Abstract

The conditions for validity of the Causal Entropy Bound (CEB) are verified in the context of non-singular cosmologies with classical sources. It is shown that they are the same conditions that were previously found to guarantee validity of the CEB: the energy density of each dynamical component of the cosmic fluid needs to be sub-Planckian and not too negative, and its equation of state needs to be causal. In the examples we consider, the CEB is able to discriminate cosmologies which suffer from potential physical problems more reliably than the energy conditions appearing in singularity theorems.

I. INTRODUCTION

The validity of entropy bounds in Bekenstein’s non-singular cosmological model [1] has been recently challenged [2]. In the course of the analysis some of the energy conditions that sources in the Einstein equations are assumed to obey [3,4] were questioned. Here we determine the conditions that guarantee the validity of the CEB [5] for non-singular cosmologies with classical sources, and discuss their relation to the energy conditions of the classic singularity theorems [6].
CEB is a covariant entropy bound which is applicable, in principle, to any space-like region [5] in an arbitrary space-time dimension $D$ [7]. It is an improvement of the Hubble Entropy Bound (HEB) [8] (see also [9–11]), which was motivated by the following reasonable assumptions ($i$) entropy is maximized by the largest stable black hole that can fit in a given region of space. ($ii$) the largest stable black hole in a cosmological background is typically of size comparable to that of the Hubble horizon (this assumption is qualitatively supported by previous calculations [12]). In cosmological backgrounds, the CEB refines HEB by defining the “horizon” concept through the identification of a critical (“Jeans”-like) causal connection scale $R_{CC}$, above which perturbations are causally disconnected, so that black holes of larger size are unlikely to form.

In homogeneous and isotropic $D$ dimensional cosmological backgrounds $R_{CC}$ depends on the Hubble parameter $H(t)$, its time-derivative $\dot{H}(t)$, and the scale factor $a(t)$ [7],

$$R_{CC}^2 = \frac{D-2}{2} \max \left[ \frac{\dot{H}}{H^2} + \frac{D-2}{2} \frac{k}{a^2} , -\frac{\dot{H}}{H^2} + \frac{D-4}{2} H^2 + \frac{D-2}{2} \frac{k}{a^2} \right]$$

$$= \frac{4\pi G_N}{D-1} \max \left[ \rho - (D-1)p , (2D-5)\rho + (D-1)p \right],$$

(1.1)

where $k = 0, \pm 1$ determines the spatial curvature. To derive the second equality we have used Einstein’s equations, $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$ and a perfect-fluid form for the energy-momentum tensor. Notice that $R_{CC}$ is well defined if $\rho$ is positive because the maximum in Eq.(1.1) is larger than the average of the two entries in the brackets, and the average is equal to $2(D-2)\rho$.

Previously [5] three cases which were believed to exhaust all possible types of cosmologies were considered 1:

1. $|\dot{H}| \sim H^2 \sim |k|/a^2$, or $|\dot{H}| \sim H^2 \gg |k|/a^2$. In this case effective energy density and pressure are of the same order, $\rho \sim p$, and all length scales that may be considered

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1In [5] space curvature was implicitly included in the total energy budget as a regular additional source.
in entropy bounds, such as particle horizon, apparent horizon, $R_{CC}$, and the Hubble length, are parametrically equal. This case includes non-inflationary FRW universes with matter and radiation.

2. $H^2 \gg |k|/a^2, |\dot{H}|$. In this case $|\rho + p| \ll \rho$, and the universe is inflationary. Here the naive holographic bound fails miserably, but HEB, CEB and Bousso’s modification of the holographic entropy bound [3] do well.

3. $|\dot{H}| \gg H^2, |k|/a^2$. In this case $|\rho| \ll p$. Since $\rho$ and $p$ are the effective energy density and pressure, there are no problems with causality. This case occurs, for instance, near the turning point of an expanding universe which recollapses, or near a bounce of a contracting universe which reexpands.

There is however an additional case which was not explicitly included in cases (1)-(3):

4. $k/a^2 \gg |\dot{H}|, H^2$ so that spatial curvature determines the causal connection scale. This occurs, for example, when both $H$ and $\dot{H}$ vanish as in a closed Einstein Universe, or in the static version of Bekenstein’s non-singular Universe [1].

