Virial coefficients from 2+1 dimensional QED effective actions at finite temperature and density

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Abstract

From spinor and scalar 2+1 dimensional QED effective actions at finite temperature and density in a constant magnetic field background we calculate the corresponding virial coefficients for particles in the lowest Landau level. These coefficients depend on an arbitrary parameter $\theta$ related to the time component of the gauge field potential, which plays an essential role for large gauge invariance. Existence of $\theta$ implies fractional statistics, i.e., variation of this parameter corresponds to interpolate between fermionic and bosonic virial coefficients. The obtained coefficients are remarkably close to that coming from non-relativistic anyons.
1 Introduction

Gauge invariance in 2+1 spacetime dimensions allows the existence of a mass term for
gauge fields, known as the Chern-Simons (CS) term [1]. It is now well known that this
term can be generated dynamically by radiative corrections [2] and can not preserve parity
and gauge invariance simultaneously. At finite temperature, gauge invariance takes on a
particular form being large (not infinitesimal) with respect to the compactified Euclidean
time, leaving a free parameter in the theory, usually a constant value of the time compo-
nent of the gauge field. Recently, it has been shown that a resummation of the one-loop
graphs is essential to preserve the large gauge invariance of the finite temperature 2+1
dimensional quantum electrodynamics (QED$_3$) effective action [3, 4, 5].

In this Letter we are going to show that large gauge invariance at finite temperature
(and density) in 2+1 dimensions implies fractional statistics. This will be shown explicitly
by computing the virial coefficients from spinor and scalar QED$_3$ effective actions at finite
temperature and density in a constant magnetic field background for particles in the lowest
Landau level. These virial coefficients depend on the time component of the gauge field
and interpolate continuously those coefficients of fermions and bosons and are remarkably
close those coming from non-relativistic anyons (quasi-particles with fractional statistics)
[6]. These quasi-particles are known to play an essential role in the fractional quantum
Hall effect (FQHE) [7] whose properties such as fractional charges have been measured
recently [8].

Besides, the virial coefficients obtained here are similar to those found previously by
imposing generalized boundary conditions to scalar and spinor fields in 2+1 dimensions in
the absence of magnetic fields [9]. The possibility of relating these generalized boundary
conditions to fractional statistics was envisaged before in a quantum mechanical one-
dimensional model [10]. It has also been shown in different contexts that if one considers
an imaginary part of the chemical potential (which is equivalent to a time component of
the gauge field at finite temperature, as one can check - see eq. (9) bellow) fermions can
transmute into bosons, both at finite temperature [11, 12].
Effective actions at finite temperature and density

In QED the one-loop effective action for the gauge field $A_\nu = A_\nu(x)$ is obtained by integrating out the fermion field. In 2+1 dimensions one has

$$\exp\{iS_{\text{eff}}[A_\nu]\} = \int D\psi D\bar{\psi} \exp\left\{i \int d^3x \bar{\psi} \left[ \gamma^\nu (i\partial_\nu + eA_\nu) - m \right] \psi - \frac{1}{4} F^{\rho\nu} F_{\rho\nu} \right\}, \quad (1)$$

where $\gamma^\nu$ with $\nu = 0, 1, 2$ are the Dirac matrices, $\partial_\nu \equiv \partial/\partial x^\nu$, $x^\nu = (x^0, x^1, x^2)$, and $F_{\rho\nu} = \partial_\rho A_\nu - \partial_\nu A_\rho$ is the Maxwell tensor. The effective action can be calculated exactly for some configurations of the gauge field in which cases the field is understood as a classical background.

The thermodynamics of a system in thermal equilibrium at a finite temperature $T = \beta^{-1}$ is described by its partition function $Z$ which can be obtained by a Wick rotation of the effective action from Minkowski to Euclidean space ($x^0 = i\tau, \gamma^0 = i\gamma^3, A_0 = iA_3$) with the imaginary time $\tau$ compactified into the interval $[0, \beta]$. Further, to describe the system at finite density we introduce a chemical potential $\mu$ which is identified with the imaginary part of the time component of gauge field $A_\nu$, so we are going to take $A_3 \rightarrow A_3 + i\mu$ (see eq. (9) bellow). Then, its free energy, $\Omega(\beta, \mu) = -(1/\beta) \ln Z$, within this prescription is given in terms of the real part of the effective action as [13]:

$$\Omega(\beta, \mu) = -\frac{1}{\beta} \Re \{S_{\text{eff}}(\beta, \mu)\}. \quad (2)$$

