The amplitudes and the electric field vectors of the reflected and transmitted waves, respectively, are related by the single transfer matrix (1) for a plane interface between two semi-infinite media. The complex refractive indices of the media are described by the complex amplitudes of the electric field vectors. The electric field vectors at the interface are projected onto a common plane. The transfer matrix relates the electric field vectors of the incident wave to the reflected and transmitted waves. The transfer matrix is given by

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\mathbf{M} = \begin{pmatrix}
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where \( r_{01} \) and \( t_{01} \) are the Fresnel reflection and transmission coefficients for the interface 01, and \( r_{10} \) and \( t_{10} \) refer to the corresponding coefficients for the interface 10. These Fresnel coefficients are determined by demanding that across the boundary the tangential components of \( \mathbf{E} \) and \( \mathbf{H} \) should be continuous. For nonmagnetic media they are given by

\[
\begin{align*}
    r_{01}^\parallel &= \frac{N_1 \cos \theta_0 - N_0 \cos \theta_1}{N_1 \cos \theta_0 + N_0 \cos \theta_1}, \\
t_{01}^\parallel &= \frac{2N_0 \cos \theta_0}{N_1 \cos \theta_0 + N_0 \cos \theta_1}, \\
r_{01}^\perp &= \frac{N_0 \cos \theta_0 - N_1 \cos \theta_1}{N_0 \cos \theta_0 + N_1 \cos \theta_1}, \\
t_{01}^\perp &= \frac{2N_1 \cos \theta_0}{N_0 \cos \theta_0 + N_1 \cos \theta_1},
\end{align*}
\]

for both basic polarizations. It is worth noting that, although these equations are written for electromagnetic waves, it is possible to translate all the results for particle-wave scattering, since there is a one-to-one correspondence between the propagation in an interface between two media of electromagnetic waves and of the nonrelativistic particle waves satisfying Schrödinger equation.

The linearity revealed by Eqs. (3) suggests the use of 2 \times 2 matrix methods. However, Eqs. (3) links output to input fields, while the standard way of treating this topic is by relating the field amplitudes at each side of the interface. Such a relation is expressed as

\[
\begin{bmatrix} E_{01}^{(+)} \\ E_{01}^{(-)} \end{bmatrix} = \mathbf{I}_{01} \begin{bmatrix} E_{10}^{(+)} \\ E_{10}^{(-)} \end{bmatrix}.
\]

The choice of these column vectors is motivated from the optics of layered media, since it is the only way of calculating the field amplitudes at each side of every layer by an ordered product of matrices.

We shall call \( \mathbf{I}_{01} \) the interface transfer matrix and, from Eqs. (6), is given by

\[
\mathbf{I}_{01} = \frac{1}{t_{01}} \begin{bmatrix} 1 & -r_{01} \\ r_{01} & t_{01}t_{10} - r_{01}r_{10} \end{bmatrix}.
\]

By using a matrix formulation of the boundary conditions one can factorize the interface transfer matrix \( \mathbf{I}_{01} \) in the new and remarkable form (that otherwise one can also check directly using the Fresnel formulas)

\[
\mathbf{I}_{01} = \mathbf{R}^{-1}(\pi/4) \begin{bmatrix} \cos \theta_1/\cos \theta_0 & 0 & 0 \\ 0 & N_1/N_0 \end{bmatrix} \mathbf{R}(\pi/4),
\]

where

\[
\mathbf{R}(\pi/4) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]

represents the matrix of a clockwise rotation of angle \( \pi/4 \). Now, it is straightforward to convince oneself that a diagonal matrix postmultiplied by \( \mathbf{R}(\pi/4) \) and premultiplied by its inverse is always of the form

\[
\begin{bmatrix} a & b \\ b & a \end{bmatrix},
\]

\( a \) and \( b \) being, in general, complex numbers. This result implies that \( \mathbf{I}_{01} \) in Eq. (5) must be also of this form, which turns out the constrainst

\[
r_{10} = -r_{01},
\]

\[
1 + r_{01}r_{10} = t_{01}t_{10}.
\]

This applies to both basic polarizations by the simple attachment of a label to all the coefficients and constitutes an alternative algebraic demonstration of the well-known Stokes relations without resorting to the usual time-reversal argument. Similar results can be also derived in particle scattering from the unitarity requirement on the \( S \) matrix. However, note that the equality \( |r_{10}| = |r_{01}| \) implied by Eq. (7) can become counterintuitive when applied to particle reflection, since one might expect stronger reflection for particle waves moving up in a potential gradient than for those going down. In fact, these relations, as emphasized by Lekner, ensure that the reflectivity is exactly the same in the two cases, unless there is total internal reflection.