Here we discuss this last case and show that the same conditions that guarantee validity of CEB in the first three cases suffice to guarantee its validity in the fourth case.

CEB states that the maximal entropy $S_{CEB}$ that can be contained in a space-like region of proper volume $V$ is given by (our units are such that $\hbar = c = 1$ and $G_N = M_P^{-(D-2)} = \ell_P^{D-2}$),

$$S_{CEB} = \beta n_H S_{BH} = \beta \frac{V}{V(R_{CC})} \frac{A(R_{CC})}{4\ell_P^{D-2}},$$

where $n_H \equiv \frac{V}{V(R_{CC})}$ is the number of causally connected regions in the volume considered, $V(x)$ denotes the volume of a region of size $x$, $A(x)$ denotes the area of this region, and $\beta$ is a fudge factor reflecting current uncertainty on the actual limiting size for black-hole stability. For a spherical volume in flat space we have $V(x) = \Omega_{D-2} x^{D-1}/(D - 1)$, and $A(x) = \Omega_{D-2} x^{D-2}$, with $\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma \left(\frac{D-1}{2}\right)$, but in general the result is different and depends on the spatial-curvature radius. Since $\frac{A(R_{CC})}{V(R_{CC})} \sim \frac{D-1}{R_{CC}}$, 3
\[ S_{\text{CEB}} = \alpha (D - 1) \frac{V}{G_N R_{\text{CC}}}, \tag{1.3} \]

where \( \alpha \) is a numerical parameter of order one.

Conditions for validity of CEB were determined in [5,7]. Loosely speaking, energy densities are required to be sub-Planckian, and the total energy density of the cosmic fluid is required to be positive. In particular, for a universe with a large number of fields \( N \), in thermal equilibrium at temperature \( T \), the CEB was found to be valid for temperatures not exceeding a value of order \( M_P / \sqrt{N}^{D-2} \) (see also [13,14]).

II. CEB IN NON-SINGULAR COSMOLOGIES

A. Einstein Universe with radiation

The simplest example of a non-singular cosmology is a static Einstein model in \( D \) dimensions. This model requires positive curvature, and two types of sources: cosmological constant and dust; we denote by \( \rho_\Lambda \) and \( \rho_m \) the energy densities associated with each of the two components. To provide entropy we need an additional source, which we choose to be radiation consisting of \( N \) species in thermal equilibrium at temperature \( T \). The energy density of the radiation is given by \( \rho_r = N T^D \), and the entropy density of the radiation is given by \( s_r = N T^{D-1} \) (we ignore here numerical factors since we will be interested in scaling of quantities). The total entropy of the system is given entirely by the entropy of the radiation \( S_r = s_r V \).

In term of these sources, Einstein’s equations can be written in the following way:

\[ H^2 + \frac{1}{a^2} = \frac{16 \pi G_N}{(D-2)(D-1)} \rho_{\text{tot}} = \frac{16 \pi G_N}{(D-2)(D-1)} (\rho_\Lambda + \rho_m + \rho_r), \tag{2.1} \]

\[ \dot{H} - \frac{1}{a^2} = -\frac{8 \pi G_N}{(D-2)} (\rho_{\text{tot}} + p_{\text{tot}}) = -\frac{8 \pi G_N}{(D-2)(D-1)} [D \rho_r + (D-1) \rho_m], \tag{2.2} \]

where we have used in Eq. (2.2) the equations of state relating pressure to energy density:

\( p_\Lambda = -\rho_\Lambda, \rho_m = 0, \) and \( (D-1) \rho_r = \rho_r. \)
For given $\rho_m$ and $\rho_r$, one can choose $\rho_\Lambda$ and the scale factor $a$ such that $H$ and $\dot{H}$ vanish in Eqs. (2.1) and (2.2), and thus obtains a static solution. In particular, the condition given by Eq. (2.2) determines the scale factor in terms of $\rho_m$ and $\rho_r$,

$$a^2 = \frac{(D - 2)(D - 1)}{8\pi G_N} \frac{1}{D \rho_r + (D - 1)\rho_m}.$$  \hspace{1cm} (2.3)

Note that since both $H$ and $\dot{H}$ vanish identically, $R_{CC}$ is determined solely by the scale factor $a$ given in Eq.(2.3), as discussed previously.