This free energy can be expanded in terms of the fugacity $z = \exp(e\beta\mu)$ as

$$\Omega(\beta, \mu) = -\frac{S}{\beta} \sum_n b_n z^n, \quad (3)$$

where $S$ is the two-dimensional area and $b_n \equiv b_n(\beta)$ are the cluster coefficients related to integrals corresponding to the interaction of $n$ particles.

Then, the virial coefficients which characterize the statistics of the system can be found from the cluster coefficients using the relations ($a_1 = 1$) [14]:

$$a_2 = -\frac{b_2}{(b_1)^2}, \quad (4)$$

$$a_3 = -2\frac{b_3}{(b_1)^3} + \frac{(b_2)^2}{(b_1)^4}, \quad (5)$$

$$a_4 = -3\frac{b_4}{(b_1)^4} + 18\frac{b_2b_3}{(b_1)^5} - 20\frac{(b_2)^3}{(b_1)^6}, \quad (6)$$

$\vdots$
3 The virial coefficients from fermions

We are going to consider the QED$_3$ effective action for a configuration where the space-components of the gauge field depend only on the space-coordinates $A_j = A_j(\vec{x})$ with $j = 1, 2$ and the time-component depends only on the Euclidean time $A_3 = A_3(\tau)$, and choose $A_j = \frac{1}{2}F_{jk}x^k$ corresponding to a constant magnetic field background ($F_{12} = B$). In this case it is possible to obtain an exact result for the fermion propagator and consequently for the effective action [2] and generate the CS term dynamically preserving large gauge invariance [4, 5].

Further, we work with a ($2 \times 2$) irreducible representation of Dirac matrices in 2+1 dimensions implying a CS term that explicitly breaks parity invariance. Besides, there is a parity invariant contribution that comes from the Maxwell term which in this case corresponds to a constant magnetic field $B$.

Then, the effective action at finite temperature and density that comes from the parity violating Lagrangian density can be found to be [15, 16]

$$S_{\text{eff}}^{PV}(\beta, \mu) = \frac{eBS}{4\pi} \frac{m}{|m|} \left\{ i\epsilon\beta\Xi + G_+(|m| + i\epsilon\Xi) - G_+(|m| - i\epsilon\Xi) \right\},$$

(7)

where we defined

$$G_+(x) = \ln \left[ 1 + \exp (-\beta x) \right] ,$$

(8)

and included the chemical potential contribution through

$$\Xi = \tilde{A}_3 + i\mu,$$

(9)

where $\tilde{A}_3$ is a constant related to $A_3(\tau)$ by the large gauge transformations [4]

$$\tilde{A}_3 = \frac{2\pi k}{e\beta} + \frac{1}{\beta} \int_0^{\beta} d\tau A_3(\tau)$$

(10)

with $k = 0, \pm 1, \pm 2, ...$ being the winding number. Since the finite temperature and density QED effective actions that we discuss here depend on $A_3(\tau)$ through $\tilde{A}_3$ they are gauge invariant under large gauge transformations.

The effective action at finite temperature and density that comes from parity invariant Lagrangian can be written as [16]

$$S_{\text{eff}}^{PI}(\beta, \mu) = \frac{e|B|S}{4\pi} \sum_{\ell=0}^{\infty} \sum_{s=1}^{2} \left\{ \beta E_{\ell,s} + G_+(E_{\ell,s} + i\epsilon\Xi) + G_+(E_{\ell,s} - i\epsilon\Xi) \right\} ,$$

(11)
where $E_{\ell,s} = \sqrt{m^2 + 2e|B|(|\ell + s - 1|)}$ is the energy of the Landau levels.

