Formation of Nuclear "Pasta" in Cold Neutron Star Matter

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density of about $10^{14} \text{ g cm}^{-3}$, rather small compared with the saturation density of symmetric nuclear matter, $\rho_s \approx 2.7 \times 10^{14} \text{ g cm}^{-3}$, the spatial structure changes from a bcc Coulomb lattice of roughly spherical nuclei to a two-dimensional triangular lattice of rod-like nuclei. With the density increased further, it is transformed into a layered structure composed of slab-like nuclei and bubbles. Next, a two-dimensional triangular lattice of rod-like bubbles and a bcc Coulomb lattice of roughly spherical bubbles appear in turn. Finally, at a density of about $\rho_s/2$, the system dissolves into uniform nuclear matter. These changes of the spatial structure, accompanied by a reduction of the total surface area, are governed by the competition between the electrostatic and surface energies. Since slabs and rods look something like lasagna and spaghetti, the phases with positional order of one and two dimensions are often referred to as nuclear “pasta”.

In a separate paper,\(^8\) we more extensively calculated the corresponding equilibrium phase diagrams for zero-temperature neutron star matter at subnuclear densities. In this calculation, we used a compressible liquid-drop model developed by Baym, Bethe and Pethick\(^9\) (hereafter denoted BBP), into which we incorporated uncertainties in the surface tension $E_{\text{surf}}$ and in the proton chemical potential $\mu_p^{(0)}$ in pure neutron matter. With an increase of $E_{\text{surf}}$, accompanied by an increase in the sum of the electrostatic and surface energies, the density $\rho_m$ at which the system becomes uniform decreases. There is also a tendency for $\rho_m$ to decrease with decreasing $\mu_p^{(0)}$. This is because $-\mu_p^{(0)}$ represents the degree to which the gas phase favours the presence of protons in itself. It was shown that while the phases with rod-like bubbles and with spherical bubbles can occur only for unrealistically small $E_{\text{surf}}$, the phases with rod-like nuclei and with slab-like nuclei survive almost independently of $E_{\text{surf}}$ and $\mu_p^{(0)}$.

For these two nuclear phases, there is a direction in which the system is translationally invariant. As noted by Pethick and Potekhin,\(^10\) this situation is geometrically similar to a liquid crystal rather than to a rigid solid. The elastic properties of the nuclear rods and slabs can thus be described by elastic constants used for the corresponding liquid-crystal phases, i.e., columnar phases and smectic A phases, respectively. Pethick and Potekhin expressed these constants in terms of the electrostatic and surface energies.

We predicted in Ref.~8 that the phases with rod-like nuclei and with slab-like nuclei are energetically favoured in the density regime just below $\rho_s$ and at zero temperature, and it is thus important to consider how these nuclei are formed. Such formation requires simultaneous migration of an infinite number of nucleons, in contrast to the case of ordinary chemical reactions. This prevents the nuclear system from crossing the energy barrier formed between the initial and final states in the configuration space via quantum tunnelling. Instead, “pasta” formation can be driven by instabilities with respect to fluctuations around the initial state.

In this paper, we examine the kinds of instabilities that are involved in the formation and decay of rod-like and slab-like nuclei at zero temperature. As such, we first note an instability with respect to quadrupolar deformation of spherical nuclei, as originally investigated in the context of nuclear fission by Bohr and Wheeler.\(^11\)
Pethick and Ravenhall\cite{2} predicted that this instability occurs in the bcc Coulomb lattice and creates elongated nuclear rods when the volume fraction of the liquid phase reaches about 1/8 in the course of compression. We also consider an instability with respect to proton clustering in uniform matter, as considered by BBP.\cite{9} This clustering is induced by the isospin symmetry energy. At an instability point that the system reaches during decompression, the gain due to this energy compensates for the gradient and Coulomb energies produced by the resulting inhomogeneities, and hence a phase with nuclei of some form appears. Pethick and Ravenhall\cite{2} showed that the critical densities for instabilities with respect to quadrupolar deformation and proton clustering are close to the corresponding equilibrium transition points.

