We suggest a natural and simple to implement renormalization scheme of the Hartree–Fock–Bogoliubov (HFB) equations for the case of zero range pairing interaction. This renormalization scheme proves to be equivalent to a simple energy cut–off with a position dependent running coupling constant, which qualitatively emulates a density dependent zero range pairing interaction suggested by phenomenological analyses.

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More than forty years after the pioneering work of Bohr, Mottelson and Pines [1] there is no need to reiterate again the relevance of pairing correlations in nuclei. If one can adopt the approximation of a zero range two–body interaction the HFB equations become

\[ h(R) - \mu |u_i(R)\rangle + \Delta(R) |v_i(R)\rangle = E_i |u_i(R)\rangle, \quad (1) \]
\[ \Delta^*(R) |u_i(R)\rangle - [h^*(R) - \mu |v_i(R)\rangle = E_i |v_i(R)\rangle. \quad (2) \]

In all the formulas presented here we shall not display the spin degrees of freedom. One encounters typically no insurmountable difficulties in introducing a local Hartree–Fock (HF) Hamiltonian \( h(R) \) [2,3]. However, if one takes at face value Eqs. (1,2) one can show that the diagonal part of the anomalous density \( \nu(R,R) \) diverges, since when \( r = |r_1 - r_2| \) is small the anomalous density \( \nu(r_1, r_2) \) has the singular behavior

\[ \nu(r_1, r_2) = \sum_i v_i^*(r_1) u_i(r_2) \propto \frac{1}{|r_1 - r_2|}, \quad (3) \]

and thus the local (selfconsistent) pairing field \( \Delta(R) \) cannot be defined [4–6].

In metals this type of singularity does not play a noticeable role, because the summation over the single–particle states is cut–off at energies of the order of the Debye energy \( \omega_D \ll \varepsilon_F \), where \( \varepsilon_F \) is the Fermi energy. The single–particle density of states is essentially constant in an energy window of width \( \mathcal{O}(\omega_D) \ll \varepsilon_F \) and the expression for the anomalous density has only an infrared logarithmic divergence. This logarithmic divergence is due to states near the Fermi surface and has nothing to do with the ultraviolet divergence due to states faraway from the Fermi surface, which leads to the \( 1/|r_1 - r_2| \) singularity discussed here. The infrared divergence leads to the notorious non–analytical dependence of the gap on the coupling constant, namely \( \Delta = \omega_D \exp(-1/VN) \), where \( V \) is the strength of the interaction and \( N \) is the single–particle density of states at the Fermi energy \( \varepsilon_F \).

In nuclei and especially in very dilute fermionic atomic systems, where \( k_F r_0 \ll 1 \) and \( r_0 \) is the radius of the interaction, there is effectively no well defined cut–off and one needs to regularize the theory. A finite range interaction will provide a natural cut–off at single–particle energies of the order of \( \varepsilon_c \sim \hbar^2/mr_0^2 \), when the fast spatial oscillations of the single–particle wave functions \( u_i(r), v_i(r) \) will render the pairing field \( \Delta(r_1, r_2) \) ineffective. Even though the presence of a finite range of the interaction in the pairing channel formally removes the ultraviolet divergence of the gap, it is very difficult to come to terms with the fact that a cut–off at an energy of the order of \( \hbar^2/mr_0^2 \) could be the responsible for the definition of the gap both in the case of regular nuclei and very dilute nuclear matter as well. The characteristic depth of the interaction potential, which is of the order of \( \hbar^2/mr_0^2 \), being the largest energy in the system, can be considered to be infinite in the case of dilute systems. A well defined theoretical scheme for the calculation of a local pairing field, should lead to a converged result when only single–particle states near the Fermi surface are taken into account.

Most of the calculational schemes suggested so far for infinite systems reduce, naturally, to replacing a zero range potential by a low energy expansion of the vacuum two–body scattering amplitude [7–16]. The traditional approach in the calculations of finite nuclei consists however in introducing a simple energy cut–off, while the pairing field is computed by the means of a pseudo–zero–range interaction. In this approach the effective range of the interaction is obviously determined by the value of the energy cut–off and the two–body coupling constant in the pairing channel is chosen accordingly. Such a pure phenomenological approach lacks a solid theoretical underpinning and always leaves the reader with a feeling that “the dirt has been swept under the rug”. Another solution favored by other practitioners is to use a finite range two–body interaction from the outset, such as Gogny interaction [17]. Besides the fact that the ensuing HFB equations are much more difficult to solve numerically, such an approach also lacks the elegance and transparency of a local treatment and this seemingly simple recipe is indeed as phenomenological in spirit as the treatment based on a pseudo–zero–range interaction.
with an explicit energy cut-off. Moreover, in spite of the feebly arguments often put forward in favor of a finite range interaction in HFB calculations, the only real argument is the fact that the pairing field would otherwise diverge, and there is no mean-field observable which would be noticeable different in the case of a finite range interaction. When the size of the Cooper pair is given by the nuclear size (in infinite matter the rms radius of the Cooper pair is of the order of \( \hbar^2 k_F^\mu / m \Delta \)) and when \( \Delta \ll \varepsilon_F \), it is clear that details of the two-particle interaction at distances smaller or comparable with the inverse Fermi momentum should be irrelevant and the physics should be described basically by a single constant and be derivable from a suitably chosen zero range interaction model. (In the parlance of this work gradient terms, as those present in Skyrme interaction, are zero range, and only when terms higher than the second order in momenta are significant the interaction is considered truly finite range.)

