Operator product expansion and confinement.

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Abstract

Operator product expansion technique is analyzed in abelian and nonabelian field theoretical models with confinement. Special attention is paid to the regimes where nonzero virtuality of vacuum fields is felt by external currents. It is stressed that despite the physics of confinement is sometimes considered as being caused by "soft" fields, it can exhibit the pronounced "hard" effects in OPE.
1 Introduction

Inclusion of nonperturbative contributions [1] (proportional to the gauge-invariant local condensates) in the standard perturbative OPE [2] allowed to formulate a powerful method of QCD sum rules [1] (for reviews see [3, 4, 5]). Nevertheless some questions in the method were formulated by the original authors [6, 8] and still remain unanswered.

In particular, the relation between the property of confinement and structure of the sum rules series has never been clearly established. On the one hand, one could guess that confinement appears as a result of partial summation of some OPE subseries, while, on the other hand, confinement itself might introduce some new unconventional terms in OPE series, with the structure different from the standard form.

The phenomenological implication of such new terms, e.g. \( O(1/Q^2) \), was investigated in [9], where is was related with the short distance nonperturbative physics. The authors of [10, 11, 12, 13] checked the role of confinement for QCD sum rules exploiting nonrelativistic solvable models, and exact results for Green’s functions were compared to the sum rule results.

Especially popular is the example of nonrelativistic particle in oscillator potential, with the Euclidean short-time expansion of Green’s function (in 2d, for detailed discussion see [5], cf. the 3d case in [10])

\[
G_{osc}(T) = \frac{m}{2\pi T} \left( 1 - \frac{(\omega T)^2}{6} + \frac{7}{360}(\omega T)^4 + ... \right)
\]  

(1)

Here the first term comes from the free Green’s function while the next terms play the role of ”condensates” namely identifying Borel mass \( \varepsilon = \frac{1}{T} \), one has typical OPE structures: \( \omega^2/\varepsilon^2 \) and \( \omega^4/\varepsilon^4 \).

The result (1) has widely been used as an argument that confinement (i.e. long distance soft physics) cannot modify the standard OPE and confinement effects should be looked for in the partial sums of the type \( \sum_{n=0}^{\infty} c_n(Q^2)/(D^n F(0)D^n F(0)) \).

In what follows we shall demonstrate explicitly that confinement modifies the standard OPE for relativistic quark Green’s function: new terms appear, which bring unusual power terms in OPE.

It will be shown that the expansion (1) is typical for nonrelativistic potential Green’s functions, while for relativistic particles in the confining fields (or in the confining potential) a specific long-distance instability (divergence) occurs in the perturbative expansion, which could lead to new power terms.

Let us stress from the beginning an important difference between OPE in coordinate and momentum spaces which was discussed already in original papers [1, 7] and which will be seen clearly in what follows. Studying small \( x \)-expansion of a product of two operators \( \langle T\{J(0)J(x)\}\rangle \) when \( x \to 0 \), one observes that in relativistic case (contrary to nonrelativistic one) small value of \( x \) does not confine virtualities of internal lines in the corresponding diagram in any way. In other words, virtual particles created and annihilated by operators \( J \) can travel over large distances in coordinate space whichever small \( x \) is. As a result, in confining theory the product of operators
taken at two nearby points carries information about large-distance behaviour of a theory even if \( x \) is much smaller than typical confinement scale \( \lambda^{-1} \).

To clarify the mechanism of this phenomenon we start in the next chapter with the Green’s function of relativistic quark in the linear confining potential of static antiquark, corresponding to the Dirac equation with scalar linear potential. We shall expand Green’s function in powers of string tension (or equivalently in powers of Euclidean time \( T \)) and find explicitly a new dominant term at small \( T \), and estimate other terms. Comparison with the corresponding nonrelativistic Green’s function is done and demonstrates that no unusual terms appear in the latter case, the expansion being essentially of the same type as in (1). The reason for that is traced to the structure of the nonrelativistic free Green’s function, for which spacial deflection of particle \( \Delta x \) is limited by the time elapsed \( \Delta t, \Delta x \sim \sqrt{\Delta t} \).

Situation is different however in momentum space. Large external momentum \( Q \) plays a role of infrared cut-off and if it is much greater than particle mass \( m \) and nonperturbative scale \( \lambda \), one can successfully perform systematic expansions over \( m^2/Q^2 \) and \( \lambda^2/Q^2 \). This is how the standard OPE technique works. Nevertheless the remaining problem in this case is to determine the structure of the latter, nonperturbative subseries. The problem here is that in real QCD there are several different nonperturbative scales. The best known are given by nonperturbative quark and gluon [1] condensates \( \langle \bar{\psi} \psi \rangle, \langle F_{\mu\nu} F^{\mu\nu} \rangle \). One can include in analysis higher irreducible condensates as well. Another important scale is given by condensate virtualities, see expressions (21), (22) below. So even remaining in the standard OPE framework, one can set oneself the task of summation of different subseries in full \( \lambda^2/Q^2 \)-expansion. It will be seen below how this problem is solved in particular cases.

Moreover we present a few examples in section 6 where OPE in momentum space starts from the terms, which nontrivially account for (monopole) condensate virtuality and hence would be considered as subleading in conventional expansion.

The field-theoretical models are discusses in section 3, where the QCD equations for the heavy-light system obtained in the limit of large \( N_c \) in [14] are discussed.

It is shown, in particular, basing on the subsequent results in [15],[16], that exact equations have a nonlinear kernel, which at large spacial distances reduces to the linear confining term \( \sigma|\vec{r}| \), and hence the expansion of the Green’s function reduces to the potential example considered in section 2.

We briefly consider abelian models with confinement in section 6 such as QED with monopoles and Abelian Higgs model and study influence of confinement on short-time behaviour of Green’s functions. We also discuss various approaches related to OPE such as Feynman-Schwinger proper time method (section 5) and spectral representations of Green’s functions (section 7) and study interplay between confinement and OPE in these frameworks. Finally we present short conclusion and outlook.
2 Relativistic Green’s function of a confined quark

We study in this chapter Green’s function of the Dirac equation in the Euclidean space-time

\[-i(\hat{\partial} + m + \sigma |\vec{x}|)S(x, y) = \delta^{(4)}(x - y).\]  (2)

In what follows we shall study the function \(S(\vec{x} = 0; x_4 = 0; \vec{y} = 0, y_4 = T) \equiv S(T)\) as a function of \(T\), at small values of \(T\).

The free Green’s function \(S_0(x - y)\) can be written as

\[S_0(x) = \int \frac{d^4p}{(2\pi)^4} \exp(ipx) (\gamma p + im) = i(m - \hat{\partial})(0| (m^2 - \partial^2)^{-1}|x) =
\]

\[= i(m - \hat{\partial}) \frac{m}{4\pi^2} K_1(mx) = i \left( m - \frac{\hat{x} \cdot \partial}{x \partial x} \right) \frac{m}{4\pi^2} K_1(mx)\]  (3)

where \(x = \sqrt{\vec{x}^2 + x_4^2}\) and \(K_1\) is the McDonald function. In the massless limit one obtains

\[S_0(x) \rightarrow \frac{i\hat{x}}{2\pi^2 x^4}.\]  (4)

In the first order in \(\sigma\) one obtains in the massless limit

\[S(0, T) = S_0(0, T) + i \int d^4x S_0(0, x)\sigma |\vec{x}| S_0(x, T) + \ldots \equiv S_0(0, T) + S_1(0, T) + \ldots\]  (5)

where function \(S_1\) can be written in the massless limit as

\[S_1(0, T) = \frac{i\sigma}{(2\pi^2)^2} \int \frac{\hat{x} \cdot \vec{x}}{x^4 |\vec{x}| (x - T)^4} d^4x\]  (6)

Integration in (6) yields

\[S_1(0, T) = \frac{i\sigma}{8\pi T}\]  (7)

Consider now the higher-order terms in the expansion (5). The typical \(O(\sigma^n)\) term looks like

\[S_n(0, T) = i^n \int d^4x_1 \ldots d^4x_n |\vec{x}_1| S_0 \ldots \sigma |\vec{x}_n| S_0\]  (8)

It is easy to see that the integrals are infrared divergent at large \(|\vec{x}|\) starting from the term with \(n = 2\), however for \(m \neq 0\) this divergence is eliminated and integrals are cut-off by the mass at \(x \sim \frac{1}{m}\). Therefore typical \(S_n(0, T)\) has the form for \(n > 2\)

\[S_n(0, T) \sim \left( \frac{\sigma}{m^2} \right)^n m^3\]  (9)

while the \(n = 1\) term obtains corrections of the form

\[S_1(0, T) = \frac{i\sigma}{8\pi T}(m TK_1(mT) + O(mT)).\]  (10)
It is instructive to compare (4), (7), (9) with the nonrelativistic expansion (1). One can see that apart from difference in free Green’s functions, the first dynamical term is nonsingular in the nonrelativistic case (1), \( G_{osc}^{nr} = -\frac{m_\omega^2 T}{12\pi} \), while it is singular in relativistic case (7) if \( T \to 0 \).

To clarify the origin of this difference one can compute \( S_1(0,T) \) for nonrelativistic Green’s function with linear potential. Note, that free Green’s function in 3d is

\[
G_{nr}^{0}(\vec{x}_1,t_1;\vec{x}_2,t_2) = \left( \frac{m}{2\pi(t_2 - t_1)} \right)^{3/2} \exp \left( -\frac{m(\vec{x}_2 - \vec{x}_1)^2}{2(t_2 - t_1)} \right)
\]  

(11)
a calculation similar to (6) immediately yields

\[
G_{nr}^{1r}(0,T) = \frac{\sigma m}{8\pi}
\]  

(12)
which is nonsingular at small \( T \) in contrast to \( S_1(0,T) \) in (7). It is easy to see that also all higher terms in \( \sigma^n \) are nonsingular due to the specific feature of nonrelativistic Green’s function (11): all time intervals are ordered \((t_n > t_{n-1} > t_{n-2})\) and all space intervals are cut-off by the time intervals and the mass, so that quark cannot escape far away during a short time interval – in contrast to the relativistic case, when a light quark can travel as far as \( \frac{1}{m} \gg T \) for however small \( T \). Thus crucial difference between nonrelativistic and relativistic dynamics causes the different behaviour of the Green’s functions at small distances/times.