In summary, these Stokes relations allow one to write

\[
\mathbf{I}_{01} = \frac{1}{t_{01}} \begin{bmatrix} 1 & -r_{01} \\ r_{01} & 1 \end{bmatrix}.
\]

It is worth noting that the inverse matrix satisfies \( \mathbf{I}_{01}^{-1} = \mathbf{I}_{10} \) and then describes the interface taken in the reverse order. The physical meaning of these matrix manipulations is analyzed in Section 3.

### III. Renormalization of Field Amplitudes

From Eqs. (9) one directly obtain that, for both basic polarizations, we have

\[
\det \mathbf{I}_{01} = \det \mathbf{I}_{10} = \frac{N_1 \cos \theta_1}{N_0 \cos \theta_0} \neq 1.
\]
For the reasons that will become clear in Section IV, it is adequate to renormalize the field amplitudes to ensure that the transfer matrix has always unit determinant. To this end, let us define

\[ \varepsilon^{(+)}_b = \sqrt{N_0 \cos \theta_0} \, E^{(+)}_b, \]
\[ \varepsilon^{(-)}_b = \sqrt{N_1 \cos \theta_1} \, E^{(-)}_b. \]

Accordingly, the action of the interface is described now by

\[ \begin{pmatrix} \varepsilon^{(+)}_b \\ \varepsilon^{(-)}_b \end{pmatrix} = i_{01} \begin{pmatrix} \varepsilon^{(+)}_1 \\ \varepsilon^{(-)}_1 \end{pmatrix}, \]

where the renormalized interface matrix is

\[ i_{01} = R^{-1}(\pi/4) \begin{pmatrix} 1/\xi_{01} & 0 \\ 0 & \xi_{01} \end{pmatrix} R(\pi/4) \]
\[ = \frac{1}{2} \begin{pmatrix} \xi_{01} + 1/\xi_{01} & \xi_{01} - 1/\xi_{01} \\ \xi_{01} - 1/\xi_{01} & \xi_{01} + 1/\xi_{01} \end{pmatrix}, \]

and the factor \( \xi_{01} \) has the values

\[ \xi_{01}^\parallel = \sqrt{N_1 \cos \theta_0 / N_0 \cos \theta_1} = \sqrt{\sin(2\theta_0) / \sin(2\theta_1)}, \]
\[ \xi_{01}^\perp = \sqrt{N_1 \cos \theta_0 / N_1 \cos \theta_1} = \sqrt{\tan \theta_1 / \tan \theta_0}. \]

Other way of expressing these relations is

\[ \xi_{01}^{\parallel} \xi_{01}^{\perp} = \frac{\cos \theta_0}{\cos \theta_1}, \]
\[ \xi_{01}^{\parallel}/\xi_{01}^{\perp} = \frac{N_1}{N_0}. \]

It is now evident from Eq. (15) that the renormalized interface matrix satisfies \( \det i_{01} = +1 \), as desired. Moreover, by taking into account the general form given in Eq. (11), we can reinterpret \( i_{01} \) in terms of renormalized Fresnel coefficients as

\[ i_{01} = \frac{1}{\hat{t}_{01}} \begin{pmatrix} 1 & \hat{\gamma}_{01} \\ \hat{\gamma}_{01} & 1 \end{pmatrix}, \]

where

\[ \hat{\gamma}_{01} = \frac{\xi_{01} - 1/\xi_{01}}{\xi_{01} + 1/\xi_{01}}, \]
\[ \hat{t}_{01} = \frac{2}{\xi_{01} + 1/\xi_{01}}, \]

which satisfy

\[ \hat{\gamma}_{01}^2 + \hat{t}_{01}^2 = 1. \]

This relation does not trivially reduce to the conservation of the energy flux on the interface, because the complex reflection and transmission coefficients appear in the form \( \hat{\gamma}_{01}^2 + \hat{t}_{01}^2 \) instead of \( |\hat{r}_{01}|^2 \) and \( |\hat{t}_{01}|^2 \). In fact, it can be seen as a consequence of the renormalization factors appearing in the definition (3) that project the direction of the corresponding wave vector onto the normal to the boundary.