We now wish to determine under which conditions (if any) some violations of CEB may occur in this model. Recall that according to Eq.(1.3) the CEB bounds the total entropy of a region contained in a comoving volume $V$ by $S_{CEB} = \alpha(D - 1)\frac{V}{G_N R_{CC}}$, and that in the static case under consideration $R_{CC} = 2a/(D - 2)$. The square of the ratio of $S_{CEB}$ and the entropy of the system $S_r$, is given by

$$\left( \frac{S_{CEB}}{S_r} \right)^2 = \left( \frac{\alpha(D - 1)}{s_r R_{CC} G_N} \right)^2 = \left[ 2\pi \alpha^2 (D - 1)(D - 2) \right] \left[ D + (D - 1)\frac{\rho_m}{\rho_r} \right] \left[ \frac{1}{N} \left( \frac{M_P}{T} \right)^{D-2} \right].$$  \hspace{1cm} (2.4)

Since the second factor in expression (2.4) is larger than unity if $\rho_m$ and $\rho_r$ are positive, and neglecting the overall prefactor which is independent of the sources in the model, we conclude that the CEB is valid provided that

$$N \left( \frac{T}{M_P} \right)^{D-2} \approx 1.$$  \hspace{1cm} (2.5)

This is the same condition discussed in [7], and should be interpreted as a requirement that temperatures are sub-Planckian, in the case of many number of species $N$ (see also [13,14]).

We therefore conclude that, as long as the temperature of radiation stays well below Planckian, CEB is upheld. The fact that the model is gravitationally unstable to matter perturbations does not seem to be particularly relevant to the issue of validity of the CEB.
A non-singular cosmological model which can describe time-dependent cosmologies was found years ago by Bekenstein [1]. This is a 4D Friedman-Robertson-Walker universe which is conformal to the closed Einstein Universe. It contains dust, consisting of \( N \) particles of mass \( \mu \) (\( N \) is constant and \( \mu \) is positive), coupled to a classical conformal massless scalar field \( \psi \), and \( N \) species of radiation in thermal equilibrium. The action for the dust-\( \psi \) system is given by

\[
S = -\frac{1}{2} \int \sqrt{-g} \left[ (\nabla \psi)^2 + \frac{1}{6} \psi^2 R \right] d^4x - \int (\mu + f \psi) d\tau. \tag{2.6}
\]

It includes in addition to the usual action for free point particles of rest mass \( \mu \), a dust-scalar field interaction whose strength is determined by the coupling \( f \). Accordingly, we may define the effective mass of the dust particles: \( \mu_{\text{eff}} = \mu + f \psi \).

The total energy density and pressure in Bekenstein’s Universe are given by

\[
\rho_{\text{tot}} = \rho_r + \rho_\psi + \rho_m, \quad p_{\text{tot}} = p_r + p_\psi + p_m, \tag{2.7}
\]

where \( \{\rho_r, p_r\} \), \( \{\rho_\psi, p_\psi\} \), and \( \{\rho_m, p_m\} \) are the energy densities and pressures associated with the radiation, scalar field and dust respectively. They depend on the scale factor in the following way

\[
\rho_r = CNa^{-4} = NT^4, \\
\rho_\psi = \frac{1}{2} f^2 N^2 a^{-4}, \tag{2.8}
\]

\[
\rho_m = N \mu_{\text{eff}} a^{-3} = N \mu a^{-3} - 2p_\psi,
\]

and their equations of state \( \gamma_r = p_r/\rho_r \), \( \gamma_\psi = p_\psi/\rho_\psi \), \( \gamma_m = p_m/\rho_m \) are the following

\[
\gamma_r = 1/3, \\
\gamma_\psi = -1/3, \tag{2.9}
\]

\[
\gamma_m = 0.
\]
The dependence of $\psi$ on $a$ is $\psi = -f Na^{-1}$, yielding $\mu_{\text{eff}} = \mu - f^2 Na^{-1}$. $C$ is an integration constant and the only source of entropy is the radiation whose entropy density is given by $s_r = NT^3$.