### 3 Relativistic equation for the heavy-light system

In this chapter we shall discuss the situation for the field-theoretical model, namely for the two-body system made of a spinor particle with the mass \( m \) and heavy scalar antiparticle whose mass is considered as infinite. We assume that this "meson" interact with confining gauge-field background, which is characterized by the Gaussian field strength correlator (see review [41] and references therein)

\[
\Delta^{(2)}_{\mu_1\nu_1,\mu_2\nu_2} = \langle tr_c(F_{\mu_1\nu_1}(z_1)\Phi(z_1,z_2)F_{\mu_2\nu_2}(z_2)\Phi(z_2,z_1)) \rangle =
\]

\[
= \frac{1}{2} \left( \frac{\partial}{\partial z_{\mu_1}} (z_{\mu_2}\delta_{\nu_1\nu_2} - z_{\nu_2}\delta_{\mu_1\mu_2}) + \frac{\partial}{\partial z_{\nu_1}} (z_{\nu_2}\delta_{\mu_1\mu_2} - z_{\mu_2}\delta_{\nu_1\nu_2}) \right) D_1(z_1 - z_2) +
\]

\[
+ (\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2} - \delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1}) D(z_1 - z_2)
\]  

(13)
where \( \Phi(x,y) \) stays for the phase factor

\[
\Phi(x,y) = \text{Pexp} \left( i \int_x^y A_\mu(u)du_\mu \right)
\]  

(14)
The Green’s function of such system can be represented as follows

\[
\langle \bar{\psi}(x)\Phi(x,y)\psi(y) \rangle = S_0(x,y) + S_2(x,y) + ..
\]  

(15)
where $S_0$ is given by (3) while the first nontrivial interaction term has the following form

$$\text{Tr}S_2(x, y) = \left\langle \int d^4u \int d^4w \text{Tr} \left( S_0(x, u)i\hat{A}(u)S_0(u, w)i\hat{A}(w)S_0(w, y) \right) \right\rangle \quad (16)$$

where $\text{Tr} = tr_c tr_L$ is a product of traces over color and Lorentz indices. We adopt the Fock-Schwinger gauge condition with the base point $x_0 = x$: $A^a_\mu(w - x_0)_\mu = 0$. In this gauge the Green’s function of the heavy particle is proportional to unity while the gauge field propagator takes the form

$$\langle \text{tr}A_\mu(u)A_\nu(w) \rangle = D(0) \cdot [(u - x)(w - x)\delta_{\mu\nu} - (u - x)_\nu(w - x)_\mu] \cdot f(u, w) \quad (17)$$

where dimensionless function $f(u, w)$ is given by the following expression

$$f(u, w) = \frac{1}{D(0)} \int_0^1 d\alpha \int_0^1 d\beta D(\alpha u - \beta w) \quad (18)$$

Functions of this kind are often used in the formalism of coordinate gauges, one can find in Appendix of the present paper detailed analysis of $f(u, w)$ for particular choice of Gaussian ansatz $D(z) = D(0) \exp(-z^2 / T_g^2)$. We shall keep only the function $D(z)$ in what follows since the function $D_1(z)$ is not responsible for confinement effects. It was also found on the lattice that nonperturbative part of $D_1(z)$ is significantly smaller than that of $D(z)$ in QCD, see [41] and references therein.

In momentum space (16) takes the form:

$$\text{Tr}S_2(x, y) = 4im \int \frac{d^4l}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \int_0^1 d\beta \exp(il(x - y)) \cdot \frac{D(k^2)}{l^2 + m^2} \cdot \left[ \frac{\partial}{\partial r_{\rho}} \frac{\partial}{\partial s_{\sigma}} \right]_{r = k, s = k, \beta} \left\{ \frac{1}{(l - s + r)^2 + m^2} \frac{1}{(l - s)^2 + m^2} \right\} \cdot (\delta_{\rho\sigma} (3m^2 - l^2 - 2lr + sr + s^2) + 4l_{\rho}l_{\sigma} - 4l_{\rho}s_{\sigma} + 2l_{\rho}r_{\sigma} - 2l_{\sigma}s_{\rho} + 2s_{\rho}s_{\sigma} - r_{\sigma}s_{\rho} - r_{\rho}s_{\sigma}) \right\} \quad (19)$$

The properties of the expression (19) are determined by the interplay of external parameters such as particle mass $m$ and distance $|x - y|$ and properties of the confining background encoded in the function $D(z)$. In case of QCD the latter is usually found on the lattice [45]. It decays with distance and has some typical correlation length scale which we denote as $T_g$ throughout the paper. The exact dependence of $D(z)$ on $z$ is of no principal importance, one usually takes exponential fits (see [41]). At the origin $D(z)$ is normalized to the nonperturbative gluon condensate, according to

$$D(0) = \frac{1}{12} \langle \text{tr}c F_{\mu\nu}F_{\mu\nu} \rangle$$

It is worth mentioning that the actual numerical value of $T_g$ in gluodynamics and QCD is rather small: it is estimated as 0.22 Fm for quenched $SU(3)$ and as 0.34 Fm for full QCD with four flavours [45, 43, 41]. As it will be clear from what follows this circumstance bounds region of applicability of conventional OPE based on local condensates.
We study first the heavy quark case, i.e. we assume that \( m_T \gg 1 \). The integrals in (16), (19) are saturated at momenta \( l^2 \) of the order of the mass \( m^2 \) and one can make systematic expansion over \( 1/m_T \). Straightforward although rather lengthy calculation leads to the following answer for the heavy quark condensate:

\[
\text{Tr} S_2(x, x) = \frac{-i(\text{tr}_c F_{\mu \nu} F_{\mu \nu})}{24\pi^2 m} \left[ 1 - \frac{44}{45} \frac{1}{m^2 T_g^2} + O \left( \frac{1}{m^4 T_g^4} \right) \right]
\]  

where \( T_g \) is defined as

\[
\frac{1}{T_g^2} = \frac{1}{4D(0)} \int \frac{d^4 k}{(2\pi^4)} D(k^2) k^2 = \frac{\langle \text{tr}_c (F D^2 F) \rangle}{\langle \text{tr}_c F^2 \rangle}
\]

where the last relation is valid in Gaussian approximation when all contributions from higher correlators are neglected. Let us mention that virtuality of quark condensate usually measured by the quantity

\[
\lambda_q^2 = \frac{\langle \bar{\psi} D^2 \psi \rangle}{\langle \bar{\psi} \psi \rangle}
\]

in the sum rule approach is comparable with that of the gluon condensate (21), indeed, \( \lambda_q^2 = 0.4 \pm 0.1 \text{GeV}^2 \) according to [49], while \( T_g \) was found on the lattice to be \( 0.34 \pm 0.02 \text{Fm} \) in \( SU(3) \) with 4 dynamical flavours [43], i.e. \( \lambda_q T_g \) is of the order of one. It could be instructive therefore to reconcile our approach with the method of nonlocal quark condensates worked out in [37, 38].

For \( D(z) \propto \exp(-z^2/T_g^2) \) with the correlation length \( T_g \) (as is used in Appendix) one has \( T_g = \sqrt{2} T_g \). The first term in the expansion (20) is well known OPE–result for the heavy quark condensate [48], see also [47]. The second term is the first nonlocal correction. It is worth mentioning that due to the smallness of \( T_g \) (see above) it can be omitted as numerically small correction for \( b, t \) quarks only, while for \( s, c \) quarks keeping only the first term in the expansion in \( 1/mT_g \) is not to be considered as good approximation.

Equations of the form (21), (22) account for nonzero virtuality of vacuum lines in standard OPE language – one considers quantum averages, which contain derivatives. As we shall see in what follows, this language is not universal and implicitly assumes small averaged virtuality corresponding to the vacuum state, i.e. large \( T_g \) limit. Another essential ingredient of this language is the use of equations of motion for such averages. Although it is rather easy to justify the validity of this component of the approach in abelian case, to the best of author’s knowledge, this procedure has never been proved for nonabelian theories with the level of rigour adopted in the field. Since we are discussing nonlocal correlators, the following remark is of importance. Consider parallel transported field strength tensor \( F_{\mu \nu} \), i.e.

\[
G_{\mu \nu}(x, x_0) = \Phi(x_0, x) F_{\mu \nu}(x) \Phi(x, x_0)
\]

and nonlocal gauge-invariant two-point correlator

\[
\langle \text{tr}_c G(x, x_0) G(y, x_0) \rangle
\]

The above correlator depends on the positions of the points \( x, y, x_0 \) and on profiles of the contours used in factors \( \Phi \). However, if \( x \to y \) all these dependences disappear (phase factors
cancel each other, while \( x \)-dependence is prohibited by translational invariance) and the resulting local average coincide with \((\text{tr}_cF^2)\). Let us consider now expansion of (23) if \(|x-y|\) is small. In principle one might consider two different expansions, with correlators involving derivatives in both cases. In the first case it reads:

\[
\langle \text{tr}_cG_{\mu\nu}(x_0)G_{\rho\sigma}(y, x_0) \rangle \approx \langle \text{tr}_cF^2 \rangle + (y-x)_\alpha \cdot \left\langle \text{tr}_cG_{\mu\nu}(x, x_0) \left[ \frac{\partial G_{\rho\sigma}(y, x_0)}{\partial y_\alpha} \right] \right|_{y=x} \right\rangle + \ldots \tag{24}
\]

where the derivative is given by

\[
\frac{\partial G_{\rho\sigma}(y, x_0)}{\partial y_\alpha} = \Phi(x_0, y) \left( D_\alpha F_{\rho\sigma}(y) + i(y-x)_\beta \int_0^1 sds [G_{\beta\alpha}(z, y), F_{\rho\sigma}(y)] \right) \Phi(y, x_0) \tag{25}
\]

and \([\ldots\ldots]\) in (25) denotes commutator with respect to the color indices. The second term (and all higher terms) in the r.h.s. of (24) contains nonlocal part and depends on contour profiles and on the position of the points \(x, x_0\) unless \(x = y = x_0\). On the other hand the expansion goes in powers of the quantity \((y-x)\) which is assumed to be small.