We extend these considerations to other changes in nuclear shapes. The possible instability with respect to proton clustering in planar and cylindrical nuclei tends to divide each nucleus into nuclei of lower dimension, while the possible fission-like instability of slab-like and rod-like nuclei tends to lead to the formation of uniform matter and slab-like nuclei, respectively. Using a typical nuclear model, we find that planar and cylindrical nuclei are stable with respect to deformation-induced fission and proton clustering. This, together with the finding that these nuclei are thermodynamically stable and do not exhibit proton drip, suggests the possibility that they persist beyond the equilibrium transition points, e.g., up (down) to a critical point at which the stable uniform (bcc) phase nucleates via quantum tunnelling of the energy barrier in the configuration space. (Note that a critical droplet of the stable phase considered here, which, after forming, develops into bulk material, has a finite size, in contrast to the critical droplets of the “lasagna” and “spaghetti” phases.) Finally, the implications of such persistency for neutron star structure are discussed.

We begin by recalling the mechanism of nuclear fission investigated by Bohr and Wheeler.\cite{11} They regarded a nucleus as a spherical liquid drop of radius $R$ and total charge $Z e$ in which neutrons and protons are distributed uniformly, and then calculated the change in the sum of the surface and electrostatic energies induced by various kinds of deformations. Since we are interested in the limiting case in which the fission barrier vanishes, it is sufficient to consider a slight quadrupolar deformation of a spherical drop, characterized by the distance from the center of the drop to an arbitrary point on the surface with polar angle $\theta$:

$$X(\theta) = R [1 + a_0 + a_2 P_2(\cos \theta) + \cdots].$$

(1)

Here, $a_0$ is the fractional change in the mean radius of the surface, and $a_2 P_2(\cos \theta)$ represents the degree of the quadrupolar deformation. Nuclear saturation leads us to assume the volume of the drop to be invariant under the deformation (1); we may thus write $a_0 = -(1/5)a_2^2$. From the condition that the resulting change in the sum of the surface and electrostatic energies is zero, we can derive the well-known fission-instability relation

$$E_C^{(0)} = 2E_s^{(0)}.$$  

(2)

Here $E_C^{(0)} = (3/5)Z^2e^2/R$ is the Coulomb self energy, and $E_s^{(0)} = 4\pi E_{\text{surf}}R^2$ is the surface energy with the surface tension $E_{\text{surf}}$. 

In neutron star matter at zero temperature, these drops form a bcc lattice embedded in a roughly uniform, neutralizing background of electrons and, at densities above neutron drip, in a sea of neutrons. We assume that the neutron and proton number densities \( n_n \) and \( n_p \) are flat outside and inside the drop, whereas the electron number density \( n_e \) is everywhere constant. (We ignore here the thickness of the surface layer of the drop.) In the Wigner-Seitz approximation, the electrostatic energy of the Wigner-Seitz cell of radius \( R_c \) reads\(^4\)

\[
E_C = E_C^{(0)} \left( 1 - \frac{3}{2} u^{1/3} + \frac{1}{2} u \right),
\]

where \( u = (R/R_c)^3 \) is the volume fraction of the drop. Equilibrium with respect to \( R \) with fixed number densities \( n_i \) \((i = n, p, e)\) leads to

\[
2E_C = E_C^{(0)}.
\]

By using conditions (3) and (4) up to \( O(u^{1/3}) \), we obtain the fission-instability criterion, \( u = 1/8 \), derived by Petrich and Ravenhall. \(^3\) Here we have noted that corrections to condition (2) due to the lattice are of order \( u^{1/3} \). \(^1\) \(^2\) Since \( u \) generally increases with increasing density, we can uniquely determine the density at which the drop becomes unstable with respect to the deformation (1).

We next derive a fission-instability condition appropriate for a nuclear rod, composed of uniformly distributed neutrons and protons (proton charge density \( \rho \)) as well as having a circular section (sectional radius \( R \)) and an infinitely long axis. For this rod, we consider a small sectional deformation of the quadrupole type that is uniform in the direction of the axis. This deformation is characterized by the distance, measured on a horizontal section, from the axis to a point located on the surface with a given angle \( \theta \):

\[
X(\theta) = R(1 + \alpha_0 + \alpha_2 \cos 2\theta + \cdots).
\]

Here, \( \alpha_0 \) is the fractional change in the mean sectional radius of the rod, and \( \alpha_2 \cos 2\theta \) represents the degree of the quadrupole-type deformation. Subject to the condition of constant sectional area, we obtain \( \alpha_0 = -(1/4) \alpha_2^2 \).