The only attempt to implement a consistent regularization scheme for finite systems that we are aware of is that of Ref. [6]. In agreement with the analysis of Ref. [4] the authors of Ref. [6] conclude that in the case of a zero range two-body interaction the anomalous density has a \( 1/[r_1 − r_2] \) singularity. However the approach suggested in Ref. [6] has some unclear elements and as is formulated is suitable for systems in a harmonic trap only, which does not apply to atomic nuclei or other self-sustaining systems. Bruun et al. advocate the use of the following calculational procedure for the anomalous density. First of all one represents the anomalous density as

\[
\nu(r_1, r_2) = \sum_i \left[ v_i^\nu(r_1) u_i(r_2) + \frac{\Delta(R)}{2} \frac{\psi_i^*(r_1) \psi_i(r_2)}{\mu - \varepsilon_i} \right] \]

\[
= \frac{\Delta(R)}{2} G_0(r_1, r_2, \mu),
\]

\[
[h(R) - \varepsilon_i] \psi_i(R) = 0,
\]

\[
[m - h(r_1)] G_0(r_1, r_2, \mu) = \delta(r_1 - r_2),
\]

where \( R = (r_1 + r_2)/2 \). In the limit \( r_1 \to r_2 \) the sum over single-particle states in Eq. (4) is converging now and one has only to extract the “regulated” part of the propagator \( G_0(r_1, r_2, \mu) \), using the pseudo-potential approach [19]

\[
\nu(R) := \nu^{\text{reg}}(R) = \lim_{r_1 \to r_2} \nu(r_1, r_2)|^{r_1=r_2}
\]

\[
= \sum_i \left[ v_i^\nu(R) u_i(R) + \frac{\Delta(R)}{2} \frac{\psi_i^*(R) \psi_i(R)}{\mu - \varepsilon_i} \right] \]

\[
= \frac{\Delta(R)}{2} G_0^{\text{reg}}(R, \mu),
\]

\[
G_0^{\text{reg}}(R, \mu) = \lim_{r_1 \to r_2} G_0(r_1, r_2, \mu) + \frac{m}{2 \pi \hbar^2 |r_1 - r_2|}
\]

and obtain for the local pairing field

\[
\Delta(R) = \frac{4\pi \alpha |\hbar^2|}{m} \sum_i \left[ v_i^\nu(R) u_i(R) + \frac{\Delta(R) \psi_i^*(R) \psi_i(R)}{2(\mu - \varepsilon_i)} \right]
\]

\[
= \frac{4\pi \alpha |\hbar^2|}{2m} G_0^{\text{reg}}(R, \mu),
\]

where \( \alpha \) is the two-particle scattering length, which is negative. We shall work here in the lowest approximation in \( k_F \alpha \) and we shall not consider the next order correction due to the effective range [20]. The renormalization procedure and the extraction of the regulated part from various diverging quantities is completely analogous to the familiar procedures in QFT. Only in this case everything is performed in coordinate space and one simply “throws away” the diverging terms and retains the non-vanishing finite contributions, see e.g. Eq. (8) above and Eq. (10) below.

In a trap one can easily establish a one-to-one correspondence between the spectra of the HF and of the HFB equations. This one-to-one correspondence is needed as one has to specify each term of the regulated sum in Rel. (4). There is no such obvious one-to-one correspondence for self-sustaining systems such as nuclei [4,21], where sufficiently deep bound hole states lie in the continuum and where there is no one-to-one correspondence between the HF and HFB spectra. The approach suggested by these authors also requires the determination of the regular part of the single-particle Green function \( G_0^{\text{reg}}(R, \mu) \), for which there is so far no clear computational scheme in the case of an arbitrary self-consistent field.