In the second case one expands each \(G\) in (23) in the vicinity of the point \(x_0\):

\[
\langle \text{tr}_cG_{\mu\nu}(x_0)G_{\rho\sigma}(y, x_0) \rangle \approx \langle \text{tr}_c(F_{\mu\nu}(x_0) + (x-x)_\alpha D_\alpha F_{\mu\nu}(x_0) + \ldots)(F_{\rho\sigma}(x_0) + (y-x)_\alpha D_\alpha F_{\rho\sigma}(x_0) + \ldots) \rangle \approx \langle \text{tr}_cF^2 \rangle + (y-x)_\alpha \cdot (\text{tr}_cF_{\mu\nu}(x_0)D_\alpha F_{\rho\sigma}(x_0)) + (x-x)_\alpha \cdot (\text{tr}_cD_\alpha F_{\mu\nu}(x_0)F_{\rho\sigma}(x_0)) + \ldots \tag{26}
\]

This is an expansion adopted in conventional OPE. In contrast with (24) nonlocal parts are absent, the price to pay however is that the expansion goes in \(x-x_0\), \(y-x_0\) instead of \(y-x\). Needless to say that in many physical applications \(|y-x|\) can be small whereas \(|x-x_0|\) and \(|y-x_0|\) are very large. Notice also that opposite situation is impossible: smallness of \(|x-x_0|, |y-x_0|\) implies smallness of \(|y-x|\).

After this rather academic discussion we come back to the limit of small quark mass and/or correlation length \(mT_g \ll 1\), which is opposite to what has been explored in (20). As it was already mentioned, in real QCD parameter \(\langle \text{tr}_cF^2 \rangle T_g^4\) can be considered as small, even in presence of dynamical quarks. In particular, typical momenta \(l^2\) in (19) can be rather large in comparison with nonperturbative scale given by the condensate \(\sqrt{\langle \text{tr}_cF^2 \rangle}\) but still small when compared to nonlocality scale \(~ T_g^{-2}\). Test particle resolves nonlocality of vacuum field correlations in this regime.

The Green’s function \(S_2(x, y)\) we are interested in is defined in (16). We are working in coordinate representation here and choose the Fock-Schwinger gauge reference point \(x_0\) at the origin \(x_0 = x = 0\). We rewrite \(S_2(x = 0, y)\) using (17) as

\[
S_2(0, y) = \frac{iD(0)}{64\pi^6} \int d^4u \int d^4w \left[ \left( \frac{m}{u^2 - 2 \frac{\hat{u}}{u^4}} \right) \cdot (4(uw) - \hat{u}\hat{w}) \cdot f(u, w) \cdot \left( 2 \frac{\hat{u} - \hat{w}}{(u - w)^4} + \frac{m}{(u-w)^2} \right) \right] \tag{27}
\]
where we have kept only linear in mass $m$ terms in propagators since we consider small mass limit. The kernel $f(u, w)$ is defined in (18) and $\hat{u} = u_{\mu} \gamma_{\mu}$.

The actual value of this integral is defined by the properties of the function $f(u, w)$ which encodes all nonperturbative dynamics in the chosen Gaussian approximation. They are rather peculiar however (see Appendix) and this circumstance precludes one to obtain exact analytic answer. On the other hand, (27) can be calculated numerically for any particular ansatz for $D(x)$. Let us investigate general structure of $S_2$. In massless limit one immediately obtains

$$\lim_{y \to 0} S_2(0, y) = 0$$

due to absence of chiral symmetry breaking in the problem in question. It is seen that $S_2$ is UV-finite (small $u, w$ domain) because nonperturbative background is soft:

$$\lim_{u, w \to 0} f(u, w) = 1/4.$$ 

In infrared domain $|u|, |w| \gg T_g$ the integral is convergent due to the properties of $f(u, w)$ (see Appendix). One obtains in massless case the following leading term at small $|y|$:

$$S_2(0, y) = -ic \cdot D(0) \cdot \hat{y} + \mathcal{O}(y^2)$$

Numerical constant $c$ is determined by the function $f(u, w)$, but is $T_g$-independent. The massive parts of $S_2$ provide finite contribution at $y = 0$:

$$S_2(0, y) \sim imD(0)T_g^2$$

If mass is increasing and reaches values of the order of $T_g^{-1}$, it begins to work as IR cutoff instead of $T_g$ and one comes back to (20). However if mass is small then light quarks essentially feel the virtuality distribution of vacuum gluon fields (i.e. the profile of $f(u, w)$).

It is instructive to show how the potential problem considered in section 2 appears from field-theoretical framework invoking by us here. To this end one is to consider equation for heavy-light system which was obtained from the QCD Lagrangian in [14] in the limit of large $N_c$. Keeping only the Gaussian field correlator one has instead of (2) the equation for the quark Green’s function (made gauge-invariant due to phase factor coming from the heavy source propagator)

$$-i(\hat{\partial} + m)S(x, y) - i \int M(x, z) d^4 z S(z, y) = \delta^{(4)}(x - y)$$

where the nonlocal kernel $M(x, z)$ depends on the exact Green’s function $S(x, z)$, making Eq. (28) nonlinear. Till the end of this section we are working in the so called modified Fock-Schwinger gauge (see all details in [20]) where the temporal axis is singled out. We have retained for simplicity only color electric part of the correlator as defined in [21] $\langle E_i(x) E_k(z) \rangle \sim \delta_{ik} D(x - z)$. Assuming for $D(x - z)$ Gaussian ansatz, one arrives at the following form of nonlocal kernel $M(x, y)$,

$$M(x, y) = D(0)(\vec{x}\vec{y})f(\vec{x}, \vec{y})S(x, y) \exp\left(\frac{-(x_4 - y_4)^2}{T_g^2}\right)$$

where $f(\vec{x}, \vec{y})$ is given in Appendix. Notice that $\vec{x}, \vec{y}$ are three-dimensional vectors here and not four-dimensional as in (27).

As it was shown in [15] using the relativistic WKB method developed in [14], the function $S(x, y)$ at large $\vec{x}, \vec{y}$, i.e. if $|\vec{x}|, |\vec{y}| \gg T_g$ can be written in the following form

$$S(h, \vec{x}, \vec{y}) = ie^{-\sigma|\vec{x}| + m}g(\vec{x}, \vec{y})\begin{pmatrix} \theta(h) \\ \theta(-h) \end{pmatrix}$$

(30)
where \( h \equiv x_4 - y_4 \) and \( g(x, y) \) is a smeared \( \delta \)-function

\[
g(\vec{x}, \vec{y}) = \frac{S^{(3)}(\vec{x} - \vec{y})}{\sigma x^2} \sim \sigma y^2 \gg 1, \tag{31}
\]
mOREover for large \( \vec{x} \) and \( \vec{y} \), and \( |\vec{x} - \vec{y}| \ll |\vec{x}| \sim |\vec{y}| \) (see [14] and also Appendix of the present paper).

\[
f(\vec{x}, \vec{y}) \sim \frac{T_g}{|\vec{f}|} \tag{32}
\]
and in this region one can integrate in (28) over \( d^4z \), since

\[
M(x, z) \cong \sigma |\vec{x}| \delta^{(4)}(x - z) \tag{33}
\]
where the string tension \( \sigma = (\pi/2)D(0)/T^2_g \) for Gaussian ansatz (see (62) below). Thus at large spacial arguments the kernel \( M \) coincides with linear potential considered in the previous chapter. Therefore all estimates for terms in the expansion proportional to \( M^n, n \geq 2 \) are in agreement with those for the local case, eq. (9), since integrals in these terms are essentially saturated by large spacial distances, \( |\vec{x}| \gg T_g \).

Although potential behaviour (33) is typical for large– \( T \) - regime, it is instructive to show how the nonlocality cures \( 1/T \) behaviour found in local potential problem. We shall demonstrate now that Green's functions in question have finite limit when \( T \to 0 \) either for small or for large \( T_g \). Let us briefly analyse the situation with the nonlocal equivalent of (6), i.e.

\[
S^{(M)}_1(0, T) = \frac{iD(0)}{(2\pi)^2} \int d^4x \int d^4y \frac{\tilde{x}(\vec{y} - T)(\vec{x}^2) f(\vec{x}, \vec{y}) S(h, \vec{x}, \vec{y})}{x^4(y - T)^4} \exp \left( -\frac{\hbar^2}{T_g^2} \right) \tag{34}
\]
with \( S \) given in (30). It is convenient to introduce dimensionless quantities

\[
\tilde{x}_\mu = x_\mu \sqrt{\sigma}, \quad \tilde{y}_\mu = y_\mu \sqrt{\sigma}, \quad \tilde{T}_g = T_g \sqrt{\sigma}, \quad \tilde{T} = T \sqrt{\sigma}. \tag{35}
\]
Rewriting (34) in terms of tilde variables, one immediately realizes, that \( S \) tends to 3-dimensional delta-function only in the limit when \( |\tilde{x}|, |\tilde{y}| \gg 1 \) and otherwise the integral is defined by the region \( |\tilde{x}| \sim |\tilde{y}| \sim 1 \), when the nonlocality is at work, i.e. \( |\tilde{x} - \tilde{y}| \sim |\tilde{x}|, |\tilde{y}| \). Imposing the limit of small \( T_g \), i.e. \( \tilde{T}_g \ll 1 \) one reduces two powers of \( \tilde{x}, \tilde{y} \) in the numerator of (34), but the integral is still defined by large values of \( \tilde{x}, \tilde{y} \) of the order of unity and one finally obtains

\[
S^{(M)}_1(0, T) \sim \text{const} \cdot \sigma^{3/2} \tag{36}
\]
One can also show that the same estimate holds true also for higher terms \( O(m^n) \) and write

\[
S^{(M)}_n(0, T) \sim c_n \sigma^{3/2}. \tag{37}
\]
Consider now the opposite limit \( \tilde{T}_g \gg 1 \), i.e. \( T_g \gg 1/\sqrt{\sigma} \). In this case \( \tilde{T}_g \) does not confine the differences \( \tilde{x} - \tilde{y} \) in \( f(\tilde{x}, \tilde{y}) \) and \( \tilde{x}_4 - \tilde{y}_4 \) in the exponent in (34) to small values as compared to \( |\tilde{x}|, |\tilde{x}_4| \) or \( |\tilde{y}|, |\tilde{y}_4| \). Therefore the integration over \( d(\tilde{x}_4 - \tilde{y}_4) \) is limited only by the exponent in (30). As a result one obtains for \( S_1 \) the following estimate (as always, we assume mass \( m \) to be not large, \( m \ll \sqrt{\sigma} \))

\[
S^{(M)}_1(0, T) \sim \text{const} \cdot \frac{\sqrt{\sigma}}{T_g^2} \tag{37}
\]
Thus in both cases the normal procedure of OPE, based on the analysis of subsequent terms of perturbative expansion (with separating soft and hard parts of diagrams) is not applicable and one should sum up the whole series or else solve the nonlinear equation (28) exactly.
4 Another field-theoretical example: how the linear confinement is built up out of condensates