Proceeding with the argument for the rod in a vacuum, we are inevitably led to an infinite value of the electrostatic energy per unit length. For this reason, we include the effect of a triangular lattice formed by the rods immersed in a uniform neutralizing background of electrons and in a sea of neutrons. We here again utilize the Wigner-Seitz approximation. The electrostatic energy for a Wigner-Seitz cell of sectional radius \( R_c \) can then be expressed per unit length as

\[
E_C = \frac{(\pi \rho R_c^2)^2}{2} (- \ln u - 1 + u),
\]

where \( u = (R/R_c)^3 \) is the volume fraction of the rod. The deformation (5) produces a change in the electrostatic energy from this \( E_C \) given by

\[
\delta E_C = (\pi \rho R_c^2)^2 \left( -\frac{1}{2} + u \right) \alpha_2^2.
\]
Next, we can obtain the surface energy per unit length of the undeformed rod and its change due to the deformation (5) as

\[ E_s^{(0)} = 2\pi E_{\text{surf}} R, \]

\[ \delta E_s = \frac{3}{4} E_s^{(0)} \alpha_s^2. \]

By combining the instability condition \( \delta E_C + \delta E_s = 0 \), determined by Eqs. (7) and (9), with the size-equilibrium condition (4), in which \( E_C \) and \( E_s^{(0)} \) are now given by Eqs. (6) and (8), we conclude that the rod is stable with respect to the deformation (5) for \( 0 \leq u \leq 1 \). Recall that this deformation is uniform in the direction of the axis. Any small deviation from this uniformity with a fixed nucleon density and volume of the rod, however, results in further energy loss. This is because an increase in the surface energy, which is proportional to the deviation, cannot be compensated for by a change in the Coulomb energy, which is of second order in the deviation.

We remark in passing that deformations of the liquid-crystal type as considered by Pethick and Potekhin\(^{10}\) are elastic, and are not accompanied by the sectional distortion considered here.

In the case of a nuclear slab of thickness \( 2R \) contained in a cell of thickness \( 2R_s \), there is no quadrupole-type deformation corresponding to (1) and (5). All the deformations that hold fixed the nucleon density and volume of the slab have already been examined in the context of elastic deformations by Pethick and Potekhin;\(^{10}\) the slab experiences a restoring force for these deformations.

We next turn to the condition for instability with respect to proton clustering in uniform nuclear matter neutralized and \( \beta \) equilibrated by electrons. This condition was obtained by BBP\(^{9}\) by expanding the energy density functional \( E[n_i(r)] \) \( (i = n, p, e) \) of the system with respect to small density fluctuations \( \delta n_i(r) \) around the homogeneous state. The contribution of first order in \( \delta n_i \) vanishes due to the equilibrium of the unperturbed homogeneous system, while the second order contribution can be described in the spirit of the extended Thomas-Fermi model for finite nuclides\(^{9}\)

\[ E = E_0 = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} v(q) |\delta n_p(q)|^2, \]

where \( E_0 \) is the ground-state energy, \( \delta n_p(q) \) is the Fourier transform of \( \delta n_p(r) \), and \( v(q) \) is the potential of the effective interaction between protons, as given by

\[ v(q) = v_0 + \beta q^2 + \frac{4\pi e^2}{q^2 + k_{TF}^2}. \]

