The early stages of a relativistic heavy-ion collision are examined in the framework of an effective classical SU(3) Yang-Mills theory in the transverse plane. We compute the initial energy and number distributions, per unit rapidity, at mid-rapidity, of gluons produced in high energy heavy ion collisions. We discuss the phenomenological implications of our results in light of the recent RHIC data.

The Relativistic Heavy Ion Collider (RHIC) is currently colliding beams of gold nuclei at the highest center of mass energies, per nucleon, $\sqrt{s_{NN}} = 200$ GeV. The goal of these experiments is to explore strongly interacting matter, in particular the quark gluon plasma (QGP), with one notable exception. Specifically, a new procedure had to be devised in order to determine the initial condition for the transverse components of the gauge fields. To this end, one must solve Eq. 37 of [11]:

$$z_\mu \equiv i \text{Tr} \lambda_\mu \left( [U^{(1)} + U^{(2)}](I + U^\dagger) - \text{h.c.} \right) = 0 , \quad (1)$$

where $U^{(1,2)}$ are SU(3) group elements corresponding to CGC of the two nuclei before the collision, $U$ is the sought group element to the initial gauge field, and $\lambda_\mu$ are the Gell-Mann matrices. In other words, an Hermitian matrix $M(U) \equiv i([U^{(1)} + U^{(2)}](I + U^\dagger) - \text{h.c.})$ must be proportional to the unit matrix. In order to solve (1)
The relaxation equation has the form
\[ F(U) \equiv z \mu z = \text{Tr}(M^2) - \frac{1}{3}(\text{Tr} M)^2. \] (2)

If \( U \) satisfies (1), \( F(U) \) attains its absolute minimum, \( F(U) = 0 \). Next, we minimize \( F(U) \) by relaxation. The relaxation equation has the form

\[ \dot{U} = -i \lambda_{\mu} \partial_{\mu} F(e^{i \tau \lambda_{\mu} U})|_{\gamma_{\mu}=0} U \] (3)

where \( \gamma_{\mu} \) are real variables, and \( \dot{U} \) is the derivative with respect to the relaxation time \( t \). The explicit expression for the right-hand side of (3) is somewhat lengthy and will be presented elsewhere. The relaxation equation is then integrated numerically to yield the initial condition we seek.

In this work we will determine two observables: the energy and the number distribution of produced gluons. In doing so, we closely follow the procedure developed for the SU(2) case. In the continuum limit the theory contains two dimensional parameters: \( \Lambda_s \) and the nuclear radius \( R \). Any observable can therefore be expressed as a power of \( \Lambda_s \), times a function of the dimensionless product \( \Lambda_s R \) and of the coupling constant \( g \).

In Figure 1 (a), we plot the Hamiltonian density, for \( \Lambda_s R = 83.7 \) (on a 512x512 lattice) in dimensionless units as a function of the proper time in dimensionless units. We note that in the SU(3) case, as in SU(2), \( \varepsilon \tau \) converges very rapidly to a constant value. The form of \( \varepsilon \tau \) is well parametrized by the functional form \( \varepsilon \tau = a + \beta \exp(-\gamma \tau) \). Here \( dE_T/d\eta/\pi R^2 = \alpha \) has the proper interpretation of being the energy density of produced gluons, while \( \tau_D = 1/\gamma/\Lambda_s \) is the “formation time” of the produced glue.

In Figure 1 (b), the convergence of \( \alpha \) to the continuum limit is shown as a function of the lattice spacing in dimensionless units for two values of \( \Lambda_s R \). In Ref. [12], this convergence to the continuum limit was studied extensively for very large lattices (up to 1024x1024 sites) and shown to be linear. The trend is the same for the SU(3) results—thus, despite being further from the continuum limit for SU(3) (due to the significant increase in computer time) a linear extrapolation is justified. We can therefore extract the continuum value for \( \alpha \). We find \( f_E(25) = 0.537 \) and \( f_E(83.7) = 0.497 \). The RHIC value likely lies in this range of \( \Lambda_s R \). The formation time \( \tau_D = 1/\gamma/\Lambda_s \) is essentially the same for SU(2)-for \( \Lambda_s R = 83.7, \gamma = 0.362 \pm 0.023 \). As discussed in Ref. [12], it is \( \sim 0.3 \) fm for RHIC and \( \sim 0.13 \) fm for LHC (taking \( \Lambda_s = 2 \) GeV and 4 GeV respectively).

We now combine our expression in Eq. (4) with our non-perturbative expression for the formation time to obtain a non-perturbative formula for the initial energy density,

\[ \varepsilon = \frac{0.17}{g^2} \Lambda_s^4 \] (5)

This formula gives a rough estimate [19] of the initial energy density, at a formation time of \( \tau_D = 1/\gamma/\Lambda_s R \) where we have taken the average value of the slowly varying function \( \gamma \) to be \( \bar{\gamma} = 0.34 \).