In this chapter we consider another example: a scalar (Higgs-type) particle interaction with the background Yang-Mills field, and again calculate the heavy-light Green’s function, where light particle is the color fundamental Higgs, while heavy source is like an antiquark. The Lagrangian and the Green’s function are given by

\[ L = \frac{1}{4} (F_{\mu \nu})^2 + |D_\mu \varphi|^2 - \frac{m^2 |\varphi|^2}{2}, \quad G^{(\varphi)}(0, T) = \langle \bar{\varphi}(0) \Phi(0, T) \varphi(T) \rangle \]  

(38)

One can rewrite \( G^{(\varphi)} \) as

\[ G^{(\varphi)}(0, T) = \langle (m^2 - D_\mu^2)^{-1} \Phi(0, T) \rangle_B \]  

(39)

where we have introduced the external (background) field \( B_\mu: D_\mu = \partial_\mu - ig B_\mu \). For simplicity of consideration we shall confine ourselves to the vertices

\[ L_4 \equiv g^2 (B^{ab}_\mu \varphi^b)(B^{ac}_\mu \varphi^c) \]  

(40)

and choose the gauge [20] to write equation for \( G^{(\varphi)}(x, y) \) analogous to (28),

\[ (m^2 - \partial_\mu^2)G^{(\varphi)}(x, y) + \int I(x, z)G^{(\varphi)}(z, y)d^4z = \delta^{(4)}(x - y) \]  

(41)

where \( I(x, z) = I^{(1)}(x, z) + I^{(2)}(x, z) \), and \( I^{(1)} \) refers to the kernel with one power of \( B_\mu \), while \( I^{(2)} \) corresponds to the Lagrangian (40) and can be written as

\[ I^{(2)}(x, y) = \delta^{(4)}(x - y)g^2 B^2_\mu(x) \]  

(42)

The contributions from \( I^{(1)} \) not analyzed by us here are of the same general structure as that of \( I^{(2)} \) apart from nonlocality controlled by the particle mass \( m \). Our consideration in this section is of illustrative purpose only.

Let us take now the first order graph in \( g^2 \) (keeping only \( I^{(2)} \) in (41)). In Euclidean space-time one has

\[ G^{(\varphi)}_1(0, T) = g^2 \left\{ \int d^4x \ G^{(\varphi)}_0(0, x)B^2_\mu(x)G^{(\varphi)}_0(x, T) \right\}_B \]  

(43)

where

\[ G^{(\varphi)}_0(x) = \frac{m}{4\pi^2 x} K_1(mx), \quad x = \sqrt{x^2}. \]  

(44)

We shall be interested in the vacuum averaged expression for \( G^{(\varphi)}_1 \) and to this end one should express \( B^2_\mu(x) \) in terms of field correlators (one way) or in terms of condensates (another – standard way). In the gauge [20] one can write e.g. for \( B^2_\mu \):

\[ \langle B^2_\mu(x) \rangle = \int_0^x du \int_0^x dv \langle F_{\mu 4}(u) \Phi(u, v) F_{\nu 4}(v) \rangle \]  

(45)

and using [21] and keeping as in section 3 only the confining part \( D \), one has

\[ \langle B^2_\mu(x) \rangle = \int_0^x du \int_0^x dv D(u - v), \quad i = 1, 2, 3. \]  

(46)
By way of example let us consider exponential ansatz for \(D(u - v)\). As it was already said, this behaviour of \(D(x)\) was observed in lattice simulations at distances larger than some typical correlation length \(T_g\). So one has

\[
\langle B_1^2(x) \rangle = D(0) \bar{x}^2 \int_0^1 d\alpha \int_0^1 d\beta \exp \left( \frac{|\bar{x}|}{T_g} \cdot |\alpha - \beta| \right)
\]  

(47)

Notice absence of additional multipliers \(\alpha, \beta\) in (47) contrary to (18), it is a property of modified Fock-Schwinger gauge [20] (temporal axis is singled out) used by us in this section, instead of usual one, used in (18) (one point is singled out).

Straightforward integration leads to the following result:

\[
\langle B_1^2(x) \rangle = 2D(0)|\bar{x}|T_g \left( 1 - \frac{T_g}{|\bar{x}|} \cdot \left[ 1 - \exp \left( -\frac{|\bar{x}|}{T_g} \right) \right] \right)
\]  

(48)

At large \(|\bar{x}| \gg T_g\), one has from (48):

\[
\langle B_1^2(x) \rangle \approx 2D(0)T_g(|\bar{x}| - T_g)
\]  

(49)

Notice that nonlocality enters (49) in explicit way. At small \(|\bar{x}|\), when \(|\bar{x}|/T_g \sim 1\) the linear behaviour of (49) is replaced by the quadratic one

\[
\langle B_1^2(x) \rangle \approx D(0) \cdot \bar{x}^2
\]  

(50)

It is clear that vacuum field correlator method implemented in (45)-(49) demonstrates the creation of the string between the Higgs particle at the point \((\bar{x}, x_4)\) and static source at the point \((0, x_4)\).

Now let us look at the same problem from the point of view of standard OPE. According to general rules [1], [3]-[5], one should expand \(F_{\mu 4}(u), F_{\nu 4}(v)\) in (45) in the vicinity of a point \(\bar{u} = \bar{v} = 0, u_4 = v_4 = x_4\) (in the Fock-Schwinger gauge that would be the point \(z_0\) usually chosen at the origin, \(z_0 = 0\)). In this way one obtains

\[
\langle B_1^2(x) \rangle = \sum_{n,m} \frac{1}{(n+1)!(m+1)!} x_{i_1}x_{i_2}\ldots x_{i_n}x_{k_1}\ldots x_{k_m} \langle D_{i_1}\ldots D_{i_n}F_{4i}(0)D_{k_1}\ldots D_{k_m}F_{4k}(0) \rangle
\]  

(51)

In this form (51) the appearance of the string is not visible, and one should rearrange the derivatives in nontrivial way, so as to separate out the correlator \(D(u - v)\) as in (46). Derivatives of the latter produce powers of \(T_g^{-1}\), while dependence on the sum \(\frac{1}{2}(u_i + v_i)\) in the integral (46) is separated out to yield linear confinement in (49).

Now we consider the expansion of \(G^{(\varphi)}\) in powers of \(g^2\). From (49) and (43), (44) it is clear that one obtains

\[
G_1^{(\varphi)}(0, T) \sim g^2 \frac{c_4}{m}, \quad G_n^{(\varphi)} \sim g^{2n} \left( \frac{c_4}{m^3} \right)^n m^2
\]  

(52)

where nonlocal constant \(c_4 \sim D(0)T_g \sim \int d\bar{z}D(\bar{z})\). All integrals like (43) are diverging at large distances for \(m = 0\) and cut-off at \(x \sim \frac{1}{m}\) when \(m \neq 0\).
It is instructive to turn to the momentum space and define the following one-dimensional Fourier-transformed Green’s function:

\[ G^{(\phi)}_{1}(Q) = \int_{-\infty}^{\infty} dT G^{(\phi)}_{1}(0, T) \exp(-iQT) \]  

(Our problem is 3+1 dimensional, we do not perform 4-dimensional transformation however since the temporal axis was separated by our gauge choice from the beginning). Since \( \langle B^{2}_{4}(x) \rangle \) does not depend on the temporal coordinate \( x_{4} \), integration in (43) is trivial:

\[ G^{(\phi)}_{1}(M) = \frac{g^{2}}{16\pi} \frac{D(0)}{M^{3}} \cdot \frac{2MT_{g}}{1 + 2MT_{g}} \]  

where \( M^{2} = Q^{2} + m^{2} \). Expression (54) has the following asymptotic expansions:

\[ G^{(\phi)}_{1}(M) = \frac{g^{2}}{16\pi} \frac{D(0)}{M^{3}} \cdot \left(1 - \frac{1}{2MT_{g}} + \frac{1}{4M^{2}T^{2}_{g}} + \mathcal{O}\left(\frac{1}{M^{3}T^{3}_{g}}\right)\right), \quad MT_{g} \gtrsim 1 \]  

and

\[ G^{(\phi)}_{1}(M) = \frac{g^{2}}{8\pi} \frac{D(0)T_{g}}{M^{2}} \cdot \left(1 - 2MT_{g} + 4M^{2}T^{2}_{g} + \mathcal{O}(M^{3}T^{3}_{g})\right), \quad MT_{g} \lesssim 1 \]  

It is clearly seen that the actual answer is given by different series in regions \( QT_{g} \ll 1 \) and \( QT_{g} \gg 1 \) (we assume that \( Q \gg m \) and \( M \approx Q \)). The expansion (55) is associated with the standard OPE (51), while (56) goes essentially in powers of nonlocal quantity. It is also worth mentioning that both expansions (55) and (56) are model dependent beyond the leading condensate term and actual coefficients in (55), (56) are determined by the profile of \( D(z) \).