Now, we can calculate the cluster and virial coefficients from the above spinor effective actions. Let us start with the action (7) associated with the parity violating (PV) part Lagrangian. Then, using eq. (9) and expanding (8) as a power series of the fugacity $z$, we have that the corresponding contribution to the free energy, accordingly to its definition (2), is given by:

$$
\Omega_{\text{PV}}(\beta, \mu) = \frac{eBS}{4\pi \beta |m|} \left\{ e\beta \mu + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp(-n\beta|m|) \cos(en\beta \tilde{A}_3) \right\} \times [\exp(en\beta \mu) - \exp(-en\beta \mu)]. \quad (12)
$$

The corresponding cluster coefficients can be obtained by comparing this expression with the expansion (3). Note that in a non-relativistic system one usually finds only positive (or negative) powers of the fugacity $z$. However, in a relativistic system due to the presence of particles and antiparticles one finds both positive and negative powers of $z$, as in the above equation. Also the chemical potentials for particles and antiparticles differ only by its sign, as a consequence of charge conservation [17]. The cluster coefficients for the PV Lagrangian are then given by:

$$
b_{\text{PV},n}^\pm = \mp \frac{(-1)^n}{n} \frac{eBm}{4\pi |m|} \exp(-n\beta|m|) \cos(en\beta \tilde{A}_3); \quad (n = 1, 2, ...), \quad (13)
$$

where the signs $\pm$ refers to particles and antiparticles respectively, with opposite cluster coefficients while $b_0 = -e^2B\beta \mu m / 4\pi |m|$ does not contribute to the thermodynamics of the system once it depends linearly on $\beta$.

The action that comes from the parity invariant (PI) Lagrangian is more involved than the PV part since it includes the summation over all Landau levels. It can be split into the lowest Landau level (LLL) contribution ($E_{0,1} = |m|$) and the excited states ones as

$$
\Omega_{\text{PI}}(\beta, \mu) = -\frac{|B|S}{4\pi \beta} \Re \left\{ \beta|m| + G_+ (|m| + ie \Xi) + G_+ (|m| - ie \Xi) \right\} + 2 \sum_{k=1}^{\infty} \beta E_k + G_+ (E_k + ie \Xi) + G_+ (E_k - ie \Xi) \right\}, \quad (14)
$$

where $E_k = \sqrt{m^2 + 2e|B|k}$. Then, the cluster coefficients in this case can be written as

$$
b_n^{\text{PI}} = b_n^{\text{PI}, \text{LLL}} + b_n^{\text{PI}, \text{excited}} \quad (15)
$$
and the LLL case follows similarly to the PV case so that

\[ b^{PI}_{\pm n|LLL} = (-1)^{n-1} \frac{e|B|}{4n\pi} \exp(-n\beta|m|) \cos(en\beta\tilde{A}_3); \quad (n = 1, 2, ...), \]  

(16)

where again \( \pm \) refers to particles and antiparticles but in this case they have the same cluster coefficients and \( b^{PI}_{0|LLL} = -e\beta|mB|/4\pi \) again does not affect the thermodynamics since it also depends linearly on \( \beta \). The contribution of the excited states for particles and antiparticles can obtained as

\[ b^{PI}_{\pm n|excited} = (-1)^{n-1} \frac{e|B|}{2n\pi} \sum_{k=1}^{\infty} \exp(-n\beta E_k) \cos(en\beta\tilde{A}_3); \quad (n = 1, 2, ...). \]  

(17)

Now we want to find the virial coefficients corresponding to the sum of PI and PV actions. Assuming that the magnetic field is strong and the temperature low enough to keep the particles in the LLL state we can take the contributions from the cluster coefficients eqs. (13), (16), so that (choosing \( B > 0 \)) we have

\[ b^{LLL}_{\pm n} = (-1)^{n-1} \frac{eB}{4n\pi} \left( 1 \pm \frac{m}{|m|} \right) \exp(-n\beta|m|) \cos(en\beta\tilde{A}_3); \quad (n = 1, 2, ...). \]  

(18)

Up to this point the distinction of particles and antiparticles (\( \pm \)) is completely arbitrary. However, due to the factor \( 1 \pm m/|m| \) one can see that only particles or antiparticles will contribute depending on the sign of \( m \) that we choose. This result is closely related to the choice of the non-equivalent irreducible representation of Dirac matrices that we pick up for the PV action. Have we chosen the other inequivalent irreducible representation we would have found the PV action with reversed signs so that the role of particles and antiparticles would be reversed. From now on we take \( m > 0 \) and \( n > 0 \), so that

\[ b^{LLL}_{n} = (-1)^{n-1} \frac{\rho_L}{n} \exp(-n\beta m) \cos(n\theta), \]  