To determine the gluon number per unit rapidity, we first compute the gluon transverse momentum distributions. The procedure followed is identical to that described in Ref. [13]—we compute the number distribution in Coulomb gauge [18], \( \nabla \cdot A_\perp = 0 \). In Fig. 2(a), we plot the normalized gluon transverse momentum distributions versus \( k_T/\Lambda_s \) with the value \( \Lambda_s R = 83.7 \), together with SU(2) result. Clearly, we see that the normalized result for SU(3) is suppressed relative to the SU(2) result in the low momentum region. In Fig. 2(b), we plot the same quantity over a wider range in \( k_T/\Lambda_s \) for two values of \( \Lambda_s R \). At large transverse momentum, the modes are nearly those of non-interacting harmonic oscillators. At smaller momenta, the suppression is due to non-linearities, whose

\[ \frac{1}{\pi R^2} \frac{dE_T}{d\eta}|_{\eta=0} = \frac{1}{g^2} f_E(\Lambda_s R) \Lambda_s^3, \] (4)

The function \( f_E \) is determined non-perturbatively as follows. In Figure 1 (a), we plot the Hamiltonian density, for a particular fixed value [17] of \( \Lambda_s R = 83.7 \) (on a 512x512 lattice) in dimensionless units as a function of the proper time in dimensionless units.

For the transverse energy of gluons we get
effects, we have confirmed, are greater for larger values of the effective coupling \( \Lambda_s R \).

\[
\frac{1}{\pi R^2} \frac{dN}{d\eta} \bigg|_{\eta=0} = \frac{1}{g^2} f_N(\Lambda_s R) \Lambda_s^2.
\] (8)

We find that \( f_N(83.7) = 0.3 \). The results for a wide range of \( \Lambda_s R \) vary on the order of 10% in the case of SU(2).

The broad features of the CGC picture have recently been compared to the RHIC data [20,21]. We shall here discuss the phenomenological implications of our specific model in light of the recent RHIC data on multiplicity and energy distributions. The final multiplicity of hadrons [22] is related to the initial gluon multiplicity by the relation \( dN^h/d\eta = \kappa_{\text{inel}} dN^g/d\eta \). Here \( \kappa_{\text{inel}} \) is a factor accounting for the transition from \( 2 \to n \) gluon number changing processes which may occur at late times beyond when the classical approach is applicable [23]. Moreover, if partial or full thermalization does occur [23,24], the initial transverse energy is reduced-both due to inelastic collisions prior to thermalization and subsequently due to hadronic expansion-by a factor \( \kappa_{\text{work}} \). We then have

\[
\frac{dE^h_T}{d\eta} \bigg|_{\eta=0} = \frac{\pi}{g^2} f_E(\Lambda_s R) \Lambda_s (\Lambda_s R)^2,
\]

\[
\frac{dN^h}{d\eta} \bigg|_{\eta=0} = \frac{\pi \kappa_{\text{inel}}}{g^2} f_N(\Lambda_s R) (\Lambda_s R)^2.
\] (9)

From the RHIC data at \( \sqrt{s_{NN}} = 130 \) GeV, we have \( dN^h/d\eta \big|_{\eta=0} \sim 1000 \) for central collisions [25–28]. For \( g = 2 \) (\( \alpha_s = 0.33 \)), \( \pi R^2 = 148 \text{ fm}^2 \), and \( f_N = 0.3 \), we have \( \kappa_{\text{inel}} \Lambda_s^2 = 3.5 \text{ GeV}^2 \). Now, from Eq. (9), the ratio \( R^h = dE^h_T/d\eta / dN^h/d\eta \) is, since \( f_E / f_N = 1.66 \), \( R^h = 1.66 \Lambda_s / \kappa_{\text{work}} / \kappa_{\text{inel}} \). The experimental value [26] for \( \sqrt{s_{NN}} = 130 \) GeV is \( R^h = 0.5 \) GeV. Now, if we assume that there is no work done due to thermalization, \( \kappa_{\text{work}} = 1 \), we obtain from the two conditions \( \Lambda_s = 1.02 \) GeV and \( \kappa_{\text{inel}} = 3.4 \) as the values that give agreement with the data. The latter value is the maximal amount of inelastic gluon production possible. Alternatively, if we assume hydrodynamic work is done, one obtains \( \kappa_{\text{work}} = (\tau_f / \tau_i)^{1/3} \) where \( \tau_f \) and \( \tau_i \) are the final and initial times of hydrodynamic expansion respectively. This gives us \( \kappa_{\text{work}} \approx 2 \). Following the same analysis as previously, we obtain \( \Lambda_s = 1.28 \) GeV and \( \kappa_{\text{inel}} = 2.13 \). Thus, within the CGC approach, we are able to place bounds on both the saturation scale and on the amount of inelastic gluon production at RHIC energies. An independent method to extract \( \Lambda_s \) directly from the data (albeit assuming parton-hadron duality) is to compute the relative event-by-event fluctuations of the gluon number [29].

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For recent analytical work, see Y. Kovchegov, hep-ph/0011252.

For a gold nucleus, \( R = 6.87 \text{ fm} \), so this value of \( \Lambda_s R \) corresponds to a \( \Lambda_s \sim 2.44 \text{ GeV} \), somewhat larger than most estimates for RHIC.

Though there are potentially gauge artifacts in this method, it was shown empirically in Ref. [13] that the results for the integrated number per unit rapidity in this method agree well with results from a gauge invariant cooling method.

The number 0.17 = \( f_E \cdot \gamma \). The function \( f_E \) is a slowly varying function of \( \Lambda_s R \), varying in the SU(2) case in the entire RHIC-LHC range by 20%. The function \( \gamma \) was shown to be constant within error bars for the RHIC-LHC range. Therefore, in the regime of interest, we estimate the systematic uncertainty of \( f_E \cdot \gamma \) to be \( \sim 10\% \).

We assume that hadronization is isentropic, in accord with the “parton-hadron duality” seen empirically.


For a gold nucleus, \( \Lambda_s \sim \Lambda_{\text{RHIC}} \), varying in the SU(2) case in a slowly varying function of \( \Lambda_s R \).