5 Feynman-Schwinger formalism and OPE

We start with the same Green’s function \( G^{(\phi)}(0, T) \) and write the Feynman-Schwinger representation (FSR) for it (see [22],[23] where refs. to earlier papers are given, for more discussion see [24]).

\[ G^{(\phi)}(0, T) = \int_{0}^{\infty} ds \int (Dz)_{0,T} \exp(-K) \langle W(C_{z}) \rangle. \]  

Here \( K = m^{2}s + \frac{1}{2} \int_{0}^{s} \frac{\Delta z^{2}}{\mu^{2}} d\tau \), \( s \) is Schwinger proper time and \( \langle W(C_{z}) \rangle \) is a Wilson loop consisting of a straight line \((0, T)\) and the trajectory of the Higgs particle from 0 to \( T \). Notice also, that

\[ (Dz)_{0T} = \prod_{n=1}^{n} \frac{d^{4}\Delta z(n)}{(4\pi \varepsilon)^{2}} \int \frac{d^{4}p}{(2\pi)^{4}} \exp \left(i p \left(\sum_{n} \Delta z(n) - T\right)\right) \]  

In Gaussian approximation \( \langle W(C_{z}) \rangle \) can be written as

\[ \langle W(C_{z}) \rangle = \exp \left(-\frac{g^{2}}{2} \int_{S} d\sigma_{\mu\nu}(u) \int_{S} d\sigma_{\rho\lambda}(v) \langle F_{\mu\nu}(u) F_{\rho\lambda}(v) \rangle \right) \]  

and we have omitted for simplicity the parallel transporters inside \( \langle FF \rangle \). Here \( S \) is the prescribed minimal area surface in the loop \( C_{z} \) (there is no sensitivity on the choice of \( S \) when
all higher cumulants are kept; with the choice of the Gaussian correlators and minimal surface, the contribution of all higher correlators was estimated to be around few percents, see [41] and references therein).

For small contour $C_z$ (which means that not only $T$ is small but also spatial size of the contour is small), one has from (59) [21]

$$\langle W(C_z) \rangle = \exp \left( -\frac{g^2 S^2}{24N_c} \langle F^a_{\mu\nu}(0) F^a_{\mu\nu}(0) \rangle \right)$$  \hspace{1cm} (60)

For a rectangular contour $C_z$ of an arbitrary size $R \times T$ one can write

$$\langle W(C) \rangle = \exp \left( -\frac{1}{2} \int d^2 x \int S d^2 y D(x - y) \right)$$  \hspace{1cm} (61)

Choosing for simplicity $D(z) = D(0) \exp(-z^2/T_g^2)$, one has

$$\sigma = \frac{1}{2} \int D(z)d^2 z = \frac{\pi D(0) T_g^2}{2}$$  \hspace{1cm} (62)

and finally

$$\langle W(C) \rangle = \exp \left( -\frac{\sigma RT}{\pi^2} L \left( \frac{T_g^2}{R^2} \right) L \left( \frac{T_g^2}{T^2} \right) \right)$$  \hspace{1cm} (63)

where we have defined

$$L(u) = \int_{-\infty}^{\infty} dt e^{-t^2 u} \sin^2 \frac{t}{u}$$  \hspace{1cm} (64)

with the expansions

$$L(u) = \sqrt{\frac{\pi}{u}} \left\{ 1 - \frac{11}{72u^2} + \frac{1}{80u^4} + \ldots \right\}, \quad u \gg 1$$  \hspace{1cm} (65)

$$L(u) = \pi + O(\sqrt{u}), \quad u \ll 1.$$  \hspace{1cm} (66)

Now we consider $\langle W \rangle$ inside the integral (57). If one assumes, that for small $T$ one can indeed use the approximation of small area of the loop $C_z$, i.e. Eq. (60), then one has in the relativistic case, but considering $T$ small, $T \ll 1/m$ and expanding (60)

$$G^{(\varphi)}(0, T) = G_0^{(\varphi)}(0, T) + G_1^{(\varphi)}(0, T) + \ldots$$  \hspace{1cm} (67)

where

$$G_1^{(\varphi)}(0, T) \sim g^2 \langle S^2 \rangle \langle F^a F^a \rangle \frac{1}{T^2} \sim g^2 T^2 \langle F^a F^a \rangle.$$  \hspace{1cm} (68)

(we have assumed according to what was said before (60) that $\langle S^2 \rangle \sim T^4$. This is a standard result of OPE analysis with a local condensate accompanied by higher powers of $T$ (or higher powers of $1/Q^2$ in the momentum representation).

Let us now consider again the term $O(g^2)$, but now taking into account the nonlocal character of the correlators $\langle F(x)F(y) \rangle$. To this end we expand the Wilson loop in (57) and making use of two simple identities

$$(Dz)_{xy} = (Dz)_{xu} d^4 u (Dz)_{uv} d^4 v (Dz)_{vy}$$
\[ \int_0^\infty ds \int_0^s d\tau_1 \int_0^{\tau_1} d\tau_2 f(s, \tau_1, \tau_2) = \int_0^\infty ds \int_0^{s+\tau_1 + \tau_2, \tau_1 + \tau_2, \tau_2} d\tau_2 f(s + \tau_1 + \tau_2, \tau_1 + \tau_2, \tau_2) \]  

(69)

one can write

\[ G^{(\varphi)}_1(0, T) = \int d^4 x \int d^4 y G_0^{(\varphi)}(0, x) \langle A_\mu(x) A_\nu(y) \rangle \dot{x}_\mu G^{(\varphi)}_0(x, y) \dot{y}_\nu G^{(\varphi)}_0(y, T) \]  

(70)

Notation used in (70) implies, that \( \dot{x}_\mu(\tau) = \frac{dx_\mu}{d\tau} \), and \( \langle A_\mu(x) A_\nu(y) \rangle \) is expressed through a vacuum average of field strength \( \langle F(u) F(v) \rangle \), e.g. as in the coordinate gauge [20]

\[ \langle A_4(x) A_4(y) \rangle = \int_0^x du_i \int_0^y dv_k \langle F_{i4}(u) F_{k4}(v) \rangle. \]  

(71)

One can show (see, for example, Appendix B of [24]) that \( \dot{x}_\mu \to \vec{\partial} / \partial x_\mu \) and one recovers in (70) the usual perturbation expansion for \( G^{(\varphi)} \), where now in contrast to the chapter 4 only the linear vertices \( \varphi^2 A_\mu \vec{\partial}_\mu \) are taken into account (the term \( \varphi^2 A^2_\mu \) would also appear in (57) when one takes into account term with \( x = y \)).

From (70) one can deduce that the r.h.s. stays constant at large \( |\vec{x} - \vec{y}| \), while it decreasing for large \( |x_1 - y_1| \) (this is especially clear when one uses for the correlator \( D(u-v) \) the Gaussian form). Therefore the integral (70) is convergent both at large \( x, y \), and at small \( x, y \). Integrating (70) one obtains

\[ G^{(\varphi)}_1(0, T) \sim T^2 g \langle g F^a g F^a \rangle \sim \sigma, \]  

(72)

since \( D(0) \sim \langle g F^a g F^a \rangle \sim \sigma / T^2 g \). Thus one obtains a nonlocal constant for small \( T \ll T_g \). Comparing (68) and (72) one can see that at small \( T \) the correct (nonlocal) procedure yields a larger (dominant) term as compared with the standard OPE estimation. The reason again lies in the fact that relativistic trajectories occupy larger area for the Wilson loop when treated perturbatively and nonlocally, whereas in standard OPE treatment one attributes to this term the local condensate, implying that the Higgs particle (or quark) does not go far from the static source.

6 Remarks on OPE in abelian theories with confinement

We are going to discuss OPE in abelian confining models in this section. The complications due to path ordering are absent in abelian case and one may consider general expression for the two–point correlator of the field strengths in the form (13) where the functions \( D(z), D_1(z) \) depend entirely on \( z = x - y \). We assume that confining properties of the theory are caused by condensate of monopoles, hence the equations of motion take the form:

\[ \partial_\mu F_{\mu\nu} = j_\nu \quad ; \quad \partial_\mu \tilde{F}_{\mu\nu} = J_\nu, \]  

(73)

where \( \tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \) and \( j_\mu \). \( J_\mu \) are electrically and magnetically charged currents, respectively. We define polarization operator \( \Pi(q^2) \) of the electric currents \( j_\mu \) as

\[ \int d^4 x \langle j_\mu(0) j_\nu(x) \rangle \exp(i q x) = (\delta_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2) \]  

(74)
Differentiating abelian analog of (13) and taking into account equations of motion, it is straightforward to obtain the following relation:

$$\Pi(q^2) = \int d^d x \left( D(x) + \frac{d}{2} D_1(x) + x^2 \frac{d^2 D_1}{dx^2} \right) \exp(iqx) \tag{75}$$

In $d = 4$ case it can be rewritten in symmetric form as

$$\langle j_\mu(0) j_\nu(x) \rangle + \langle J_\mu(0) J_\nu(x) \rangle = -\frac{1}{6} (\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) \langle F_{\alpha\beta}(0) F_{\alpha\beta}(x) \rangle \tag{76}$$

where for the condensate one has

$$\langle F_{\alpha\beta}(0) F_{\alpha\beta}(x) \rangle = 6 \left( D(x) + \tilde{D}(x) \right) \tag{77}$$

The function $\tilde{D}(x) = D(x) + 2 D_1(x) + x^2 \frac{d D_1}{dx}$ corresponds to the confining part of the correlator of dual field strengths $\tilde{F}$ in the same way as $D(x)$ corresponds to the correlator of $F$ in (13). In case of $d = 4$ QED without monopoles, one easily finds [42]

$$D(x) \equiv 0 ; \quad D_1(x) = \frac{1}{\pi^2} \left( \frac{e^2(x)}{x^4} - \frac{1}{x^2} \frac{de^2(x^2)}{dx^2} \right) \tag{78}$$

For polarization operator one obtains

$$\Pi(q^2) = -\frac{1}{\pi^2} \int d^d x \exp(iqx) \frac{d^2 e^2(x^2)}{(dx^2)^2} \tag{79}$$