The Fresnel coefficients can be obtained from the renormalized ones as

\[ r_{01} = \hat{r}_{01}, \]
\[ t_{01} = \frac{\sqrt{N_1 \cos \theta_1}}{N_0 \cos \theta_0} \hat{\gamma}_{01}. \]

It is clear from Eqs. (10) that the single parameter \( \xi_{01} \) gives all the information about the interface, even for absorbing media or when total reflection occurs. We have \( i_{01}^{\parallel} = i_{01}^{\perp} \); that is, the inverse also describes the interface taken in the reverse order. Thus, \( \xi_{01} = 1/\xi_{01} \) and it follows that

\[ \hat{r}_{01} = -\hat{r}_{01}, \]
\[ \hat{t}_{01} = \hat{t}_{01}. \]

In Fig. 2 we have plotted the behavior of \( \xi_{01}^{\parallel} \) and \( \xi_{01}^{\perp} \) as a function of the angle of incidence \( \theta_0 \), for an interface air-glass \( (N_0/N_1 = 2/3) \) and, for the purpose of comparison, the corresponding values of \( r_{01}^{\parallel} \) and \( r_{01}^{\perp} \). The discussion about these amplitude coefficients and the corresponding phase shifts can be developed much in the same way as it is done in most of the undergraduate optics textbooks.

**IV. THE INTERFACE AS A HYPERBOLIC ROTATION**

The definition of the renormalized transfer matrix for an interface in Eq. (11) may appear, at first sight, rather artificial. In this Section we shall interpret its meaning by recasting it in an appropriate form that will reveal the origin of the rotation matrices \( R(\pi/4) \).

To simplify as much as possible the discussion, let us assume that we are dealing with an interface between two transparent media when no total reflection occurs. In this relevant case, the Fresnel reflection and transmission coefficients, and therefore \( \xi_{01} \), are real numbers. Let us introduce a new parameter \( \xi_{01} \) by

\[ \xi_{01} = \exp(\xi_{01}/2). \]

Then, the action of the interface can be expressed as

\[ \begin{pmatrix} \varepsilon^{(+)}_b \\ \varepsilon^{(-)}_b \end{pmatrix} = \begin{pmatrix} \cosh(\xi_{01}/2) & \sinh(\xi_{01}/2) \\ \sinh(\xi_{01}/2) & \cosh(\xi_{01}/2) \end{pmatrix} \begin{pmatrix} \varepsilon^{(+)}_1 \\ \varepsilon^{(-)}_1 \end{pmatrix}, \]

where the renormalized Fresnel coefficients can be written now as
\[ \hat{r}_{01} = \tanh(\xi_{01}/2), \]
\[ \hat{t}_{01} = \frac{1}{\cosh(\xi_{01}/2)}. \]

Given the importance of this new reformulation of the action of an interface, some comments seem pertinent: it is clear that the reflection coefficient can be always expressed as a hyperbolic tangent, whose addition law is simple. In fact, such an important result was first derived by Khashan and is the origin of several approaches for treating the reflection coefficient of layered structures, including bilinear or quotient functions, that are just of the form \( \frac{1}{\cosh(x)} \). However, the transmission coefficient for these structures seems to be (almost) safely ignored in the literature, because it behaves as a hyperbolic tangent, whose addition law is more involved.

Now the meaning of the rotation \( R(\pi/4) \) can be put forward in a clear way. To this end, note that the transformation \( \xi_{01}/2 \) acting on the complex field variables \( [e(\pm), e(-\pm)] \). As it is usual in hyperbolic geometry, it is convenient to study this transformation in a coordinate frame whose new axes are the bisecting lines of the original one. In other words, in this frame whose axes are rotated \( \pi/4 \) respect to the original one, the new coordinates are
\[ \left( \begin{array}{c} \hat{e}(\pm) \\ \hat{e}(\pm) \end{array} \right) = R(\pi/4) \left( \begin{array}{c} e(\pm) \\ e(-\pm) \end{array} \right) \]
for both 0 and 1 media, and the action of the interface is represented by the matrix
\[ \left( \begin{array}{cc} 1/\xi_{01} & 0 \\ 0 & \xi_{01} \end{array} \right) \left( \begin{array}{c} \hat{e}(\pm) \\ \hat{e}(\pm) \end{array} \right) , \]
which is a squeezing matrix that scales \( \hat{e}_0(\pm) \) down to the factor \( \xi_{01} \) and \( \hat{e}_1(\pm) \) up by the same factor.

Furthermore, the product of these complex coordinates remains constant
\[ e_0(\pm)e_0(-\pm) = \hat{e}_0(\pm)\hat{e}_0(-\pm) , \]
or
\[ |e_0(\pm)|^2 - |e_0(-\pm)|^2 = |\hat{e}_0(\pm)|^2 - |\hat{e}_0(-\pm)|^2 , \]
which appears as a fundamental invariant of any interface. In these renormalized field variables it is nothing but the hyperbolic invariant of the transformation. When viewed in the original field amplitudes it reads as
\[ N_0 \cos \theta_0 \{|E_0(\pm)|^2 - |E_0(-\pm)|^2\} = N_1 \cos \theta_1 \{|E_1(\pm)|^2 - |E_1(-\pm)|^2\} , \]
which was assumed as a basic axiom by Vigoureux and Grossel.

