Abstract

We consider a version of Palais’ Principle of Symmetric Criticality (PSC) that is applicable to the Lie symmetry reduction of Lagrangian field theories. PSC asserts that, given a group action, for any group-invariant Lagrangian the equations obtained by restriction of Euler-Lagrange equations to group-invariant fields are equivalent to the Euler-Lagrange equations of a canonically defined, symmetry-reduced Lagrangian. We investigate the validity of PSC for local gravitational theories built from a metric. It is shown that there are two independent conditions which must be satisfied for PSC to be valid. One of these conditions, obtained previously in the context of transverse symmetry group actions, provides a generalization of the well-known unimodularity condition that arises in spatially homogeneous cosmological models. The other condition seems to be new. The conditions that determine the validity of PSC are equivalent to pointwise conditions on the group action alone. These results are illustrated with a variety of examples from general relativity. It is straightforward to generalize all of our results to any relativistic field theory.
1. Introduction

An important approach to studying properties of the solution space of gravitational field equations is to restrict attention to metrics and matter fields that admit a specified group of symmetries. It was noted quite some time ago by Hawking [1], and subsequently discussed in some detail by MacCallum and Taub [2], that with homogeneous cosmological models (“Bianchi models”) one cannot always impose the symmetry on the fields in the Einstein-Hilbert action functional since varying the action in this restricted class of fields may not yield the correct field equations. In particular, it was noted that only the Bianchi class A groups would, in general, allow for a successful symmetry reduction of the Einstein-Hilbert action. Many others have elaborated on this issue in homogeneous cosmology, see, for example, [3,4,5,6,7] and references therein. As Hawking points out in [1], the difficulty which arises with the Bianchi class B models is due to the presence of a non-trivial boundary term in the restricted variational principle. Such a difficulty does not appear in many other symmetry reductions. For example, Pauli restricts the Einstein-Hilbert action to a class of static, spherically symmetric metrics and obtains the reduced Einstein equations by Hamilton’s principle [8] (he attributes this approach to Weyl). In addition, Lovelock has shown that a variety of Lagrangians for fourth-order field equations allow for reduction by spherical symmetry [9]. In light of such examples, it is natural to ask whether there exist general criteria that allow one to decide for a given symmetry group when one can successfully reduce a generic Lagrangian or action functional. Our goal in this paper is to give a systematic account of the symmetry reduction of gravitational Lagrangians and field equations, and to completely characterize the symmetry group actions that guarantee the reduced Lagrangian produces the reduced field equations.

Viewing these issues strictly from the point of view of action functionals and variational principles Palais has arrived at the Principle of Symmetric Criticality (PSC) [10]. Given a group action on a space of fields, one can consider the restriction of an action functional $S$ to the group invariant fields to obtain the reduced action $\hat{S}$. Palais’ PSC asserts that, for any group invariant functional $S$, critical points of $\hat{S}$ within the class of group invariant fields are (group invariant) critical points of $S$. As Palais emphasized, PSC need be neither well-defined nor valid. Under hypotheses that guarantee PSC makes sense, he goes on to give necessary and sufficient conditions for the validity of PSC in a variety of settings.

Unfortunately, a straightforward application of these results to general relativity (and, more generally, classical field theory) is somewhat awkward since one must decide at the outset what class of spacetimes to consider in the variational principle, what asymptotic and/or boundary conditions to impose, what to do about spacetime singularities, etc. Moreover, different group actions may necessitate different choices in this regard. All these issues, which are fundamentally global in nature, will arise when using PSC formulated in terms of the action integral, viewed as a functional on the infinite-dimensional space of metrics.

These difficulties can be avoided by using a purely local formulation of PSC that is based on the Lagrangian rather than on the action integral. The version of PSC adopted in this paper asserts that, for a given group action and for any group invariant Lagrangian, the reduced field equations obtained by restriction of Euler-Lagrange equations to group invariant fields are equivalent to the Euler-Lagrange equations of a canonically defined, reduced Lagrangian. This formulation of PSC was studied in [11] under the hypothesis of a spacetime without spacetime singularities. Using a development in [12], this formulation of...
The condition stated in Proposition 5.4 for the existence of a cochain map can be reformulated in terms of the relative Lie algebra cohomology $\mathcal{H}^*(\Gamma, G_x)$ of the Lie algebra $\Gamma$ of $G$ relative to its isotropy subgroups $G_x$. The cohomology $\mathcal{H}^l(\Gamma, G_x)$ at degree $l$ is defined as follows. Fix a basis $e_a$, $a = 1, 2, \ldots, \dim G$ for the Lie algebra. The structure constants of the Lie algebra are then defined by the Lie bracket:

$$[e_a, e_b] = C_{ab}^c e_c.$$ 

The Lie algebra $\Gamma$ can be viewed as the space of left-invariant vector fields on the manifold $G$ and the Lie bracket as the vector field commutator. The dual basis of left-invariant 1-forms on $G$, $\omega^a$, $a = 1, 2, \ldots, \dim G$, satisfy

$$d\omega^a = -\frac{1}{2} C_{bc}^a \omega^b \wedge \omega^c. \quad (5.10)$$

The Lie algebra cohomology of $G$ at degree $l$ is defined as the space of closed $l$-forms on $G$ modulo the exact $l$-forms, with all forms being left-$G$-invariant. It can be computed using the exterior derivative formula (5.10). The relative Lie algebra cohomology $\mathcal{H}^l(\Gamma, G_x)$ is defined as the set of closed $l$-forms on the group $G$ modulo the set of exact $l$ forms, where all forms are left-$G$-invariant and right $G_x$-basic. This last condition means that, with $\Gamma_x = \text{Lie}(G_x)$, all forms are required to be invariant under the right action of $G_x$ on $G$ and to satisfy

$$X \cdot \omega = 0, \quad \forall X \in \Gamma_x.$$ 

A $G_x$-basic form on $G$ is the pull-back of a form on $G/G_x$. The relative Lie algebra cohomology $\mathcal{H}^*(\Gamma, G_x)$ computes the $G$-invariant de Rham cohomology of the orbit $G/G_x$ through $x \in M$.

Let $l$ be the dimension of the group orbits in $M$. Then we have the following necessary and sufficient condition for PSC1.

Proposition 5.5. Around each point $x \in M$ there is a $G$-invariant neighborhood $U$ such that a cochain map exists on $U$ if and only if

$$\mathcal{H}^l(\Gamma, G_x) \neq 0, \quad \forall x \in M. \quad (5.11)$$

Proof: See [11] for a proof of existence of a cochain map in a neighborhood of each point $x \in M$ when (5.11) holds. It is straightforward to check that this neighborhood can always be chosen to be $G$-invariant. □

We call (5.11) the Lie algebra condition for PSC. If $x$ and $x'$ lie in the same $G$ orbit then $\mathcal{H}^*(\Gamma, G_x)$ is isomorphic to $\mathcal{H}^*(\Gamma, G_x')$, so one need only check the Lie algebra condition along a cross section of the group action.

Finally, we note that Proposition 5.4 can be used to show the necessity of PSC1 in Theorem 5.2. Suppose that PSC1 does not hold, then there is no cochain map and according to Proposition 5.4 there will exist a $G$-invariant vector field $S$ such that $L_S \chi \neq 0$ for any $G$-invariant chain $\chi$. In fact, as shown in [11], $S$ can be chosen tangent to the group orbits and there exists a smooth, non-trivial $f$ such that

$$L_S \chi = f \chi, \quad (5.12)$$

where both $S$ and $f$ can be chosen independently of the $G$-invariant $l$-chain $\chi$. Consider the trivial Lagrangian

$$\lambda(g) = d(S \cdot \epsilon(g)) = L_S \epsilon(g) = \frac{1}{2} g^{\mu\nu} (L_s g_{\mu\nu}) \epsilon(g), \quad (5.13)$$

where $\epsilon(g)$ is the volume form of $g$. Being exact, this Lagrangian has identically vanishing Euler-Lagrange form. On the other hand, if we compute the pull back of the reduced Lagrangian $\tilde{\lambda}$ to $M$, we have

$$\pi^* \tilde{\lambda}(\hat{q}) = \chi \cdot \lambda(g(\hat{q})) = L_S (\chi \cdot \epsilon(g(\hat{q}))) - (L_S \chi) \cdot \epsilon(g(\hat{q})). \quad (5.14)$$