Abelian projected monopoles - to be or not to be*

Pierre van Baal*

*aInstituut-Lorentz for Theoretical Physics, University of Leiden, PO Box 9506, NL-2300 RA Leiden, The Netherlands.

We analyse what happens with two merging constituent monopoles for the SU(3) caloron. Identified through degenerate eigenvalues (the singularities or defects of the abelian projection) of the Polyakov loop, it follows that there are defects that are not directly related to the actual constituent monopoles.

1. Introduction

Finite temperature instantons (calorons) have a rich structure if one allows the Polyakov loop, \( P(\vec{x}) = P \exp(\int_0^\beta A_0(x, t) dt) \) in the periodic gauge \( A_0(t, \vec{x}) = A_0(t + \beta, \vec{x}) \), to be non-trivial at spatial infinity (specifying the holonomy). It implies the spontaneous breakdown of gauge symmetry. For a charge one SU(1) caloron, the location of the constituent monopoles can be identified through: i. Points where two eigenvalues of the Polyakov loop coincide, which is where the \( U^{n-1}(1) \) symmetry is partially restored to \( SU(2) \times U^{n-2}(1) \). ii. The centers of mass of the (spherical) lumps. iii. The Dirac monopoles (or rather dyons, due to self-duality) as the sources of the abelian field lines, extrapolated back to the cores. If well separated and localised, all these coincide [1,2]. Here we study the case of two constituents coming close together for \( n \geq 3 \), with an example for SU(3).

The eigenvalues of \( P_\infty \equiv \lim_{|\vec{x}| \to \infty} P(\vec{x}) \) can be ordered by a constant gauge transformation \( W_\infty \)

\[
W_\infty^\dagger P_\infty W_\infty = P_\infty^0 = \exp[2\pi i \text{diag}(\mu_1, \ldots, \mu_n)],
\]

\[
\mu_1 \leq \ldots \leq \mu_n \leq \mu_{n+1} = 1 + \mu_1,
\]

with \( \sum_{\mu=1}^n \mu_m = 0 \). The constituent monopoles have masses \( 8\pi^2 \nu_\mu \), where \( \nu_\mu = \mu_{\mu+1} - \mu_\mu \) (using the classical scale invariance to put the extent of the euclidean time direction to one, \( \beta = 1 \)). In the same way we can bring \( P(\vec{x}) \) to this form by a local gauge function, \( P(\vec{x}) = W(\vec{x}) P_0(\vec{x}) W^\dagger(\vec{x}) \). We note that \( W(\vec{x}) \) (unique up to a residual abelian gauge rotation) and \( P_0(\vec{x}) \) will be smooth, except where two (or more) eigenvalues coincide.

2. Puzzle

To analyse the lump structure when two constituents coincide, we recall the simple formula for the SU(n) action density [4].

\[
\text{Tr} F^2_{\mu\nu}(x) = \partial^2_\mu \partial^2_\nu \log \left[ \frac{1}{n!} \text{tr}(A_n \cdots A_1) - \cos(2\pi t) \right],
\]

\[
\mathcal{A}_m \equiv \frac{1}{r_m} \begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} c_{m} & s_{m} \\ s_{m} & c_{m} \end{pmatrix},
\]

with \( \vec{y}_m \) the center of mass location of the \( m \)th constituent monopole. We defined \( r_m \equiv |\vec{x} - \vec{y}_m| \), \( c_m = \cosh(2\pi \nu_m r_m) \), \( s_m = \sinh(2\pi \nu_m r_m) \), as well as \( \vec{y}_{n+1} \equiv \vec{y}_1 \), \( r_{n+1} \equiv r_1 \). We are interested in the case where the problem of two coinciding constituents in SU(n) is mapped to the SU(n-1) caloron. For this we restrict to the case where \( \vec{y}_m = \vec{y}_{m+1} \) for some \( m \), which for SU(3) is always the case when two constituents coincide. Since now \( r_m = r_{m+1} \), one easily verifies that \( \mathcal{A}_{m+1} \mathcal{A}_m = \mathcal{A}_{m+1}[\nu_{m+1} \rightarrow \nu_m + \nu_{m+1}] \), describing a single constituent monopole (with properly combined mass), reducing eq. (2) to the action density for the SU(n-1) caloron, with \( n-1 \) constituents.