The solution for the scale factor $a$ is given in terms of the conformal time $\eta$ by

$$a(\eta) = a_0 (1 + B \sin \eta).$$

(2.10)

We assume that $a_0$, the mean value of the scale factor, is macroscopic, so it is large in our Planck units. If $B = 0$ the solution describes a static universe very similar to the closed Einstein Universe discussed previously. For $0 < B < 1$ the solution describes a “bouncing universe”: the universe bounces off at $\eta = 3\pi/2$ when the scale factor is minimal $a = a_{\text{min}} = a_0 (1 - B)$, expands until it turns over at $\eta = 5\pi/2$ when its scale factor is maximal $a = a_{\text{max}} = a_0 (1 + B)$, and continues to oscillate without ever reaching a singularity. The equations of motion require that the energy densities of the sources obey the following equalities at all times [1]:

$$2 a a_0 \left( \frac{\rho_\psi - \rho_r}{2\rho_\psi + \rho_m} \right) = 1 - B^2 = \frac{a_{\text{min}} a_{\text{max}}}{a_0^2}.$$  

(2.11)

Since $2\rho_\psi + \rho_m = N \mu a^{-3} > 0$, $\rho_r > 0$, and $B^2 < 1$, it follows that a necessary condition for a bounce is that $\rho_r < \rho_\psi$. This implies that the total pressure $\frac{1}{3}(\rho_r - \rho_\psi)$ is always negative. Moreover, Eq.(2.11) for $a = a_{\text{min}}$ implies that $\rho_m \leq -2\rho_r < 0$ there. But then, the conclusion must be that in order to avoid a singularity, $\mu_{\text{eff}} < 0$ at least at the bounce.

It is possible, however, to find a range of initial conditions and parameters such that $\mu_{\text{eff}}$ is positive near the turnover.

The result that $\rho_r$ and $\rho_\psi$ are manifestly positive definite, but $\rho_m$ can (and in fact must) be negative some of the time, suggest that it might be possible to parametrically decrease $\rho_{\text{tot}}$ by lowering $\mu_{\text{eff}}$ (making it large and negative) by increasing the coupling strength $f$, so that the amounts of radiation and entropy are kept constant. As it turns out this is exactly the case in which the CEB can be potentially violated. Using Einstein’s equations to express $R_{CC}$ in terms of the total energy density and pressure, we find the ratio $(S_{\text{CEB}}/S_r)^2$:  

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\[
\left( \frac{S_{\text{CEB}}}{S_t} \right)^2 \sim G_N^{-2} \left( \frac{\rho_t}{N} \right)^{-3/2} \frac{1}{N^2} G_N \text{Max} \left[ \frac{\rho_{\text{tot}}}{3} - p_{\text{tot}}, \rho_{\text{tot}} + p_{\text{tot}} \right].
\] (2.12)

A system for which the ratio above is smaller than one would violate the CEB. Recalling that the maximum on the r.h.s. of (2.12) is always larger than the mean of the two entries and rearranging we find
\[
\left( \frac{S_{\text{CEB}}}{S_t} \right)^2 \gtrsim \left[ \frac{1}{N} \frac{M_P^2}{T^2} \right] \frac{\rho_{\text{tot}}}{\rho_t}.
\] (2.13)

Since we assume that the model is sub-Planckian, namely that the first factor is larger than one as in Eq.(2.5), the only way in which CEB could be violated is if somehow the second factor was parametrically small. As discussed above, it does seem that the second term \( \rho_{\text{tot}}/\rho_t \) can be made arbitrarily small by decreasing \( \rho_{\text{tot}} \) while keeping \( \rho_t \) constant. Consequently, it is apparently possible to make the ratio \( S_{\text{CEB}}/S_t \) smaller than one and obtain a CEB violating cosmology. But this can be achieved only if the effective mass of the dust particles is negative (and large) as can be seen from Eq. (2.7).

Violations of the CEB (and as a matter of fact, of any other entropy bound such as Bekenstein’s [15], or Bousso’s [3]) go hand in hand with large negative energy densities in the dust sector. In the model under discussion, this manifests itself in the form of dust particles with highly negative effective masses. Occurrence of such negative energy density would most probably render the model unstable (see below). We argue that any analysis of entropy bounds should be performed for stable models. This is particularly relevant for the CEB, whose definition involves explicitly the largest scale at which stable black holes could be formed. Note, however, that instability does not necessarily lead to violations of the CEB as in the previous case.