It is evident that free field term $\sim e_0^2/x^4$ does not contribute to $\Pi(q^2)$ and the only nonzero contributions to the r.h.s. of (75) comes from either the running of the charge $e^2(x^2)$ (as in perturbation theory) or from nonperturbative parts of $D(x), D_1(x)$, if they are nonzero. Let us examine the latter contributions to $\Pi(q^2)$. Standard OPE reasoning would suggest to look for leading term of this kind in the form $\langle F^2 \rangle/q^4$. It is easy to see that for functions $D(x)$, which are smooth at the origin (for example, $D(x) = D(0) \exp(-x^2/T_g^2)$), the corresponding contribution to $\Pi(q^2)$ is exponentially suppressed at large $q^2$ (i.e. for $q^2 T_g^2 \gg 1$), it means that power corrections are absent in this case, in other words nonperturbative background is “too soft”. In particular, there is no $D(0)/q^4$ term. For $D(x)$ such that it is not smooth but finite at the origin (e.g. for often used exponential fit $D(x) = D(0) \exp(-|x|/T_g)$, one has as a leading large-$q$ nonperturbative asymptotics

$$\Delta \Pi(q^2) \sim \frac{D(0)}{T_g} \left( \frac{1}{q^2} \right)^{\frac{5}{2}} \sim \frac{\langle F^2 \rangle}{T_g q^6} \tag{80}$$

The situation becomes even more dramatic if $D(x)$ is singular at the origin (as it happens, for example, in the London limit of Abelian Higgs model [44]), where $D(x) \sim M^2/x^2$ if $x \to 0$, one has in this particular case

$$\Delta \Pi(q^2) = \frac{M^2}{q^2 + M^2} \propto \frac{M^2}{q^2} \text{ then } q^2 \gg M^2 \tag{81}$$

This $1/q^2$ regime in AHM is bounded from above, however, by the Higgs mass $m_H$: if $q^2 \gtrsim m_H^2$, the Ginzburg-Landau description of the condensate is not valid, broken symmetry is restored
and microscopic degrees of freedom come into play. Presumably the same reasoning in applicable to "thin" strings scenario, proposed in [46]: at distances much smaller than coherence length neither "thick" nor "thin" strings can be formed. Notice that the string tension $\sigma$ depends on $m_H$ logarithmically in the London limit: $\sigma \sim \log(m_H/M)$.

It is interesting to compare the result (81) with an answer for massive photon propagator. It can be obtained from (78) taking $e^2(x) = m_x K_1(m_x)$ which corresponds to massive vector field propagator $\langle A_\mu(0)A_\nu(x) \rangle$. Differentiation in (79) yields

$$\Pi(q^2) \propto -\frac{m^2}{q^2} \quad \text{then} \quad q^2 \gg m^2$$

This result is obvious from the form of propagator in momentum space. Notice the sign difference between (82) and (81). It can be said, following [39] that leading power correction $\Delta P(q^2)$ in confining theory is caused by exchange of massive particle with tachyonic mass. This interesting point will be discussed elsewhere.

7 Spectral representation of Green’s functions and OPE

In this chapter we shall look at OPE from another point of view, trying to calculate terms of OPE using the known properties of spectrum of gauge-invariant Green’s function.

This type of analysis was done most extensively for the 'tHooft model (1+1 QCD at large $N_c$) [26] where exact results for the spectra and Green’s functions are known. (For details of analysis the reader is referred to [27, 28, 29, 30, 31]. We follow most closely notation and the line of reasoning of [31]. We consider again the heavy-light system but now in the $d = 1 + 1$.

The Green’s function can be written as

$$G^{(Qq)}(x) = \langle 0 | \bar{q}(x) \Phi(x,0) q(0) | 0 \rangle = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \langle \bar{q}(x) D_\mu(2n) q \rangle$$

Defining on the other hand the correlation function

$$P(q^2) = i \int e^{iqx} d^2 x \langle 0 | T \{ \bar{q}Q(x), Qq(0) \} | 0 \rangle \sim i \int G^{(Qq)}(x) e^{iqx} d^2 x,$$

one can write and expansion in inverse powers of $E = m_Q - q_0$

$$P(E) = \frac{1}{E} \left[ \langle \bar{q}q \rangle - \frac{1}{E^2} \langle \bar{q}P^4_0 q \rangle - \ldots \right] + \text{pert.part}$$

where $P_0 = iD_0$.

On the other hand one can write a spectral decomposition (dispersion relation) for $P(E)$

$$P(E) = \frac{N_c}{2\pi} m_0 \sqrt{\pi} \sum_n \frac{f_n^2}{E + E_n} \sim \frac{N_c}{\pi} \sum_n \frac{1}{\sqrt{n(\sqrt{n + \varepsilon})}}$$
where we have used notation \( m_0^2 \equiv g^2 N_c / \pi \), \( 2m_0 \sqrt{\pi \varepsilon} = E \) and relations for the heavy-light spectrum \[31\]

\[
E_n = 2m_0 \sqrt{\pi n} \left( 1 + O \left( \frac{\log n}{n} \right) \right), \quad f^2_n = \sqrt{\pi n} \left( 1 + O \left( \frac{\log n}{n} \right) \right). \tag{87}
\]

Now one can compare (85) and (86) and expanding the latter in powers of \( \frac{1}{E_n} \sim \frac{1}{\varepsilon} \), one obtains \[31\] for coefficients in (85)

\[
\langle \bar{q} P^{2n}_0 q \rangle \sim \langle \bar{q} q \rangle (\pi m_0)^{2n} n! \tag{88}
\]

The factorial growth of coefficients in (88) is typical both for 1+1 and 3+1, as will be shown below in this chapter.

One can do another derivation of the coefficients (88) starting from equations of motion in which case instead of (88) one obtains

\[
\langle \bar{q} (x_{\mu} D_{\mu})^{2n} q \rangle \sim x^{2n} n! \langle \bar{q} q \rangle \left( -\frac{g^2 \langle \bar{q} q \rangle}{2m_q} \right)^n \tag{89}
\]

Thus appears another feature (or a puzzle, as it was formulated in \[31\]): condensates computed from the spectrum or from microscopic equations of motion have drastically different scales: \( m_0^{2n} \) in the first case and \( \left( \frac{m_3}{m_q} \right)^{2n} \) in the second case, where \( m_q \) tends to zero in the chiral limit.

We shall now show that in the 3+1 QCD at least for \( N_c \to \infty \) the situation is very similar to that of the 'tHooft model:

- a) OPE coefficients of the \( \frac{1}{Q^{2n}} \) expansion ("condensates") have factorially growing behaviour.

- b) Condensates calculated from spectrum and from diagrams (plus equations of motion) are different.

Consider now the 3+1 problem – description of the selfenergy part \( \Pi(q^2) \). For two light quarks the standard OPE of \( \Pi(Q^2) \) in the Euclidean region is well known [1]

\[
\Pi(Q^2) = -\frac{1}{4\pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) \ln \frac{Q^2}{\mu^2} + \frac{6m^2}{Q^2} + \frac{2m \langle \bar{q} q \rangle}{Q^4} + \frac{\alpha_s \langle FF \rangle}{12\pi Q^4} + ... \tag{90}
\]

Following [32] one can use the background perturbation theory for the calculation of \( \Pi(Q^2) \) and represent it in the form

\[
\Pi(Q^2) = \Pi^{(0)}(Q^2) + \alpha_s \Pi^{(1)}(Q^2) + \alpha_s^2 \Pi^{(2)}(Q^2). \tag{91}
\]

Let us first consider \( \Pi^{(0)}(Q^2) \) (for details of computations the reader is referred to \[32\] and papers quoted therein).
In the large $N_c$ limit $\Pi^{(0)}(Q^2)$ has the form

$$\Pi^{(0)}(Q^2) = \frac{1}{12\pi^2} \sum_{n=0}^{\infty} \frac{C_n}{Q^2 + M_n^2}. \quad (92)$$

The masses $M_n$ can be taken as the eigenvalues of the well-known Hamiltonian, which was derived from QCD with the assumption of area law for minimal surface and was shown to be valid for small angular momentum $L = 0, 1, 2$ [33], while for larger $L$ the string rotation should be taken into account, $\Delta H_{str}$, yielding the correct Regge slope $(2\pi\sigma)^{-1}$ for masses $M_n$ [33, 34, 35]

$$H^{(0)} \Psi_n = M_n \psi_n; \quad H^{(0)} = 2\sqrt{\vec{p}^2 + m_f^2 + \sigma r + \Delta H_{str}} \quad (93)$$

Solutions to (93) can be written in the form

$$M_n^2 = 2\pi\sigma(2n_r + L) + M_0^2 \quad (94)$$

where $M_0^2 \approx m_0^2$. For $C_n$ one has

$$C_n(L = 0) = \frac{2}{3}Q_f^2 N_c m_0^2, \quad C_n(L = 2) = \frac{1}{3}Q_f^2 N_c m_0^2. \quad (95)$$

Here $m_0^2 = 4\pi\sigma$, and $Q_f$ is the electric charge of quark of flavour $f$. Taking into account degeneracy of masses with $L = 0, n_r = 1$ and $L = 2, n_r = 0$ the total $C_n$ is the sum

$$\bar{C}_n = C_n(L = 0) + C_n(L = 2) = Q_f^2 N_c m_0^2. \quad (96)$$

Since $\bar{C}_n$ does not depend on $n$ in this approximation, one obtains the sum

$$\sum_{n=0}^{\infty} \frac{1}{M_n^2 + Q^2} = -\frac{1}{m_0^2} \psi \left( \frac{Q^2 + M_0^2}{m_0^2} \right) + \text{const} \quad (97)$$

where the constant term is divergent and is eliminated by renormalization of $\Pi(Q^2) \rightarrow \Pi(Q^2) - \Pi(0)$.

Here $\psi(z)$ is the Euler function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \psi(z)|_{z \to \infty} = \ln z \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kz^{2k}} \quad (98)$$

where $B_n$ are Bernoulli numbers. Hence at large $Q^2$ the leading term in (98) yields

$$\Pi^{(0)}(Q^2) = -\frac{Q_f^2 N_c}{12\pi^2} \ln \frac{Q^2 + M_0^2}{\mu^2} + O \left( \frac{m_0^2}{Q^2} \right). \quad (99)$$

For $Q^2 \gg M_0^2$ this term coincides with the leading term in the OPE (90) (the latter is written for $Q_f = 1$).
From (92) and (97) one can compute also the next terms of the expansion in $\frac{1}{Q^2}$

$$\Pi^{(0)}(Q^2) = -\frac{Q^2 N_c}{12 \pi^2} \ln \frac{Q^2 + M_0^2}{\mu^2} + \sum_{n=1}^{\infty} \frac{\lambda_{2n} m_{2n}^2}{Q^{2n}}. \quad (100)$$

It is clear that $\lambda_n$ at large $n$ grow factorially due to the asymptotics of Bernoulli numbers, $B_{2n} = \frac{(-1)^{n-1} (2n)!}{2^{2n-1} n! \pi^{2n}} \zeta(2n)$.