The topological charge can be reduced to surface integrals near the singularities with the use of \( \text{tr}(P^1dP)^3 = d \text{tr}((P_1^A W P_0 + 2P_0^A dP_0) \wedge A W) = d \text{tr}(AW \wedge (2A P_0, W P_0)) \), where \( A W \equiv W^\dagger dW \). If one assumes all defects are pointlike, this can be used to show that for each of the \( n \) types the (net) number of defects has to equal the topological charge, the type being selected by the branch of the logarithm (associated with the \( n \) elements in the center) \cite{3}. One might expect the defects to merge when the constituent monopoles do. A triple degeneracy of eigenvalues for \( SU(3) \) implies the Polyakov loop takes a value in the center. Yet this can be shown not to occur for the \( SU(3) \) caloron with unequal masses. We therefore seem to have (at least) one more defect than the number of constituents, when \( \vec{y}_m \rightarrow \vec{y}_{m+1} \).

3. Example

We will study in detail a generic example in \( SU(3) \), with \( (\mu_1, \mu_2, \mu_3) = (-17, -2, 19) / 60 \). We denote by \( \vec{z}_m \) the position associated with the \( m \)th constituent where two eigenvalues of the Polyakov loop coincide. In the gauge where \( P_\infty = P_\infty^0 \) (see eq. (1)), we established numerically \cite{2} that

\[
\begin{align*}
P_1 &= P(\vec{z}_1) = \text{diag}(e^{2\pi i \mu_2}, e^{2\pi i \mu_2}, e^{-4\pi i \mu_2}), \\
P_2 &= P(\vec{z}_2) = \text{diag}(e^{-\pi i \mu_2}, e^{2\pi i \mu_2}, e^{-\pi i \mu_2}), \\
P_3 &= P(\vec{z}_3) = \text{diag}(-e^{-\pi i \mu_2}, e^{2\pi i \mu_2}, -e^{-\pi i \mu_2}).
\end{align*}
\]

This is for any choice of holonomy and constituent locations (with the proviso they are well separated, i.e. their cores do not overlap, in which case to a good approximation \( \vec{z}_m \approx \vec{y}_m \)). Here we take \( \vec{y}_1 = (0, 0, 10 + d) \), \( \vec{y}_2 = (0, 0, 10 - d) \) and \( \vec{y}_3 = (0, 0, -10) \). The limit of coinciding constituents is achieved by \( d \rightarrow 0 \). With this geometry it is simplest to follow for changing \( d \) the location where two eigenvalues coincide. In very good approximation, as long as the first two constituents remain well separated from the third constituent (carrying the Taubes winding), \( P_3 \) will be constant in \( d \) and the \( SU(3) \) gauge field \cite{2} of the first two constituents will be constant in time (in the periodic gauge). Thus \( P(\vec{z}_m) = \exp(A_0(\vec{z}_m)) \) for \( m = 1, 2 \), greatly simplifying the calculations.

When the cores of the two approaching constituents start to overlap, \( P_1 \) and \( P_2 \) are no longer diagonal (but still block diagonal, mixing the lower \( 2 \times 2 \) components). At \( d = 0 \) they are diagonal again, but \( P_2 \) will be no longer in the fundamental Weyl chamber. A Weyl reflection maps it back, while for \( d \neq 0 \) a more general gauge rotation back to the Cartan subgroup is required to do so, see fig. 1. At \( d = 0 \), each \( P_m \) (and \( P_\infty \)) lies on the dashed line, which is a direct consequence of the reduction to an \( SU(2) \) caloron.

To illustrate this more clearly, we give the expressions for \( P_m \) (which we believe to hold for any non-degenerate choice of the \( \mu_i \)) when \( d \rightarrow 0 \):

\[
\begin{align*}
P_1 &= P(\vec{z}_1) = \text{diag}(e^{2\pi i \mu_2}, e^{2\pi i \mu_2}, e^{-4\pi i \mu_2}), \\
P_2 &= P(\vec{z}_2) = \text{diag}(e^{-\pi i \mu_2}, e^{2\pi i \mu_2}, e^{-\pi i \mu_2}), \\
P_3 &= P(\vec{z}_3) = \text{diag}(-e^{-\pi i \mu_2}, e^{2\pi i \mu_2}, -e^{-\pi i \mu_2}).
\end{align*}
\]