To support this argument let us outline possible instabilities in the dust scalar field system when the dust particles mass is negative. To do this we need to be more specific about the model. Consider a possible field theoretic model for the dust as a fermionic field \( \chi \). In this case the dust-scalar field action is given by
\[
S = -\frac{1}{2} \int d^4x \sqrt{-g} \left[ (\nabla \psi)^2 + \frac{1}{6} \psi^2 R + i\bar{\chi} \Gamma \chi + \mu \bar{\chi} \chi + f \bar{\psi} \chi \chi \right].
\] (2.14)
The equations of motion determine the constant non-vanishing values of $\psi$ (for simplicity consider the static case only) and $\bar{\chi}\chi$. We see that when the effective mass $\mu + f\psi$ becomes negative the model becomes unstable due to $\chi$ pair production, and will prefer a state with a $\bar{\chi}\chi$ condensate, which will feed back into $\psi$. Correspondingly, such rapid creation of pairs would be accompanied by strong fluctuations. It is not clear whether under these circumstances the condition for bounce $\rho_t < \rho_\psi$ will continue to hold indefinitely, or whether a collapse to a singularity will ensue after a finite number of cycles of the universe. A complete discussion of the time-dependent situation is beyond the scope of this paper but it is clear that violations of CEB are related with a potential instability in the dust sector, and cannot be simply taken as a bona-fide example of CEB violation.

The fact that Bekenstein’s universe is non-singular indicates that the singularity theorems of Penrose and Hawking [6] are somehow eluded. And indeed the Strong Energy Condition (SEC) is violated in the model: $\rho_{\text{tot}} + 3p_{\text{tot}} = 2\rho_t + \rho_m$ is negative at the bounce, positive at the turnover and changes continuously in between. As we show later, violation of some energy conditions does not necessarily mandate a violation of the CEB. We will argue that in this sense the CEB has a better discriminating power than energy conditions (see below).

III. CONDITIONS FOR VALIDITY OF CEB
WITH GENERAL CLASSICAL SOURCES

We may summarize the lessons of the previous examples by imposing conditions on sources in a generic cosmological setting such that CEB is obeyed. This analysis is not restricted to a static universe, nor to a closed one, and contains the previous examples as particular cases.

We consider a cosmic fluid consisting of radiation, an optional cosmological constant, and additional unspecified classical dynamical sources which do not include any contributions from the cosmological constant or radiation. For simplicity we assume that the additional sources have negligible entropy. This is the most conservative assumption: if some of the
additional sources have substantial entropy our conclusions can be strengthened. We use the previous notations for the total, cosmological, and radiation energy densities, \( \rho_{\text{tot}} \), \( \rho_\Lambda \) and \( \rho_r \) respectively, and denote by \( \rho^* \) the combined energy density of the additional sources. Thus

\[
\rho_{\text{tot}} = \rho_r + \rho_\Lambda + \rho^*.
\] (3.1)

We use the same notation for the relative pressures, and for the equation of state \( \gamma^* \equiv \rho^*/p^* \), which may be time-dependent.

In term of these sources, the causal connection scale can be written as

\[
R_{\text{CC}}^2 = \frac{4\pi G_N}{D-1} \times \Max \left\{D\rho_\Lambda + \left[1 - (D-1)\gamma^*\right]\rho^*, (D - 4)\rho_\Lambda + \left[2(D - 5) + (D-1)\gamma^*\right]\rho^* + 2(D - 2)\rho_r \right\}. \tag{3.2}
\]

We may now express the ratio of \((S_{\text{CEB}}/S_r)^2\), neglecting as usual prefactors of order one

\[
\left( \frac{S_{\text{CEB}}}{S_r} \right)^2 \approx \frac{1}{N} \left( \frac{M_P}{T} \right)^{D-2} \times \Max \left\{D\frac{\rho_\Lambda}{\rho_r} + \left[1 - (D-1)\gamma^*\right]\frac{\rho^*}{\rho_r}, (D - 4)\frac{\rho_\Lambda}{\rho_r} + \left[2(D - 5) + (D-1)\gamma^*\right]\frac{\rho^*}{\rho_r} + 2(D - 2)\right\}. \tag{3.3}
\]

As was already pointed out in the previous section, a condition for any CEB violations is that this ratio be parametrically smaller than one. Notice that the first factor is larger than one by our requirement that the radiation energy density be sub-Planckian. Thus the only remaining possibility for violating CEB is that the second factor be parametrically smaller than unity. As we show below, this can occur only if at least one of the additional sources has negative energy density.