Two properties are clearly visible in the expansion (100)

a) the ”condensates” $m_{2n}^2$ are large, $m_0 \approx 2.5$ GeV, as compared to the standard OPE condensates, e.g. $(\langle FF \rangle) \sim 0.1 \div 0.2$ GeV$^4$.

b) the coefficients $\lambda_n$ grow factorially, which is in agreement with discussion in [29] and analysis of the ’tHooft model in [30, 31], signifying that the OPE series is asymptotic.

Thus in both cases, 1+1 and 3+1, when confinement is present and spectrum contains nondecreasing probabilities $C_n$ (which is the feature of linear confining interaction) the OPE is a factorially diverging series, implying renormalon singularities in the Borel plane [32]. Another feature which is general to both 1+1 and 3+1 theories, is the mismatch between condensates calculated via spectrum (as in (100) and via diagrammatic analysis (as in (90)). In [31] a possible solution of this mismatch for the 1+1 case was suggested, which introduces the notion of ”effective condensates”, which may differ from actual condensates (defined, for example, on the lattice) due to the asymptotic character of the OPE series.

In 3+1 case there is another possibility to explain the mismatch, namely one should take into account that coefficients $\lambda_n$ of all higher condensates get contribution not only of the leading terms in $n$ of $M_n$ and $c_n$, but also subleading terms, and the final result for say $\lambda_4$ could be two orders of magnitude smaller due to cancellation between different terms, thus removing the mismatch. However this requires a mechanism of fine tuning between the subleading coefficients, the physical reason for which is still not known.

One could leave discussion of the mismatch at this point, if another check were not possible. Indeed, let us take the OPE with large (spectral) condensates and do a sum rule analysis of experimental data for $e^+e^- \to$ hadrons with $I = 1$ (see [36]).

This analysis was done in [32] using the hadronic ratio $R^I(s) = 12\pi \text{Im}\Pi^I(S)$. For $I = 1$ adding the perturbative terms with known coefficients as in [1, 36], but taking the background modified coupling constant [32] e.g. in one loop

$$\alpha_B(Q^2) = \frac{4\pi}{b_0 \ln \left( \frac{Q^2 + M_B^2}{\Lambda_B^2} \right)}$$

where $M_B \approx 1.5$ GeV, $\Lambda_B^{(3)} \approx 482$ MeV, one has

$$R^{I=1}(s) = \frac{3}{2} \sum_{n=0}^{\infty} C_n^{I=1} \delta(s - M_n^2) + \frac{3}{2} \left( 1 + \frac{\alpha_B(s)}{\pi} + 1.64 \left( \frac{\alpha_B}{\pi} \right)^2 \right) \quad (101)$$

$$C_n^{I=1} = m_0^2, \quad M_n^2 = m_0^2 + n m_0^2, \quad n = 1, 2, \ldots; \quad C_0 = \frac{2}{3} m_0^2.$$
and the corresponding Borel transform is
\[
\tilde{I}_0(M) = \frac{2}{3M^2} \int_0^\infty ds e^{-s/M^2} R^{l=1}(s). \tag{102}
\]
Substituting (101) into (102) yields
\[
\tilde{I}_0(M) = \frac{m_0^2}{M^2} \left\{ \frac{2}{3} e^{-m_0^2/M^2} + \sum_{n=1}^\infty e^{-(m_0^2+nm_0^2)/M^2} \right\}
+ \frac{\alpha_B(M)}{\pi} + 2.94 \left( \frac{\alpha_B(M)}{\pi} \right)^2, \quad m_0^2 = 4\pi\sigma. \tag{103}
\]
This should be compared to the standard result [1] with standard (small) condensates
\[
\tilde{I}_0^{st}(M) = 1 + \frac{\alpha_s(M)}{\pi} + 2.94 \left( \frac{\alpha_s(M)}{\pi} \right)^2 + \frac{\pi^2 G_2}{3 M^4} + \frac{448\pi^3\alpha_s |\langle 0|\bar{q}q|0\rangle|^2}{81 M^6}. \tag{104}
\]
In (104) \(\alpha_s(M)\) is standard, i.e. obtained from \(\alpha_B\) by setting \(M_B \equiv 0\).

It is clear that (103) contains in the Borel plane a set of poles at \(M^2 = M_k^2 = \pm \frac{m_0^2}{2\pi k}, \ k = 1, 2, \ldots\) and an essential singularity at \(M = 0\). These features imply presence of renormalons and are connected to the factorial growth of coefficients \(\lambda_{2n}\) in (100).

Now remarkably both Borel transforms lie inside the corridor of experimental errors, thus describing satisfactorily data with very different values of condensates (for details of comparison see [32]). Thus situation is becoming even more unclear: not only one has two sets of condensates (and consequently two sets of sum rules) but in addition experimental data cannot give preference to one of them.

While leading perturbative large-\(M\) asymptotics of \(\tilde{I}_0(M)\) and \(\tilde{I}_0^{st}(M)\) coincide, there is an important difference at small \(M\): while \(M^2 \tilde{I}_0(M)\) is defined for all \(M\), \(M^2 \tilde{I}_0^{st}(M)\) is diverging for \(M \to 0\) due to higher condensates and higher powers of \(\alpha_s(M)\).

### 8 Conclusions and outlook

The main emphasis of the present paper is the influence of confinement on the behaviour of Green’s functions in their dependence on momentum and the behaviour of Borel transforms. We stressed above everywhere the importance of large distances working in coordinate representation, especially for light quarks in presence of confinement. As a first and most clear example the Green’s function of Dirac equation with linear scalar potential was considered and it was demonstrated that the Euclidean time expansion (equivalent of Borel transform for heavy-light systems) looks completely different from the nonrelativistic case, and from the template oscillatory Green’s function. In this way it was shown that large distances may be important even for small Euclidean times and bring about new terms in the OPE in coordinate space.
As a second example we have treated the nonlinear equations for a quark in the heavy-light system—nonlocal equivalent of the Dirac case, and found that again the result is different from what one would expect in standard OPE, but the terms of expansion turn out to be constant, \( S_1(T) \sim \text{const} \cdot \sigma^{3/2} \). Translating this contribution into the form of the usual correlation function \( \Pi(Q^2) \) of vector currents (like it is done in the reaction \( e^+e^- \rightarrow \text{hadrons} \)) one would have the contribution \( \Delta \Pi(Q^2) \sim \frac{m\sigma^{3/2}}{Q^4} \), which is similar to the standard term \( \frac{m\langle \bar{q}q \rangle}{Q^4} \), and is presumably one term in the subseries generating \( m\langle \bar{q}q \rangle \). In this example large distances, explicitly accounting for in our analysis, do not produce new OPE terms but give some path to calculating chiral condensate through confinement characteristics (i.e. string tension \( \sigma \)).

In section 4, in contrast to that, another problem was elucidated: how linear confinement is built up out of higher condensates of OPE, and the answer is given by comparison of Eqs. (46)-(49) and (51). Indeed, the infinite sum of derivatives of field correlators in (51) is equivalent to the linear confinement term in (49), and to extract it explicitly one needs to rearrange all derivatives.

We have analyzed abelian confining models in section 6 and described different possible sources of nonstandard OPE terms, e.g. \( \frac{1}{q^2} \).

Finally, the last problem considered in the paper concerns the derivation of OPE from the spectral representation of the meson Green function. When the spectrum and coefficients \( c_n \) (equivalent of quark decay constants \( f_n \)) are known, the OPE is calculated automatically and can be compared with that obtained "microscopically"—i.e. via Feynman diagrams in the external fields and equations of motion.

In the \( d = 1 + 1 \) QCD this program was fully investigated in a series of papers (see e.g. [31] and refs. therein) and a mismatch between condensates obtained in those two ways was found.

In the 3+1 QCD situation is similar and as shown in [32] and in the present paper, the mismatch of condensates in scales and order of magnitude also is evident. The situation is sharpened by the fact, that the QCD sum rules for \( e^+e^- \rightarrow \text{hadrons} \) reproduce experimental data for both choices of condensates.

We have not tried here to resolve this puzzle, and leave it for the future. There are two important topics in OPE we have not discussed. First, this is the partial summation of the OPE terms which can be done by introduction of nonlocal condensates in OPE, initiated and studied in [37, 38]. It would be interesting to find the link between our treatment of long-distance nonperturbative physics and the method of nonlocal condensates worked out in [37, 38].

Second, the problem of perturbative-nonperturbative interference, which may produce new singular OPE terms, like \( 1/Q^2 \), which was discussed in [39, 40, 9], is touched in section 6 only briefly. This set of problems certainly deserves further study.
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We discuss in this Appendix the properties of the kernel (18), which we used in the main text. The reader is referred to the Appendix 3 of the paper [14] where 3d counterpart of (18) was analysed. Since the kernels of the form (18) play important role in the discussed formalism we present an independent detailed analysis here both for possible future applications and for reader’s convenience.

We are interested in the properties of the following function:

$$f(\vec{\eta}, \vec{\rho}) = \int_0^1 d\alpha \int_0^1 d\beta \exp \left( -\frac{(\alpha \vec{\eta} - \beta \vec{\rho})^2}{T_g^2} \right)$$

where $\vec{\eta}, \vec{\rho}$ are $d$-dimensional vectors with angle $\theta$ between them. We denote absolute values of the arguments as $\eta = |\vec{\eta}|$, $\rho = |\vec{\rho}|$ and assume in what follows without loss of generality that $\eta \geq \rho$. The symmetry of formulas below with respect to the exchange $\rho \leftrightarrow \eta$ (which is manifest in the definition (A.1)) is to be restored by replacements $\rho \rightarrow \min\{\rho, \eta\}$ and $\eta \rightarrow \max\{\rho, \eta\}$.

