where the quantum number $j$ is the number of the multiplet, and $j_1$ and $j_2$ are positive integers. The Hamiltonian

\begin{equation}
H = \frac{1}{2m} \sum_{n=1}^{\infty} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) + \frac{\epsilon}{2} \sum_{n,m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

is called the Hamiltonian of the system. The minimal action $\mathcal{S}_m$ is given by

\begin{equation}
\mathcal{S}_m = \int_0^\beta \left( p_x \frac{dx}{d\tau} + p_y \frac{dy}{d\tau} + p_z \frac{dz}{d\tau} - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) \right) d\tau
\end{equation}

where $\beta$ is the total time. The action is stationary with respect to variations of the functions $\psi_n(x,y,z)$.

The equations of motion are

\begin{equation}
\frac{d^2\psi_n}{d\tau^2} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) = \frac{\epsilon}{m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

where the quantum number $n$ is the number of the multiplet, and $j_1$ and $j_2$ are positive integers. The Hamiltonian

\begin{equation}
H = \frac{1}{2m} \sum_{n=1}^{\infty} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) + \frac{\epsilon}{2} \sum_{n,m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

is called the Hamiltonian of the system. The minimal action $\mathcal{S}_m$ is given by

\begin{equation}
\mathcal{S}_m = \int_0^\beta \left( p_x \frac{dx}{d\tau} + p_y \frac{dy}{d\tau} + p_z \frac{dz}{d\tau} - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) \right) d\tau
\end{equation}

where $\beta$ is the total time. The action is stationary with respect to variations of the functions $\psi_n(x,y,z)$.

The equations of motion are

\begin{equation}
\frac{d^2\psi_n}{d\tau^2} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) = \frac{\epsilon}{m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

where the quantum number $n$ is the number of the multiplet, and $j_1$ and $j_2$ are positive integers. The Hamiltonian

\begin{equation}
H = \frac{1}{2m} \sum_{n=1}^{\infty} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) + \frac{\epsilon}{2} \sum_{n,m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

is called the Hamiltonian of the system. The minimal action $\mathcal{S}_m$ is given by

\begin{equation}
\mathcal{S}_m = \int_0^\beta \left( p_x \frac{dx}{d\tau} + p_y \frac{dy}{d\tau} + p_z \frac{dz}{d\tau} - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) \right) d\tau
\end{equation}

where $\beta$ is the total time. The action is stationary with respect to variations of the functions $\psi_n(x,y,z)$.

The equations of motion are

\begin{equation}
\frac{d^2\psi_n}{d\tau^2} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) = \frac{\epsilon}{m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

where the quantum number $n$ is the number of the multiplet, and $j_1$ and $j_2$ are positive integers. The Hamiltonian

\begin{equation}
H = \frac{1}{2m} \sum_{n=1}^{\infty} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) + \frac{\epsilon}{2} \sum_{n,m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

is called the Hamiltonian of the system. The minimal action $\mathcal{S}_m$ is given by

\begin{equation}
\mathcal{S}_m = \int_0^\beta \left( p_x \frac{dx}{d\tau} + p_y \frac{dy}{d\tau} + p_z \frac{dz}{d\tau} - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) \right) d\tau
\end{equation}

where $\beta$ is the total time. The action is stationary with respect to variations of the functions $\psi_n(x,y,z)$.

The equations of motion are

\begin{equation}
\frac{d^2\psi_n}{d\tau^2} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) = \frac{\epsilon}{m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

where the quantum number $n$ is the number of the multiplet, and $j_1$ and $j_2$ are positive integers. The Hamiltonian

\begin{equation}
H = \frac{1}{2m} \sum_{n=1}^{\infty} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) + \frac{\epsilon}{2} \sum_{n,m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}

is called the Hamiltonian of the system. The minimal action $\mathcal{S}_m$ is given by

\begin{equation}
\mathcal{S}_m = \int_0^\beta \left( p_x \frac{dx}{d\tau} + p_y \frac{dy}{d\tau} + p_z \frac{dz}{d\tau} - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) \right) d\tau
\end{equation}

where $\beta$ is the total time. The action is stationary with respect to variations of the functions $\psi_n(x,y,z)$.