To summarize this discussion at a glance, in Fig. 3 we have plotted the unit hyperbola \( [e(\pm)]^2 - [e(-\pm)]^2 = 1 \), assuming real values for all the variables. The interface action transforms then the point 1 into the point 0. The same hyperbola, when referred to its proper axes, appears as \( \hat{e}(\pm)\hat{e}(\pm) = 1 \).

V. THE PHYSICAL MEANING OF INTERFACE COMPOSITION

To conclude, it seems adequate to provide a physical picture of the matrix manipulations we have performed in this paper. First, the inverse of an interface matrix, as pointed out before, describes the interface taken in the reverse order.

Concerning the product of interface matrices, this operation has physical meaning only when the second medium of the first interface is identical to the first medium of the second one. In this case, let us consider the interfaces 01 and 12. A direct calculation from Eqs. (7) shows that
\[ I_{12}I_{12} = I_{12} , \]
for both basic polarizations, which is equivalent to the constraints
\[ r_{02} = \frac{r_{01} + r_{12}}{1 + r_{01}r_{12}} , \]
\[ t_{02} = \frac{t_{01}t_{12}}{1 + r_{01}r_{12}} . \]
Note that the reflected-amplitude composition behaves as a tanh addition law, just as in the famous Einstein addition law for collinear velocities: no matter what values the reflection amplitudes \( r_{01} \) and \( r_{12} \) (subject only to \( |r_{01}| \leq 1 \) and \( |r_{12}| \leq 1 \)) have, the modulus of the composite amplitude \( |r_{02}| \) cannot exceed the unity. Alternatively, we have
\[ \hat{r}_{02} = r_{02} = \tanh(\xi_{02}/2) = \tanh(\xi_{01}/2 + \xi_{12}/2) \]
which leads directly to the first one of Eqs. (6).

On the contrary, the transmitted amplitudes compose as a sech, whose addition law is more involved and is of little interest for our purposes here. Obviously, for this interface composition to be realistic one cannot neglect the wave propagation between interfaces. However, this is not an insuperable drawback. Indeed, let us consider a single layer of a transparent material of refractive index \( N_1 \) and thickness \( d_1 \) sandwiched between two semi-infinite media 0 and 2. Let
\[ \beta_1 = \frac{2\pi}{\lambda} N_1 d_1 \cos \theta_1 \]
denote the phase shift due to the propagation in the layer, \( \lambda \) being the wavelength in vacuum. A standard calculation gives for the reflected and transmitted amplitudes
by this layer the Airy-like functions
\[
R_{012} = \frac{r_{01} + r_{12} \exp(-i2\beta_1)}{1 + r_{01}r_{12} \exp(-i2\beta_1)},
\]
\[
T_{012} = \frac{t_{01}t_{12} \exp(-i\beta_1)}{1 + r_{01}r_{12} \exp(-i2\beta_1)}.
\]

The essential point is that in the limit \( \beta_1 = 2n\pi \) \((n = 0, 1, \ldots)\), which can be reached either when \( d_1 \to 0 \) or when the plate is under resonance conditions, then \( R_{012} \to r_{02} \) and \( T_{012} \to t_{02} \), and we recover Eqs. (32). This gives perfect sense to the matrix operations in this work.

VI. CONCLUSIONS

We have discussed in this paper a simple transformation that introduces remarkable simplicity and symmetry in the physics of a plane interface. In these new suitable variables the action of an interface appears in a natural way as a hyperbolic rotation, which is the natural arena of special relativity.

This formalism does not add any new physical ingredient to the problem at hand, but allows one to obtain previous results (like Fresnel formulas or Stokes relations) in a particularly simple and elegant way that appears closely related to other fields of physics.

REFERENCES

FIG. 1: Wave vectors of the incident, reflected, and transmitted fields at the interface 01.

FIG. 2: Plot of the factor $\xi_{01}$ and $r_{01}$ as functions of the angle of incidence $\theta_0$ (in degrees) for both basic polarizations for an interface air-glass ($N_0 = 1, N_1 = 1.5$). The marked points correspond to the Brewster angle.


FIG. 3: Schematic plot of the hyperbolic rotation performed by the interface 01 that transforms on the unit hyperbola the point 1 into the point 0.