It is instructive to consider four different asymptotic regions:

1). $\eta, \rho \sim T_g$  
2). $\eta \gg T_g; \rho \sim T_g$  
3). $\eta, \rho \gg T_g; \theta \gtrsim 1$  
4). $\eta, \rho \gg T_g; \theta \ll 1$

In the region 1) one can expand (A.1) in Taylor series with respect to both arguments, subsequent integration is straightforward:

$$f(\vec{\eta}, \vec{\rho}) = \frac{1}{4} - \frac{\rho^2 + \eta^2}{8T_g^2} + \frac{2\rho \eta \cos \theta}{9T_g^2} + \frac{\rho^4 + \eta^4}{24T_g^4} + \frac{\rho^2 \eta^2}{8T_g^2} \left( \cos^2 \theta + \frac{1}{2} \right)$$

$$- \frac{2\rho \eta \cos \theta}{15T_g^4} \left( \rho^2 + \eta^2 \right) + O(T_g^{-6})$$

In derivation of expression (20) in the main text we have used in fact the leading term of this asymptotics (i.e. 1/4).

In the regions 2), 3), 4) we will systematically omit exponentially small terms, i.e. terms proportional to $\exp(-\eta^2/T_g^2)$ and also terms $\sim \exp(-\rho^2/T_g^2)$ in the regions 3) and 4). One easily obtains the following expression in the region 2):

$$f(\vec{\eta}, \vec{\rho}) = \frac{T_g^2}{\eta^2} \left( \frac{1}{4} + \frac{\sqrt{\pi} \rho \cos \theta}{6T_g} + \frac{1}{8T_g^2} (2 \cos^2 \theta - 1) - \frac{\sqrt{\pi} \rho^3 \cos \theta \sin^2 \theta}{10T_g^3} + O(\rho^4) \right)$$

Now we come to the regions 3) and 4). It is instructive to introduce the following variables:

$$s = \frac{(\vec{\eta}\vec{\rho} - \vec{\rho}\vec{\eta})^2}{T_g^2} = \frac{4\eta^2 \rho^2}{T_g^2} \sin^2 \frac{\theta}{2} ; \quad q = \frac{(\vec{\eta}\vec{\rho} + \vec{\rho}\vec{\eta})^2}{4T_g^2} = \frac{\eta^2 \rho^2}{T_g^2} \cos^2 \frac{\theta}{2}$$

and also $\xi = \sqrt{s}/\sqrt{q} = 2 \tan^2 \frac{\theta}{2}$.
In the region 3) the upper limit of the integration in (A.1) can be shifted to infinity up to exponentially small corrections. The function \( f(\vec{\eta}, \vec{\rho}) \) in the region 3) can be written therefore as

\[
f(\vec{\eta}, \vec{\rho}) = \frac{T^4_g}{4\eta^2\rho^2} \cdot \phi(\xi)
\]

where \( \phi(\xi) \) is given by

\[
\phi(\xi) = \frac{\sqrt{\pi}}{8} \frac{(4 + \xi^2)^2}{\xi^3} \int_0^\infty dy \exp(-y^2) \left[ \left( 1 - \operatorname{Erf} \left( \frac{y\xi}{2} \right) \right) \left( 1 - \frac{y^2\xi^2}{2} \right) + \frac{y\xi}{\sqrt{\pi}} \exp \left( -\frac{y^2\xi^2}{4} \right) \right]
\]

(A.6)

The function \( \phi(\xi) \) is a monotonically decreasing function of \( \xi \). When \( \xi \) is going to infinity, \( \phi(\xi) \) is approaching the following asymptotics:

\[
\phi(\xi) = \frac{1}{3} + \frac{32}{15} \frac{1}{\xi^2} + \mathcal{O}(\xi^{-4})
\]

(A.7)

At the point \( \xi = 2 \), which corresponds to \( \theta = \pi/2 \) and hence orthogonal vectors \( \vec{\eta} \) and \( \vec{\rho} \) one finds \( \phi(2) = 1 \), in agreement with simple direct calculation from (A.1).

We are now in the position to analyse the properties of \( f(\vec{\eta}, \vec{\rho}) \) in the region 4), where \( \xi \sim \theta \ll 1 \). Making the change of variables, one gets from (A.1):

\[
f(\vec{\eta}, \vec{\rho}) = -\frac{T^2_g}{\sin \theta} \left\{ \int_0^{\frac{x^2}{s}} dy \int_0^{\frac{x^2}{s}} dx + \int_0^{\frac{x^2}{s}} dy \int_0^{\frac{x^2}{2}} dx + \int_0^{\frac{x^2}{s}} dy \int_0^{\frac{x^2}{s}} dx - \int_0^{-w} \frac{x^2 - y^2 \xi^2}{s} \exp \left( -x^2 - y^2 \right) \right\}
\]

(A.8)

where \( w = \sqrt{\eta}(1/\rho - 1/\eta) = (\eta - \rho) \cos(\theta/2)/T_g \). One can rewrite (A.8) in the following form

\[
f(\vec{\eta}, \vec{\rho}) = \frac{T^2_g}{\sin \theta} \frac{1}{s} \left[ g \left( \frac{\sqrt{s}}{\eta}, \xi \right) + g \left( \frac{\sqrt{s}}{\rho}, \xi \right) + f_2 \left( \frac{\sqrt{s}}{\eta}, w, \xi \right) - f_2 \left( \frac{\sqrt{s}}{\rho}, w, -\xi \right) \right]
\]

(A.9)

where the \( \xi \)-expansion of the functions \( g, f_3, f_4 \) can be performed systematically. It gives

\[
g(z, \xi) = \frac{\pi}{8} \kappa(z) - \frac{\exp(-z^2)z^2}{4} \cdot \xi - \frac{\pi}{32} \left( \kappa(z) + \frac{2z^3}{\sqrt{\pi}} \exp(-z^2) \right) \cdot \xi^2 + \mathcal{O}(\xi^3)
\]

(A.10)

\[
f_2(z, w, \xi) = \operatorname{Erf} \left( \frac{w}{2} \right) \frac{\pi}{8} \kappa(z) + [1 - \exp(-w^2)] \frac{\exp(-z^2)z^2}{4} \cdot \xi + \mathcal{O}(\xi^2)
\]

(A.11)

where the function \( \kappa(z) \) is defined as

\[
\kappa(z) = \operatorname{Erf}(z) - \frac{2z}{\sqrt{\pi}} \exp(-z^2)
\]

(A.12)
Extracting coefficient functions in front of higher powers in $\xi$ is a matter of straightforward algebra.

The expressions (A.10), (A.11) are exact at the given order in $\xi$ up to omitted exponentially small terms. They can be simplified in different limiting cases. If $w = 0$ (i.e. $\eta = \rho$), one has $f_2 = 0$, while the first two terms in the r.h.s. of (A.9) are equal. In the opposite limit $w \to \infty$ the following relations hold true:

$$
\lim_{w \to \infty} \left[ g \left( \frac{\sqrt{s}}{\rho}, \xi \right) - f_2 \left( \frac{\sqrt{s}}{\rho}, w, -\xi \right) \right] = 0 ; \quad \lim_{w \to \infty} f_2 \left( \frac{\sqrt{s}}{\eta}, w, \xi \right) = g \left( \frac{\sqrt{s}}{\eta}, -\xi \right) \tag{A.13}
$$

Notice that in all cases the first argument of the functions $g, f_2$ need not be small: $\sqrt{s}/\eta = (2\rho/T_g) \sin(\theta/2)$ and in the region 4) $\rho \gg T_g$, but $\theta \ll 1$.

In terms of the original variables $\eta, \rho, \theta$ the leading term in (A.9) can be represented as

$$
f(\vec{\eta}, \vec{\rho}) \approx \frac{\pi}{64 \sin^3 \frac{\theta}{2} \cos \frac{\theta}{2}} \frac{T_g^4}{\pi^2 \rho^2} \left[ \kappa \left( \frac{2\rho \sin \frac{\theta}{2}}{T_g} \right) (1 + \text{Erf}(w)) + \kappa \left( \frac{2\eta \sin \frac{\theta}{2}}{T_g} \right) (1 - \text{Erf}(w)) \right] \tag{A.14}
$$

where $w = (\eta - \rho) \cos(\theta/2)/T_g$ and $\kappa(z)$ is defined in (A.12), $\kappa(z) > 0$ if $z > 0$. This expression is valid in small $\theta$-limit.

Notice that $f$ is non-singular if $\theta \to 0$ limit (which is evident from (A.1)):

$$
\lim_{\theta \to 0} f(\vec{\eta}, \vec{\rho}) = \frac{T_g \sqrt{\pi}}{6} \left[ \frac{\rho}{\eta^2} + \frac{\eta}{\rho^2} + \text{Erf} \left( \frac{\eta - \rho}{T_g} \right) \left( \frac{\rho}{\eta^2} - \frac{\eta}{\rho^2} \right) \right] \tag{A.15}
$$

One needs some simple extrapolating representation of (A.1) for practical calculations. Notice that it is only asymptotic behaviour of $f(\vec{\eta}, \vec{\rho})$ that matters, the particular form of Gaussian kernel was taken in (A.1) just as an example. A possible expression respecting all desired properties of $f$ in the regions of large $\eta, \rho$ is as follows:

$$
f(\vec{\eta}, \vec{\rho}) \approx \frac{T_g^4}{4\eta^4 \rho^2} l(\theta) \tag{A.16}
$$

where the function $l(\theta)$ has the following ”focusing” property: being integrated with a regular function $F(\theta)$, it acts like a smoothed $\delta$-function (see [14]):

$$
\int d\theta F(\theta) l(\theta) \approx c_1 \frac{\rho^3}{T_g^3} F \left( \frac{c_0 T_g}{\rho} \right) + c_2 \int d\theta F(\theta) \tag{A.17}
$$

where $c_0, c_1, c_2$ are some constants of the order of unity. It is worth reminding that $\rho$ is the length of the smaller vector in our notation, i.e. $\rho = \min \{\rho, \eta\}$. In particular, it is seen that in the limit of large $\rho \gg T_g$ the small $\theta$ asymptotics gives dominant contribution unless $F(\theta)$ vanishes at the origin faster than $\theta^3$. 

26
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