The equations of motion are

\begin{equation}
\frac{d^2\psi_n}{d\tau^2} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_n(x,y,z) = \frac{\epsilon}{m} \langle \psi_n | \psi_m \rangle \delta(x-y) \delta(y-z) \delta(z-x)
\end{equation}
that these are consistent with the geodesic equations and so indicate the existence of a solution representing a radial time-like geodesic (RTG) emerging from the singularity. Furthermore, it is claimed that this result follows independently of the initial data \( m(r) \).

A vital part of this consistency check results in \( p = 1 + q \). To see that this condition may fail, we consider the example
\[
m(r) = m_0 r^3 + m_1 r^7.
\]
(14)

The lower power here is required for regularity of the initial data, and the higher power ensures that there are no radial null geodesics emerging from the singularity [2]. This choice is included in the class of mass functions \( m(r) \) considered in [1]. Along an RTG emerging from the singularity, we must have \( T \leq 0 \) for \( r \) sufficiently small, with equality only at \( r = 0 \). Then examining the leading order behaviour in
\[
T(r) = \frac{\sqrt{3} \sqrt{2}}{2} \left( t(r) - t_0 - br^3 \right),
\]
for the mass function given, we deduce that \( p \geq 4 \) and that
\[
T(r) \sim T_0 r^4,
\]
for some positive \( T_0 \) (the fact that \( T \geq 0 \) is vital here). The functional dependence here and below indicates evaluation along the geodesic. This asymptotic behaviour can be fed into the expressions above for \( R \) and its derivates and yields
\[
R(r) \sim R_0 r^{11/3},
\]
(15)
\[
R'(r) \sim R_1 r^{8/3},
\]
(16)
\[
R''(r) \sim R_2 r^{-5/3},
\]
(17)
\[
R'''(r) \sim R_3 r^{5/3}.
\]
(18)
Comparing with the ansatze above, we see that \( q = 8/3 \) and \( p = 4 \). However this violates the consistency condition \( p - q = 0 \), indicating that such a solution cannot in fact exist. We note that two assumptions made here played a vital role: (i) the mass function \( m(r) \) excludes radial null geodesics emerging from the singularity, and (ii) the RTG emerges into the regular region of spacetime \( T > 0 \).

The crucial point that is missing in [1], is that the parameters \( b \) and \( p \) [cf. Eq. (28) in [1]] are not independent of the initial data, as implicitly assumed therein. In fact, we must have \( p \geq n \) where \( n \) signals the first non-vanishing derivative of the initial central density distribution, \( \rho_0 \equiv (\partial^n \rho/\partial r^n)_{r = 0} \), or else the geodesic thus constructed will not belong in the spacetime. When this inequality saturates, we obtain the additional constraint \( 0 < \frac{1}{r} < \frac{1}{b} \), where \( b \) is the first non-vanishing coefficient of a MacLaurin series for \( t_0(r) \) [cf. Eq. (14) in [1]]. From Eqs. (2), (9), to leading order in \( r \) we obtain, along the RTG’s,
\[
R \sim r^{5p/11+1} + O(r^{p+2-\frac{q}{3}}),
\]
(19)
This implies \( q = 2n/3 \), and thus the consistency relation \( p = 1 + q \) reads \( p = 1 + 2n/3 \), which is formally the same as that obtained for outgoing radial null geodesics. The parameters \( p \) and \( \beta \) are then uniquely determined from the initial data, and must obey the constraint \( p = 1 + q \geq n \Rightarrow n \leq 3 \).

The statement in [1] (second paragraph) that the work of Deshingkar, Joshi and Dwivedi (DJD) [6] shows that "when one considers timelike radial geodesics, the singularity is found to be locally naked and Tipler strong for an infinite number of non-space-like geodesics, irrespective of the initial data" is partially incorrect: DJD show that only curvature strength is independent of the initial data, not visibility.

An additional comment concerns the parameters \( a_2 \) and \( c_2 \), introduced in Eqs. (28)-(34) in [1]. Since \( R'' \) is obtained from \( R' \) by differentiation with respect to \( r \) along the geodesic, \( a_2 \) is linearly dependent on \( a_1 \); \( a_3 \equiv a q_1 \). Similarly, \( c_3 = a p \). We note that the constants \( c_2 \) are not "free", since they must be fixed by consistency relations involving \( R \) and its derivatives. With the substitutions, the algebraic constraint \( C(a_2, c_2) = 0 \) reads \( a_2 c_2 + 2a q_1 = 0 \). That is, for given initial data (whereby \( a_1 \) and \( a_2 \) are fixed), there is only one degree of freedom in the specification of the two \( c_2 \) parameters (whose ratio is fixed).