(19)

where \( \rho_L = 2\beta\omega_c/\lambda^2 \) with \( \omega_c = |eB|/2m \) being half of the cyclotron frequency and \( \lambda = \sqrt{2\pi\beta/m} \) the thermal wavelength. Furthermore, we have defined

\[ \theta \equiv e\beta\tilde{A}_3, \]  

(20)

where this angular variable plays the role of an arbitrary parameter which is related to the statistics [9]. One can also define a non-relativistic chemical potential in terms of the
relativistic one up to the rest mass \( m \) (in natural units \( \hbar = c = 1 \)) so that one finds the non-relativistic cluster coefficients

\[
b^{LLL}_{n|NR} = (-1)^{n-1} \frac{\rho_L}{n} \cos(n\theta).
\]

However, this redefinition of the chemical potential does not affect the virial coefficients, as one can see from relations (4)-(6). Now, substituting the above cluster coefficients into these relations, we find that the virial coefficients are given by

\[
a^{LLL}_2 = \frac{1}{2\rho_L} [1 - \tan^2 \theta];
\]
\[
a^{LLL}_3 = \frac{1}{3(\rho_L)^2} [1 + 3 \tan^4 \theta];
\]
\[
a^{LLL}_4 = \frac{1}{4(\rho_L)^3} [1 - 3 \tan^4 \theta + 10 \tan^6 \theta];
\]
\[
a^{LLL}_5 = \frac{1}{5(\rho_L)^4} [1 + 20 \tan^6 \theta + 35 \tan^8 \theta];
\]
\[
a^{LLL}_6 = \frac{1}{6(\rho_L)^5} [1 - 10 \tan^6 \theta - 105 \tan^8 \theta + 126 \tan^{10} \theta],
\]
corresponding to the effective action \( S_{eff}(\beta, \mu) = S_{PV}^{eff}(\beta, \mu) + S_{PI}^{eff}(\beta, \mu) \) given by eqs. (7), (11) obtained from the integration over fermionic fields. We can also infer that the \( n \)-th virial coefficient is given by:

\[
a^{LLL}_n = \frac{1}{n} \left( \frac{1}{\rho_L} \right)^{n-1} \sum_{i=1}^{n} \sum_{j=0}^{n} c_{ij} \tan^{2j-2} \theta,
\]

where the constants \( c_{ij} \) can be obtained from the relations (4)-(6) and the cluster coefficients (19) or (21).

Note that a variation in \( \theta \) implies a change in the virial coefficients which determine the statistics of the particles. In particular, looking at the second virial coefficient we see that \( \theta = 0 \) gives the expected fermionic behavior while \( \theta = \pm \arctan \sqrt{2} \) gives the bosonic behavior. So any value of \( \theta \) within this interval would interpolate between fermions and bosons.

### 4 The virial coefficients from bosons

The effective action for the gauge field \( A_\nu \) in scalar QED\(_3\) can be calculated using the same choice we did (a constant magnetic field) for the fermionic case above. In this case
there is no parity violating Lagrangian once it does not involve Dirac matrices. Then the bosonic effective action is similar to the PI part of the fermionic case and we find

$$S_{B}^{\text{eff}}(\beta,\mu) = \frac{eSB}{2\pi} \sum_{l=0}^{\infty} \{ \beta E_l + G_{-}(E_l - ic\Xi) + G_{-}(E_l + ie\Xi) \}, \quad (28)$$

where $\Xi$ is still given by eq. (9), the energy of Landau levels is now given by $E_l = \sqrt{m^2 + 2eB(l + \frac{1}{2})}$ and we defined

$$G_{-}(x) = \ln \left[ 1 - \exp(-\beta x) \right]. \quad (29)$$

Now, we want to determine the bosonic virial coefficients corresponding to the above effective action. As in the PI part of the fermionic case we can split the contribution of LLL state which here is given by $E_0 = \sqrt{m^2 + eB}$ from the excited ones, so that

$$\Omega^{B}(\beta,\mu) = -\frac{eBS}{2\pi\beta} \Re \{ \beta E_0 + G_{-}(E_0 - ie\Xi) + G_{-}(E_0 + ie\Xi) \} + \sum_{l=1}^{\infty} \beta E_l + G_{-}(E_l - ie\Xi) + G_{-}(E_l + ie\Xi) \}. \quad (30)$$