Our suggestion amounts to a simple to implement approach, namely, replace in Eqs. (4,7,8,9) the subtracted single-particle wave functions and energies and the corresponding propagator by its Thomas-Fermi approximations. Thus we have

\[
G_0(r_1, r_2, \mu - U(R)) = \frac{m \exp(\alpha \hbar^2 |r_1 - r_2|)}{2 \pi \hbar^2 |r_1 - r_2|}
\]

\[
= \frac{m}{2 \pi \hbar^2 |r_1 - r_2|} + \mathcal{O}(|r_1 - r_2|),
\]

\[
\nu(R) := \nu^{\text{reg}}(R) = \sum_{E_i \leq E_c} \psi_i^*(R) u_i(R) + \frac{i \Delta(R) \hbar^2 |r_1 - r_2|}{4 \pi \hbar^2}
\]

\[
+ \frac{\Delta(R)}{4 \pi^2} \int_{k_c(R)}^{k_c(R)} \frac{k^2 dk}{\mu - \frac{\hbar^2 k^2}{2m} - U(R) + i \gamma}
\]

\[
= \sum_{E_i \leq E_c} \frac{\psi_i^*(R) u_i(R)}{E_i - U(R) + i \gamma}
\]

\[
\delta(R) = \frac{\Delta(R) k_c(R)}{2 \pi \hbar^2} \ln \frac{k_c(R)}{k_F(R)}
\]

\[
h(R) = -\frac{\hbar^2 \nabla^2}{2m} + U(R),
\]

\[
E_c = \frac{\hbar^2 k_c^2(R)}{2m} + U(R) - \mu,
\]
where the cut–off energy \( E_c \) is chosen sufficiently far away from the Fermi level to insure that the rhs of Eq. (11) has converged. As usual one has to take the limit \( \gamma \to 0+ \) at the end of the calculations. The local wave vector \( k(R) \) is real only in the physically allowed region of the Fermi level, where thus the regulated part of the propagator is imaginary. This imaginary part of the regulated propagator is, naturally, exactly canceled by the corresponding imaginary part of the momentum truncated propagator in Eq. (11). In Eq. (12) the imaginary parts canceled and the last term is always real. If the Fermi momentum becomes imaginary, \( k_F(R) = i\kappa_F(R) \) (that occurs outside nuclei for example) one can easily show using Eq. (12) that the regulated anomalous density is still real. The pairing field has thus the simple expression

\[
\Delta(R) = \sum_{E_i \leq E_c} v_i^*(R)u_i(R) 
\]

\[
(16)
\]

\[
\times \left\{ \begin{array}{c}
\frac{\text{4}\pi \hbar^2}[a] \frac{1}{\beta}
\left[ 1 + \frac{2k_c(R)[a]}{\pi} \right] - \frac{k_F(R)[a]}{\pi} \ln \frac{k_c(R) + k_F(R)}{k_c(R) - k_F(R)}
\end{array} \right\}
\]

\[
\times \left\{ \begin{array}{c}
\frac{\text{4}\pi \hbar^2}[a] \frac{1}{\beta}
\left[ 1 + \frac{2k_c(R)[a]}{\pi} \frac{2k_F(R)[a]}{\pi} \right]
\end{array} \right\}
\]

\[
\times \arctan \left( \frac{\kappa_F(R)}{k_c(R)} \right)
\]

where the upper relation should be used if \( \mu - U(R) \geq 0 \) and the lower relation otherwise. (Following usual conventions, the pairing field \( \Delta(R) \) is chosen to be positive inside the nucleus, in complete analogy with the pairing gap.) Surprisingly, these relations look very much like a simple position dependent renormalization of the coupling constant. Since for a given energy cut–off \( E_c \) the value of the cut–off momentum \( k_c(R) \) is obviously larger inside a nucleus than outside, it appears that the effective strength of the pairing interaction is smaller inside than outside, which, surprisingly again, is also in qualitative agreement with phenomenological studies, see for example Refs. [16,23].

It is instructive to apply this recipe to the case of infinite homogeneous matter. After a few simple manipulations one can show that the equation for the gap reads

\[
\frac{1}{k_F} \int_0^{k_c} dk \frac{k^2}{\sqrt{(k^2 - k_F^2)^2 + k_F^2}}
\]

\[
= \frac{\pi}{2k_F[a]} \left[ 1 + \frac{2k_c[a]}{\pi} \frac{2k_F[a]}{\pi} \ln \frac{k_c + k_F}{k_c - k_F} \right],
\]

\[
(17)
\]

where \( k_F^2 = 2m\Delta/\hbar^2 \). Using the methods described in Refs. [8–14] one would not get the term with the log–function. The technical reason is that we used \( \Delta/(\epsilon_i - \mu) \) instead of \( \Delta/\epsilon_i \) in Eqs. (4,7,9,11) respectively [22], which enhances the convergence of the corresponding sums or integrals discussed above. Parametrically we are allowed to make such a substitution as long as \( |k_Fa| \ll 1 \), otherwise one should consider effective range corrections and higher partial waves. Even though the momentum cut–off \( k_c \) appears explicitly here, once this momentum cut–off is sufficiently large, there is no dependence of the gap on the cut–off momentum. The total energy of the system can be computed using usual formulas, since there is no divergence upon taking the limit of a zero range paring interaction. The divergence in the pairing energy