The r.h.s. of (3.3) is larger than the average of the two entries, so that

\[
\left( \frac{S_{\text{CEB}}}{S_r} \right)^2 \gtrsim \frac{1}{N} \left( \frac{M_P}{T} \right)^{D-2} (D - 2)\frac{\rho_{\text{tot}}}{\rho_r} \tag{3.4}
\]

Therefore, since \( \rho_{\text{tot}} > 0 \), a necessary condition for this expression to be smaller than unity is that \( \rho_{\text{tot}} \ll \rho_r \), which we may reexpress as
\[ \frac{\rho_{\Lambda}}{\rho_r} \sim - \left( 1 + \frac{\rho^*}{\rho_r} \right). \] (3.5)

This is not a sufficient condition since the equations of motion could dictate, for example, that the first factor on the r.h.s. of eq.(3.4) could be parametrically larger than unity at the same time. By substituting condition (3.5) into Eq. (3.3), we obtain

\[
\left( \frac{S_{CEB}}{S_r} \right)^2 \sim \frac{1}{N} \left( \frac{M_P}{T} \right)^{D-2} \times \operatorname{Max} \left\{ -(D-1)(1+\gamma^*)\frac{\rho^*}{\rho_r} + D \right\}, (D-1)(1+\gamma^*)\frac{\rho^*}{\rho_r} + D \right\}. \] (3.6)

Therefore, an additional necessary condition for \( S_{CEB}/S_r \) to be smaller than one is that

\[ (1 + \gamma^*)\rho^* \simeq - \frac{D}{(D-1)} \rho_r. \] (3.7)

Condition (3.7) can be satisfied in two ways:

(i) \( 1 + \gamma^* > 0 \) and \( \rho^* < 0 \). This obviously requires that at least one of the sources has negative energy density. In this case (barring pathologies) the magnitude of \( \rho^* \) is comparable to that of \( \rho_r \).

(ii) \( 1 + \gamma^* < 0 \) and \( \rho^* > 0 \). However, for classical dynamical sources, this typically clashes with causality which requires that the pressure and energy density of each of the additional dynamical sources obey \( |p_i| < |\rho_i| \); hence if all \( \rho_i > 0 \) then necessarily \( \gamma^* = \frac{(\sum p_i)}{(\sum \rho_i)} > -1 \).

Consequently, condition (3.7) cannot be satisfied if all of the dynamical sources have positive energy densities and equations of state \( |\gamma_i| \leq 1 \). Bekenstein’s Universe discussed in the previous section fits well within our framework: the total energy density is positive, but the overall contribution to \( \rho_{\text{tot}} \) of all the sources, excluding radiation (since the cosmological constant vanishes in this case), is negative and almost cancels the contribution of radiation, leaving a small positive \( \rho_{\text{tot}} \).

To summarize, if all dynamical sources (different from the cosmological constant) have positive energy densities \( \rho_i > 0 \) and have causal equations of state \( (|\gamma_i| \leq 1) \), and if radiation temperatures are sub-Planckian, CEB is upheld.
The CEB (and entropy bounds in general) refines the classic singularity theorems in that it allows cosmologies for which the singularity theorems are not applicable because some of the energy conditions are violated, but do not seem to be problematic in any of their properties, or indicates possible problems already when the singularity theorems seem perfectly valid. For example, the scale factor for a closed deSitter Universe (i.e. a closed Universe containing a positive cosmological constant \( \Lambda \)) in \( D = 4 \) is given by \( a(t) = (\frac{\Lambda}{3})^{-1/2} \cosh \sqrt{\frac{\Lambda}{3}} t \), showing a bounce at \( t = 0 \). This is not surprising since the sources of this model violate the SEC. The reliability of the SEC as a criterion of discriminating physical and unphysical solutions is therefore questionable (as is well known in the context of inflationary cosmology). Alternatively, in a contracting 4D radiation dominated universe, the singularity theorems imply the solution will reach a future singularity, but the CEB indicates problems already when \( T \sim M_P/N^{1/2} \).