Thus, the cluster coefficients can be written as

$$b_{n}^{B} = b_{n}^{B}|_{LLL} + b_{n}^{B}|_{excited}. \quad (31)$$

Note that here, in opposition to the fermionic case, the LLL cluster coefficients depend on the magnetic field $B$ so we use the approximation

$$E_0 = \sqrt{m^2 + eB} \approx m + \frac{eB}{2m}. \quad (32)$$

valid when $eB << m^2$. Then, we find that the cluster coefficients corresponding to the LLL in this case are given by

$$b_{\pm n}^{B}|_{LLL} = \frac{\rho L}{n} \exp[-n\beta(m + \omega_c)] \cos(n\beta\tilde{A}_3); \quad (n = 1, 2, ...), \quad (33)$$

while $b_0 = -e\beta B(m + \omega_c)/2\pi$ does not contribute to the thermodynamics. Redefining the non-relativistic chemical potential in terms of the relativistic one up to the rest mass $m$ we have

$$b_{n}^{NRB}|_{LLL} = \frac{\rho L}{n} \exp(-n\beta\omega_c) \cos(n\theta); \quad (n = 1, 2, ...). \quad (34)$$
In close analogy with the fermionic case we find that the virial coefficients in the bosonic case are given by

\[ a_n^{LLL} = (-1)^{n-1} a_n^{LL}, \]  

(35)

where \( a_n^{LLL} \) are the virial coefficients based on fermions, eqs. (22)-(27). The above boson based virial coefficients also give rise to fractional statistics, as in the case of the effective action integrated out of fermions, but this time \( \theta = 0 \) gives the usual bosonic coefficients while \( \theta = \pm \arctan \sqrt{2} \) gives the fermionic behavior. Again, intermediate values of \( \theta \) interpolate between bosons and fermions.

5 Discussion and conclusion

We can compare these results with those in the literature for non-relativistic anyons in a constant magnetic field [19, 20]. In particular, our results are very close to those obtained for the LLL in [20]. This can be seen by considering the description of particles near the bosonic and fermionic points. In this case, one can approximate

\[ \tan^2 \theta \approx \sin^2 \theta \approx \theta^2. \]

Then, we can find a relation between the parameter \( \theta \) and the usual phase \( \alpha \) [18, 19, 20]

\[ \theta^2 = 2\alpha, \]

with \( \alpha \in [0, 1] \) so that the second virial coefficient found here and those from [20] coincide. In general for higher virial coefficients our results are close to those coming from non-relativistic anyons in a constant magnetic field, at least for the lowest Landau level. Further, the virial coefficients we found here also resemble the ones from non-relativistic anyons in the absence of external fields [18, 21].

So, we have shown that the virial coefficients of relativistic anyons interpolating those from bosons or fermions might be obtained from a fundamental quantum theory of electromagnetic interactions, i. e., QED, once one guarantees large gauge invariance of the theory in 2+1 dimensions at finite temperature and density, which is brought through the time component of the gauge potential.
It is important to stress that the approach used in this work to the virial coefficients which interpolate those coming from bosons and fermions it is not based on the existence of a fundamental (zero temperature) particle with fractional statistics [6] but on thermal and finite density effects on the fundamental bosonic and fermionic particles, similar to that discussed in the absence of magnetic fields [9]. In other words, in our approach fractional statistics only exists at finite temperature and density, since at zero temperature large gauge transformations become trivial and the (constant) parameter associated with the time component of the gauge field would be removed by an ordinary (infinitesimal) gauge transformation, leaving the statistics canonical.

Finally, in this work we have not discussed interactions of the fundamental fermions (or bosons) with other fields than the constant magnetic background. This implied that the constant parameter $\theta$ (or $\tilde{A}_3$) was not fixed here. It is our hope that the basic characteristics discussed here related to finite temperature and density effective fractional statistics would survive to the presence of other interactions and possibly these interactions could fix the parameter $\theta$ at some fractional value as 1/3 in FQHE [7, 8].

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