\[
E_P = - \int d^3R \sum_{E_i < E_c} v_i(R)u_i^*(R)\Delta(R)
\]

is exactly canceled by a similar term coming from the mean field [12]. The corresponding normal and anomalous densities used in the energy density are of course computed with an explicit energy cut–off. As before, even though this energy cut–off appears explicitly, the total energy is essentially independent of \( E_c \). We do not derive effective range corrections to the pairing field, even though the corresponding formulas exist [20]. The effective range corrections lead to the appearance of derivative terms in the pairing field and in principle could be useful in order to reach a better description of the nuclear pairing properties. Without proof we shall simply quote the result. In Eq. (1) one has to make the replacement

\[
\Delta(R)v_i(R) \rightarrow \Delta(R)v_i(R) + \frac{|a|\nu_0}{2} \{ \Delta(R)\nabla R^2 v_i(R) + 2\nabla R \cdot [\Delta(R)\nabla R v_i(R)] + \nabla R^2[\Delta(R)v_i(R)] \}
\]

\[
(18)
\]

and a similar replacement in Eq. (2) .

We have implemented this renormalization scheme for the pairing field for both selfconsistent and non–selfconsistent calculations of spherical nuclei. The normal and anomalous densities were computed following the complex energy integration technique extensively used by Fayans and his collaborators [16]. This technique has a number of advantages over the traditional solution of the coupled HFB differential equations (1,2). For illustrative purposes we show in Fig. 1 the neutron pairing field \( \Delta(R) \) for several tin isotopes (\( Z = 50 \)), obtained as a solution of the Eqs. (1,2,16). For the cut–off energy \( E_c \) we used various values ranging from 50 MeV to 300 MeV. The values for the pairing field obtained with all these values for the energy cut–off agreed up to 6 digits, which was also the numerical accuracy for solving the coupled differential equations Eqs. (1,2). Since in a typical selfconsistent calculation one would have to compute both the normal and anomalous densities at the same time, one would have to determine the single–particle wave functions in an energy interval up to about \( E_c \approx 50 \) MeV, or maybe even larger value anyway. The pairing field \( \Delta(R) \) seems to reach convergence however for smaller values of the cut–off energy \( E_c \).
and J. Dobaczewski for discussions.

The value of the coupling constant used was $g = 4\pi\hbar^2|a|/m = 200 \text{ MeV} \cdot \text{fm}^3$.

![FIG. 1. The neutron pairing field (16) as a function of the radial coordinate for a series of isotopes with $Z = 50$ protons.](image)

In conclusion, we have presented a renormalization procedure for the HFB equations in the case of zero range pairing interaction, which is easy to implement for any type of finite or infinite systems and which converges very fast as well. A very interesting byproduct of this scheme is its similarity with a density dependence of the pairing interaction, which leads to a weaker pairing inside nuclei, a conclusion also supported by existing analyses of nuclear masses. The numerical implementation of the present renormalization scheme is straightforward and amounts to very small changes of the existing codes.

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[5] The notations used in Ref. [4] are slightly different from those commonly used in literature, which we try to follow in this work. If one makes the following replacements in the formulas of Ref. [4]: $\rho(x, y) \rightarrow \rho(x, y)$ and $\phi(x, y) \rightarrow \nu(x, y)$, $\psi(x) \rightarrow \psi^*(x)$ one should recover the corresponding formulas in the present work.

[18] In Eq. (27) of Ref. [6] the authors use $E_i$, the eigenvalue of the HFB equations, instead of the correct value $\epsilon_i - \mu$, which is only approximately equal to $E_i$. The definition of the propagator used in Ref. [6] also differs by a sign from the commonly accepted one.
[19] K. Huang, Statistical Mechanics, John Wiley & Sons, New York (1987), pp 230–238. The procedure amounts to the replacement of the short range potential $V(r)$ according to the simple prescription $V(r)\psi(r) \rightarrow g\delta(r)\partial_\nu [r\psi(r)]$, where the coupling constant is determined by the scattering length $g = 4\pi\hbar^2/m$. One can include corrections to the leading order term of the pseudo–potential by making the replacement $a \rightarrow -f/(1+ikf)$, where $f$ is the $s$-wave scattering amplitude. If needed one can also include higher partial waves [8].
[22] The authors of Ref. [6] in an apparent slip suggest that previous authors, in particular those of Ref. [10], have used $1/(\epsilon_i - \mu)$ and not $1/\epsilon_i$ as a subtraction scheme. However, as one can easily establish, none of the previous authors (except Ref. [4]) has ever used the $1/(\epsilon_i - \mu)$ subtraction scheme. As far as we are aware, this scheme was introduced in Ref. [4] for the first time, but never really put to any meaningful use.