Here,

\[ v_0 = \frac{\partial \mu_p}{\partial n_p} - \left( \frac{\partial \mu_p / \partial n_p}{\partial \mu_n / \partial n_n} \right)^2, \]

\[ \beta = \frac{2}{n_{NM}} (B_{pp} + 2B_{np}\zeta + B_{nn}\zeta^2), \]

\[ \zeta = -\frac{\partial \mu_p / \partial n_n}{\partial \mu_n / \partial n_n}. \]
with $\mu_{n(p)}$ the neutron (proton) chemical potential, $n_{\text{NM}}$ the saturation density of symmetric nuclear matter, and $B_{ij}$ the matrix determining the gradient term in $E$, and $k_{\text{TF}} \approx 0.3n_c^{1/3}$ is the Thomas-Fermi screening length of the ultrarelativistic electrons, modifying the Coulomb term in $E$. The potential $v(q)$ takes a minimum value $v_{\text{min}}$ at $q = Q$, where

$$Q^2 = \left( \frac{4\pi e^4}{\beta} \right)^{1/2} - k_{\text{TF}}^2,$$

(15)

$$v_{\text{min}} = v_0 + 2(4\pi e^2\beta)^{1/2} - \beta k_{\text{TF}}^2.$$

(16)

In the energy expansion up to second order in $\delta n_i$, the condition that uniform nuclear matter becomes unstable with respect to proton clustering reads $v_{\text{min}} = 0$.\(^9\) Generally, $v_{\text{min}}$ is controlled by the bulk contribution $v_0$, which increases as the proton fraction (or, equivalently, the density) increases, while the gradient and Coulomb terms, which tend to suppress the instability, are small positive-definite corrections to $v_0$.\(^13\) Such a density dependence of $v_0$ ensures the presence of a density above (below) which the matter is stable (unstable) with respect to proton clustering.

For possible proton clustering in phases with nuclear rods and nuclear slabs, we may assume that the density fluctuations $\delta n_i$ leading to proton clustering are confined within the nuclei, as it requires too much energy to produce proton clustering in the surrounding neutron gas. Thus, it is useful to consider the effective interaction between protons in the nuclei, corresponding to Eq. (11) in uniform nuclear matter. The bulk term $v_0$ can now be calculated by substituting the neutron and proton number densities inside the nuclei into Eq. (12). We use the condition $v_0 = 0$ when estimating the density at which nuclear matter in the slab or rod becomes unstable with respect to proton clustering. We remark that rigorous extension of the proton effective interaction in uniform nuclear matter to that in the nuclei requires equilibrium neutron and proton distributions, which are inhomogeneous throughout the regions containing protons (see, e.g., Ref. 6), and inclusion of the gradient and Coulomb correction terms. This Coulomb term depends explicitly on the spatial scales $R$ and $R_c$.

We now proceed to calculate the densities at which instabilities arise with respect to deformation-induced fission of spherical nuclei and proton clustering in both uniform matter and nonspherical nuclei. For this purpose, we use the compressible liquid-drop model for nuclei that we constructed in Ref. 8. To obtain this model, we modified the BBP model by using $\mu_p^{(0)} = -C_1 n_n^{1/3}$ [Eq. (4) in Ref. 8] and $E_{\text{surf}} = C_2 \ln(3.5 \text{ MeV}/\mu_p^{(0)}) E_{\text{surf}}^{\text{BBP}}$ [Eq. (6) in Ref. 8], where $\mu_p^{(0)}$ is the neutron chemical potential in the neutron gas, and $E_{\text{surf}}^{\text{BBP}}$ is the BBP-type surface tension [Eq. (7) in Ref. 8]. Here, we set the parameters $C_1$ and $C_2$ as $C_1 = 400 \text{ MeV fm}^{-2}$ and $C_2 = 1$, corresponding to typical values of $\mu_p^{(0)}$ and $E_{\text{surf}}^{\text{BBP}}$ used in recent studies (see Figs. 1 and 2 in Ref. 8). This model allows for a bcc lattice of spherical nuclei or bubbles, a two-dimensional triangular lattice of cylindrical nuclei or bubbles, and a layered lattice of planar nuclei, exhibits the following first order phase
transitions in the ground-state neutron star matter: spherical nuclei \rightarrow cylindrical nuclei \rightarrow planar nuclei \rightarrow uniform matter (with increasing density). The associated discontinuities in the volume fraction $u$ and proton fraction $x$ of the nuclear matter region can be seen from Fig. 1, displaying $u$ and $x$ as functions of the baryon density $n_b$. The important point is that the transition density, $n_b \approx 0.079 \text{ fm}^{-3}$, from the phase with spherical nuclei to that with cylindrical nuclei is close to the fission-like instability point $u \approx 1/8$ or, equivalently, $n_b \approx 0.061 \text{ fm}^{-3}$. This result is consistent with the analysis of Pethick and Ravenhall.\cite{2}

![Figure 1](image1.png)

Fig. 1. The volume fraction $u$ and the proton fraction $x$ of the nuclear matter region, calculated for the ground-state neutron star matter. The solid curve represents the volume fraction, and the dashed curve represents the proton fraction.