In general, the total energy-momentum tensor of a closed “bouncing” universe violates the SEC, but it can obey the CEB. In order to see this explicitly let us consider the “bounce” condition, i.e. \( H = 0, \dot{H} > 0 \) for a closed Universe; by using the Einstein equations (2.1-2.2), we can express this condition in terms of the sources as follows:

\[
\rho_{\text{tot}} > 0, \quad (D - 3)\rho_{\text{tot}} + (D - 1)p_{\text{tot}} < 0. \tag{3.8}
\]

The second of these conditions is (in \( D = 4 \)) precisely the condition for violation of the SEC. In terms of \( \rho_r, \rho_\Lambda \) and \( \rho^* \) this reads

\[
2\rho_\Lambda - (D - 2)\rho_r - \left[ (D - 3) + (D - 1)\gamma^* \right] \rho^* > 0. \tag{3.9}
\]

In comparison, a necessary condition that the CEB is violated can be obtained from Eqs.(3.5) and (3.7),

\[
2\rho_\Lambda - (D - 2)\rho_r - \left[ (D - 3) + (D - 1)\gamma^* \right] \rho^* \sim 0, \tag{3.10}
\]

where the l.h.s of (3.10) can be either positive or negative. So we find that there is a range of parameters for which the CEB can be obeyed in some bouncing cosmologies but not in others.
In a spatially flat universe ($k = 0$), the conditions for a bounce are slightly different: $\rho_{\text{tot}} = 0$ and $\rho_{\text{tot}} + p_{\text{tot}} < 0$. At the bounce these conditions imply violation of the Null Energy Condition (NEC). As discussed previously, classical sources are not expected to violate the NEC, but effective quantum sources, such as Hawking radiation, are known to violate the NEC (see [16,17] for a more comprehensive discussion of this point). In terms of $\rho_r$, $\rho_\Lambda$ and $\rho^*$ the condition for a bounce reads

$$\left(1 + \frac{1}{D-1}\right) \rho_r + (1 + \gamma^*) \rho^* > 0.$$  \hspace{1cm} (3.11)$$

In comparison, a necessary condition that the CEB is violated can be obtained from Eq.(3.7),

$$\left(1 + \frac{1}{D-1}\right) \rho_r + (1 + \gamma^*) \rho^* \sim 0,$$  \hspace{1cm} (3.12)$$

where the l.h.s of (3.12) can be either positive or negative. So, again, we find that there is a range of parameters for which the CEB can be obeyed in some spatially flat bouncing cosmologies but not in others.

The CEB appears to be a more reliable criterion than energy conditions when trying to decide whether a certain cosmology is reasonable: taking again deSitter Universe as an example, we can add a small amount of radiation to it, and still have a bouncing model if $\rho_\Lambda$ is the dominant source, and SEC will not be obeyed (see Eq.(3.9)). Nevertheless, the general discussion in this section shows that in this case the CEB is not violated as long as radiation temperatures remain subPlanckian, despite the presence of a bounce. This happens, in part, because the CEB is able to discriminate better between dynamical and non-dynamical sources (such as the cosmological constant), and imposes constraints that involve the former ones only, such as Eq. (3.7).

IV. CONCLUSIONS

We have reached the following conclusions by studying the validity of the CEB for non-singular cosmologies:
1. Violation of the CEB necessarily requires either high temperatures $N(T_{\text{mp}})^{D-2} \gtrsim 1$, or dynamical sources that have negative energy densities with a large magnitude, or sources with acausal equation of state. Of course, neither of the above is sufficient to guarantee violations of the CEB.

2. Classical sources of this type are suspect of being unphysical or unstable, but each source has to be checked on a case by case basis. In the examples we discussed in sect. II, the sources were indeed found to be unstable or are strongly suspected to be so.

3. Sources with large negative energy density could allow, in principle, to increase the entropy within a given volume, while keeping its boundary area and the total energy constant. This would lead to violation of all known entropy bounds, and of any entropy bound which depends in a continuous way on the total energy or on the linear size of the system.

4. The CEB is more discriminating than singularity theorems. In the examples we have considered it allows non-singular cosmologies for which singularity theorems cannot be applied, but does not allow them if they are associated with specific dynamical problems.
REFERENCES


