\omega) \eta_m^i \right). \quad (32)
\]

The decoupling of the modes with different \( l \) is seen to be a direct consequence of (26), and more fundamentally, of the pure \( SU(2) \) structure of the original background
solution (12). The equation for $\eta$ is gotten by exchanging cosines for sines and $\xi$ for $\eta$
In the co-moving coordinates (30) the equation becomes a linear system with constant
coefficients:

$$i\eta_l + (l(l+1) - 2) \eta_l' - 2\omega \eta_l = \left( L_{nm}^j L_{l'm}^j + i e_{jik} L_{lnm}^k \right) \eta_m, \quad (33)$$

$$\nu_l' + (l(l+1) - 2) \nu_l' + 2\omega \nu_l = 0. \quad (34)$$

Thus we shall analyze the system of equations (31) (33), (34). The $rhs$ of (33) is a matrix
acting on a 3($2l + 1$) component vector and the eigenvalues $\Lambda$ of this block matrix are
given in the table, together with their multiplicity [25]. Choose a fixed frequency ansatz
$u_l(t) = e^{i\Omega t} u_l, \quad \nu_l(t) = e^{i\Omega t} \nu_l$. The second equation (34) can be solved as

$$\nu_l = \frac{-2\sqrt{2} i \Omega^2}{l(l+1) - 2 - \Omega^2} u_l', \quad (35)$$

and then substituted back into the first equation (33). In the basis in which the matrix
on the $rhs$ is diagonalized, it can be replaced with its respective eigenvalue
$\Lambda$, resulting in an algebraic equation for the $\Omega$: \( (l(l+1) - 2 - \Omega^2)^2 - 8\Omega^2 =
\Lambda (l(l+1) - 2 - \Omega^2) \). Finally, this quadratic equation can be solved, $\Omega_{1,2}^2 = \frac{1}{2} \Lambda + l(l+1) + 2 \pm \frac{1}{2} \sqrt{\Lambda^2 - 16\Lambda + 32(l(l+1))}$ and the corresponding modes are
given in the table

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>$\Omega_{1,2}^2$</th>
<th>$\Omega_{1,2}^2$</th>
<th>multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l(l+1) - 2$</td>
<td>0</td>
<td>$l^2 + l + 6$</td>
<td>$2l + 1$</td>
</tr>
<tr>
<td>$2l$</td>
<td>$l^2 - 3l + 2$</td>
<td>$l^2 + 3l + 2$</td>
<td>$2l + 3$</td>
</tr>
<tr>
<td>$- (2l + 2)$</td>
<td>$l^2 - l$</td>
<td>$l^2 + 5l + 6$</td>
<td>$2l - 1$</td>
</tr>
</tbody>
</table>

Note that the number of zero modes changes from 9 for the case $l = 1$ and 12 for
$l = 2$, to $2l + 1$ for arbitrary $l > 2$. Thus the total number of zero modes is the sum
$9 + 12 + \sum_{l=3}^{N-1} (2l + 1) = N^2 + 12$. Now I shall consider perturbations that are in the
directions 789, if we had oriented the original solution along 123456. The perturbations
$\delta X^k_m = \sum_n \gamma_m^l r_m^k$ for $k = 7, 8, 9$ satisfy the simple harmonic equation
$\ddot{r}_m^k + l(l+1) r_m^k = 0$. This clearly has only positive frequencies and is therefore stable. For $l = 1$
all the 9 modes have the same frequency as the original solution, corresponding to
infinitesimal global rotations of the system into the 789 hyperplane. The counting goes
as follows, there are $9 \times 2 = 18$ first order degrees of freedom here, which coincides with
the dimensionality of the grassmanian manifold of embeddings of a 6-hyperplane into
$R^9$, i.e. $SO(9) / SO(6) \times SO(3)$. From these results it follows that, zero-modes notwithstanding, all
the frequencies in the system are positive, and arbitrary small perturbation will remain
bounded for all times $^5$.

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$^5$ The same problem was considered also in the paper [35]. However the authors of [35] initially arrived
at the Mathieu equation instead of the equations (33), (34), (31) and therefore to the opposite conclusion,
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REFERENCES


namely that there exist solutions of the linearized perturbation equations which grow exponentially ([35],v1). It was found then that an algebraic mistake appeared in the calculation so that finally they ended up with the same conclusion ([35],v2).
35. M. Axenides, E. G. Floratos and L. Perivolaropoulos, Metastability of spherical membranes in supermembrane and matrix theory. hep-th/0007198 v1,v2