![Figure 2](image2.png)

Fig. 2. The minimum value $v_{\text{min}}$ and the bulk contribution $v_0$ of the effective potential between protons lying in the nuclear matter region, calculated for the phases with cylindrical nuclei and planar nuclei, as well as for the uniform phase. The solid curve represents the result for $v_{\text{min}}$, and the dashed curves represent the results for $v_0$.

Figure 2 shows the results for $v_{\text{min}}$, Eq. (16), and $v_0$, Eq. (12), calculated from the compressible liquid-drop model used here. In this calculation, we have set $n_{\text{NM}} = 0.17 \text{ fm}^{-3}$ and $B_{\text{un}} = B_{\text{np}} = B_{\text{pp}} = 8.05 \text{ MeV fm}^3$ so as to reproduce the gradient term in model I of Oyamatsu;\cite{6} this term is consistent with the experimental masses and radii of normal nuclei. From Fig. 2 we can estimate the critical density at which proton clustering occurs in uniform matter to be $\approx 0.120 \text{ fm}^{-3}$. This value agrees well with the phase-equilibrium point, $n_b \approx 0.123 \text{ fm}^{-3}$, which can be obtained from Fig. 1, a feature consistent with the result of Pethick and Ravenhall.\cite{2} The critical wavelength $2\pi Q^{-1}$ is obtained as $2\pi Q^{-1} \approx 20 \text{ fm}$. This is comparable to an internuclear spacing of $\sim 2R_c$; the cell size $R_c$, calculated for the ground state matter with the present nuclear model, is shown in Fig. 3. This suggests the eventual formation of a lattice of nuclei of some form.

We can observe from Fig. 2 that for cylindrical and planar nuclei, the bulk term $v_0$ increases with decreasing density and hence does not become negative, in contrast to the case of uniform nuclear matter. This implies that nonspherical nuclei are stable with respect to proton clustering. The observed density dependence of $v_0$ results from
the facts that $v_0$ increases with increasing proton fraction $x$ and that, as seen in Fig. 1, $x$ increases with decreasing density. Moreover, calculations of the difference between the proton chemical potentials in liquid (nuclear matter) and gas (neutron matter) regions and of the compressibility of the system employing the present nuclear model and following a line of argument used by BBP, show that there is no indication of the drip of protons out of nonspherical nuclei nor a thermodynamic instability of the system including nonspherical nuclei. Consequently, we can conclude that within the present nuclear model, no channel leading to the decay of nonspherical nuclei is opened by the possible instabilities analyzed here.

Finally, we consider a situation in which phases with nuclear rods and nuclear slabs are formed in a neutron star. Under the subsequent spin-down of the star, matter including nonspherical nuclei would be compressed in the equatorial region and decompressed in the polar region. Additionally, accretion of material from a companion star, if occurring, would act to compress the matter in the entire region of the star. During decompression, nonspherical nuclei might not separate into finite nuclei until a critical metastability with respect to quantum tunneling nucleation of the usual bcc phase with roughly spherical nuclei is realized, while during compression, they might persist up to a critical density at which uniform matter nucleates quantum mechanically in a metastable system or the volume fraction $u$ becomes sufficiently close to unity for the system to melt into uniform matter. The possible presence of such critical metastability would act to enlarge the stellar region containing the cylindrical and planar nuclei, as compared with that predicted using the equilibrium configuration. In order to estimate the size of this region, we should take into account dynamical aspects of nucleation processes and finite-temperature effects; the latter tend to reduce the critical metastability by promoting the quantum decay of nonspherical nuclei, and, as we demonstrated in Ref. 8), by melting the low-dimensional lattice of nonspherical nuclei via elastic deformations of long wavelengths.

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