As mentioned above, it can be shown that the absence of a radial null geodesic emerging from a central singularity is sufficient to guarantee censorship of the singularity, i.e., it rules out the existence of any causal geodesic emerging from the singularity. To see this, consider a general spherically symmetric space-time with line element
\[
ds^2 = -e^{2\mu} dt^2 + e^{2\nu} dr^2 + R^2(r, t) d\Omega^2,
\]
where \( \mu = \mu(r, t), \nu = \nu(r, t) \). Then the tangent to a causal geodesic satisfies
\[
-e^{2\mu} t^2 + e^{2\nu} r^2 + \frac{L^2}{R^2} = \epsilon,
\]
where the overdot represents differentiation with respect to an affine parameter, \( L \) is the conserved angular momentum, and \( \epsilon = 0, -1 \) for null and time-like geodesics, respectively. Thus, at any point on such a geodesic,
\[
e^{2\mu} t^2 \geq e^{2\nu} r^2,
\]
with equality holding only for radial null geodesics. On the \( t - r \) plane, this reads
\[
\frac{dr}{dt} \leq e^{h-\nu},
\]
(20)
where we take the positive root for future pointing outgoing geodesics (we can use coordinate freedom to guarantee that \( t \) increases into the future globally, and \( \delta, R > 0 \) in a neighbourhood of \( R = 0 \)). We can read (20) as
\[
\frac{d\rho_{CA}}{dt} \leq \frac{d\rho_{RNG}}{dt},
\]
(21)
where the subscripts represent causal (excluding radial null) geodesics and outgoing radial null geodesics respectively. Now suppose that a CG extends back to a central singularity located on the $t = r$ plane at $r = t = t_0$. Assume that the singularity is of the form $t = t_c(r)$ with $t_c(0) = t_0$ and that the regular region of space-time is $t < t_c(r)$. This is the case for the singularity studied above. Let $p$ be any point on the CG, to the future of the singularity. Applying the inequality (21) at $p$, we see that the RNG through $p$ crosses CG from below and hence points on this RNG prior to $p$ must lie to the future of points on CG prior to $p$. Thus, the RNG, which necessarily lies at $t < t_c(r)$, must extend back to $r = 0$ at time $t = t_0$, and so must emerge from the singularity. The contrapositive of this result gives the censorship result mentioned above.

We conclude by emphasising that, whereas the analysis of [2]—wherein the general solution is derived and the singularity is analysed along radial null directions—is correct, that of [1] was incomplete, which led to the incorrect claim that the singularity is always locally naked along outgoing RTG’s, regardless of the initial data. The assertion, in [1], that the singularity is always Tipler strong along RTG’s remains true, and is independent of the visibility. We have shown here that the emergence of outgoing RTG’s from the singularity is dependent on the initial data, and thus the singularity is not always locally naked along RTG’s. In particular, we have shown that initial data that precludes outgoing RNG’s also forbids outgoing RTG’s, in any spherically symmetric spacetime.

Acknowledgments

BCN acknowledges support from the DCU Albert College Fellowship scheme. FCM thanks CMAT, U.Minho and FCT (Portugal) for grant PRAXIS XXI BD/16012/98. SMCGVG acknowledges the support of FCT (Portugal) Grant PRAXIS XXI-BPD-16301-98, and NSF Grants AST-9731698 and PHY-0099568.

[3] If one shows that all outgoing null geodesics (ONG’s) are trapped upon emission—thereby rendering the singularity censored, even to local non-space-like observers—then it is obvious that any outgoing time-like geodesic (OTG), emitted at or after the time at which ONG’s do so will also be trapped, since it must lie inside the chronological future of the latter. The analysis of time-like geodesics by one of us (SG) in [1], was motivated by the fact that (i) in asymptotically flat dust collapse the central singularity is Tipler strong along time-like geodesics, regardless of the initial data, and (ii) the method used in [2] to study the existence of ONG’s provided a sufficient but not necessary criterion, in that non-radial geodesics were not accounted for. However, a recent result by two of us (FM and BN) shows that the absence of outgoing radial null geodesics precludes the existence of any non-radial null geodesics [4]. By the arguments presented here, this forbids the existence of any non-radial causal geodesics.