These can be factorised as \( P_m = \hat{P}_m Q_m \), where \( \hat{P}_m \) describes an overall \( U(1) \) factor. In terms of \( Q_1 = \text{diag}(e^{3\pi i \mu_2}, 1, e^{-3\pi i \mu_2}) \), \( Q_2 = \text{diag}(1, 1, 1) = 1 \) and \( Q_3 = \text{diag}(-1, 1, -1) \) the \( SU(2) \) embedding in \( SU(3) \) becomes obvious. It leads for \( Q_2 \) to the trivial and for \( Q_3 \) to the non-trivial element of the center of \( SU(2) \) (appropriate for the latter, carrying the Taubes winding). On the other hand, \( Q_1 \) corresponds to \( \text{diag}(e^{3\pi i \mu_2}, e^{-3\pi i \mu_2}) \), which for the \( SU(2) \) caloron is not related to coinciding eigenvalues. For \( d \rightarrow 0 \), fig. 2 shows that \( \vec{z}_1 \) gets "stuck" at a finite distance (0.131419) from \( \vec{z}_2 \).

Figure 1. The fundamental Weyl chamber with the positions of \( P_m \) indicated at \( d = -2, 1, 2, .05, .04, .03, .02, .01, .005, .001, .0005 \), and (large dots) 0. The perpendiculars point to \( P_\infty \) (center), and emanate from the values of \( P_m \) for well separated constituents. The dashed line shows the \( SU(2) \) embedding for \( d = 0 \). (\( u \equiv \exp(2\pi i/3) 1 \)
4. Resolution

The $SU(2)$ embedding determines the caloron solution for $d = 0$, with constituent locations $\vec{y}_1' = \vec{y}_2$ and $\vec{y}_2' = \vec{y}_3$, and masses $\nu_1' = \nu_1 + \nu_2 = \mu_3 - \mu_1$ and $\nu_2' = \nu_3$. The best proof for the spurious nature of the defect is to calculate its location purely in terms of this $SU(2)$ caloron, by demanding the $SU(2)$ Polyakov loop to equal $\text{diag}(e^{3\pi i \mu_2}, e^{-3\pi i \mu_2})$. For this we can use the analytic expression [7] of the $SU(2)$ Polyakov loop along the $z$-axis. The location of the spurious defect, $z_1 = (0, 0, z)$, is found by solving $3\pi \mu_2 = \pi \nu_2' - \frac{4}{5} \partial_z \text{acosh} \left( \frac{1}{2} \text{tr}(A_2' A_1') \right)$. For our example, $z = 10.131419$ indeed verifies this equation.

With the $SU(2)$ embedded result at hand, we find that only for $\mu_2 = 0$ the defects merge to form a triple degeneracy. Using $3\mu_2 = \nu_1 - \nu_2$, this is so for coinciding constituent monopoles of equal mass. For unequal masses the defect is always spurious, but it tends to stay within reach of the non-abelian core of the coinciding constituent monopoles, except when the mass difference approaches its extremal values $\pm (1 - \nu_3)$, see fig. 2 (bottom). At these extremal values one of the $SU(3)$ constituents becomes massless and delocalised, which we excluded for $d \neq 0$.

However, the limit $d \to 0$ is singular due to the global decomposition into $SU(2) \times U(1)$ at $d = 0$. Gauge rotations $U$ in the global $SU(2)$ subgroup do not affect $P_2$, and therefore any $UQ_1 U^\dagger$ gives rise to the same accidental degeneracy. In particular solving $-3\pi \mu_2 = \pi \nu_2' - \frac{4}{5} \partial_z \text{acosh} \left( \frac{1}{2} \text{tr}(A_2' A_1') \right)$ (corresponding to the Weyl reflection $Q_1 \to Q_1^\dagger$) yields $z = 9.868757$ for $\mu_2 = -1/30$ (isolated point in fig. 2 (top)). Indeed, $U \in SU(2)/U(1)$ traces out a (nearly spherical) shell where two eigenvalues of $P$ coincide (note that for $\mu_2 = 0$ this shell collapse to a single point, $z = 10$). A perturbation tends to remove this accidental degeneracy.

5. Lesson

Abelian projected monopoles are not always what they seem to be, even though required by topology. Topology cannot be localised, no matter how tempting this may seem for smooth fields.

Acknowledgements

I am grateful to Andreas Wipf for his provocative question that led to this work. I thank Jan Smit, and especially Chris Ford, for discussions.

REFERENCES